Error estimates for scalar conservation laws by a kinetic approach

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ERROR ESTIMATES FOR SCALAR CONSERVATION LAWS BY A KINETIC APPROACH

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We use the kinetic approach of Perthame and Tadmor (1991) to calculate the error estimates for general scalar conservation laws governing problems in gas dynamics or fluid mechanics in general. The Kružkov and Kuznetsov techniques are generalized to this method, and an error bound of order $\sqrt{\varepsilon}$ (where ε is the mean free path) is obtained.

1. Introduction

The kinetic approaches for scalar conservation laws have attracted attention for the last few years. Much work has been done in this field, however, important developments such as error estimates have not been totally investigated. The Perthame-Tadmor approach [16] is here used to derive the error bound for the following Cauchy problem of conservation laws:

$$\partial_t u + \sum_{i=1}^{N} \partial_{x_i}(A_i(u)) = 0, \tag{1.1}$$

$$u(0,x) = u_0(x), \tag{1.2}$$

where $(t,x) \in [0,+\infty[\times\mathbb{R}^N$, $u = u(t,x) \in \mathbb{R}$, and where the functions $A_i'(\cdot) = a_i(\cdot)$ are locally Lipschitz in $\mathbb{R}$.

It is well known that there is a lack of regularity of the solutions for $t > 0$, so that we generally look for weak solutions for (1.1)-(1.2). Moreover, there is a unique weak solution of physical interest, satisfying the additional Kružkov [10] entropy condition:

$$\partial_t(|u-\ell|) + \sum_{i=1}^{N} \partial_{x_i}\big(\operatorname{sgn}(u-\ell)(A_i(u) - A_i(\ell))\big) \leq 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N), \tag{1.3}$$

$\ell \in \mathbb{R}$ being any real scalar. For more details, see for example the classical book of Smoller [17] and the references therein. Now, the problem (1.1)–(1.3), which can be seen as

a macroscopic interpretation of gas dynamics, is generally approached by the viscosity method or by methods of numerical purpose.

In this work, we are concerned with a simple microscopic model which is not only adapted to solve the problems of existence, uniqueness, and convergence for scalar conservation laws, but adapted to calculate the error of convergence as well.

Our method consists in using the Perthame-Tadmor model which is analogous to the models of the kinetic theory of gases. This is a BGK model derived from Bhatnagar et al. [1], and contains most of the basic properties of fluid dynamics, mainly conservation of mass, momentum, and energy (see the book by Bouchut et al. [2], and/or the book by Cercignani [6]). A function $f(t,x,v)$ where v is "velocity" is considered. It represents the distribution of "molecules" located at x, having a speed v, at time t.

Let us recall that a discrete-time version of the Perthame-Tadmor approach [16] was previously and simultaneously proposed by Brenier [4, 5], by Giga and Miyakawa [9], and by Miyakawa [14]. In [16], the authors prove the convergence towards the unique solution of (1.1)–(1.3). We are concerned here with the exact error of convergence.

The paper is organized as follows. In Section 2, we introduce the method of Perthame and Tadmor, giving the main definitions and results. In Section 3, we develop an approximate entropy inequality with a precise "entropy" remainder. Then we give the error estimates for the corresponding conservation law based on the Kružkov-Kuznetsov techniques in Section 4. Section 5 concerns some parabolic case.

2. Around the kinetic model

In this section, we give the main definitions and results due to Perthame and Tadmor. In 1991, they proposed the following BGK (say pseudo-BGK) model:

$$\partial_t f_\varepsilon + \sum_{i=1}^{N} a_i(v)\partial_{x_i} f_\varepsilon = \frac{1}{\varepsilon}\left[\chi_{\int_v f_\varepsilon(t,x,v)dv} - f_\varepsilon\right], \tag{2.1}$$

$$f_\varepsilon(0,x,v) = f_0(x,v), \tag{2.2}$$

as a kinetic interpretation of the conservation law (1.1)-(1.2) when $\varepsilon \to 0$ (see [16] and also Lions et al. [13]). The scalar function $f_\varepsilon = f_\varepsilon(t,x,v)$ is the kinetic solution, $v \in \mathbb{R}$, χ_w ($w \in \mathbb{R}$) is a signature function (the pseudo-Maxwellian), and $\varepsilon > 0$ the mean free path in comparison with the Boltzmann theory. If such a solution f_ε is defined, the approximate solution for (1.1) is simply the function

$$u_\varepsilon(t,x) = \int_{v \in \mathbb{R}} f_\varepsilon(t,x,v)dv, \tag{2.3}$$

while the signature χ_w is given by

$$\chi_w(v) = \begin{cases} \operatorname{sgn} w & \text{if } (w-v)v \geq 0, \\ 0 & \text{otherwise}, \end{cases} = \begin{cases} 1 & \text{if } 0 \leq v \leq w, \\ -1 & \text{if } w \leq v \leq 0, \\ 0 & \text{else}. \end{cases} \tag{2.4}$$

Remark 2.1. We immediately deduce the following formulae:

$$u_\varepsilon(t,x) = \int_{v\in\mathbb{R}} \chi_{u_\varepsilon(t,x)}(v)dv, \tag{2.5}$$

and $|w| = \int_{v\in\mathbb{R}} |\chi_w(v)|dv$ for any $w \in \mathbb{R}$. Moreover,

$$A_i(u) - A_i(w) = \int_v a_i(v)(\chi_u(v) - \chi_w(v))dv, \quad 1 \le i \le N. \tag{2.6}$$

2.1. The well-posedness of the kinetic problem. Consider the linear version of (2.1) (without the term χ_{u_ε} at the right-hand side). The classical method of characteristics involves solving the equations

$$\frac{dX}{dt}(t,v) = a(v),$$
$$X(0,v) = X_0. \tag{2.7}$$

For $\phi(t) = f_\varepsilon(t,X(t),v)$, we have $d\phi/dt = (-t/\varepsilon)\phi$, that is, $\phi = C \cdot e^{-(1/\varepsilon)t}$. With the varying constant method, the solution of (2.1) is of the following form:

$$f_\varepsilon(t,x,v) = e^{-t/\varepsilon} f_\varepsilon(0, x - \tau a(v), v) + \frac{1}{\varepsilon} \int_{\tau=0}^t e^{(\tau-t)/\varepsilon} \chi_{u_\varepsilon(\tau, x-(t-\tau)a(v))}(v)d\tau \tag{2.8}$$

(see Omrane [15]). We also have the L^1 contraction

$$\left\|f_\varepsilon(t,x,v) - g_\varepsilon(t,x,v)\right\|_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v)} \le e^{-t/\varepsilon}\left\|f_\varepsilon(0,x,v) - g_\varepsilon(0,x,v)\right\|_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v)}$$

$$+ \frac{1}{\varepsilon}\int_v \int_{\tau=0}^t e^{(\tau-t)/\varepsilon}\left\|\chi_{u_\varepsilon(\tau,x)}(v) - \chi_{\omega_\varepsilon(\tau,x)}(v)\right\|_{L^1(\mathbb{R}_x^N)}d\tau\, dv, \tag{2.9}$$

(here $w_\varepsilon = \int_v g_\varepsilon$) where f_ε and g_ε are two solutions of (2.1). This gives

$$\left\|f_\varepsilon(t,x,v) - g_\varepsilon(t,x,v)\right\|_{L^1(\mathbb{R}^N \times \mathbb{R}_v)} \le e^{-t/\varepsilon}\left\|f_\varepsilon(0,x,v) - g_\varepsilon(0,x,v)\right\|_{L^1(\mathbb{R}^N \times \mathbb{R}_v)}$$

$$+ (1 - e^{-t/\varepsilon}) \sup_{0\le\tau\le t}\left\|f_\varepsilon(\tau,x,v) - g_\varepsilon(\tau,x,v)\right\|_{L^1(\mathbb{R}^N \times \mathbb{R}_v)}. \tag{2.10}$$

Since $0 < (1 - e^{-t/\varepsilon}) < 1$ if $f_\varepsilon(0,x,v) = g_\varepsilon(0,x,v)$, then there is a unique solution to (2.1). Moreover, by (2.10), the kinetic solution is continuously dependent on the initial data. We also have the contraction inequality

$$\left\|f_\varepsilon(t,x,v) - g_\varepsilon(t,x,v)\right\|_{L^1(\mathbb{R}^N \times \mathbb{R}_v)} \le \left\|f_\varepsilon(0,x,v) - g_\varepsilon(0,x,v)\right\|_{L^1(\mathbb{R}^N \times \mathbb{R}_v)}. \tag{2.11}$$

THEOREM 2.2. *The model (2.1) is well-posed in $L^\infty([0,T]; L^1(\mathbb{R}^N \times \mathbb{R}_v))$, and the inequality of contraction (2.11) holds true.*

Proof. See the article of Perthame and Tadmor [16] for the proof. □

2.2. Finite speed of propagation. In [16], we find the following. We first notice that for any $t \in [0, T]$,

$$\|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \int_v \|f_\varepsilon(t, \cdot, v)\|_{L^\infty(\mathbb{R}^N)} dv \leq \int_v \|f_\varepsilon(0, \cdot, v)\|_{L^\infty(\mathbb{R}^N)} dv, \qquad (2.12)$$

with the help of Jensen's inequality. Denoting $u_\infty = \|u_\varepsilon(t, x)\|_{L^\infty(\mathbb{R}^N_x)}$, the support of $f_\varepsilon(t, x, \cdot)$ is showed to be contained in K_v:

$$K_v = (\text{supp}\, (f_\varepsilon(0, x, \cdot))) \cup [-u_\infty, +u_\infty]. \qquad (2.13)$$

Consequently, the term $a(v) \cdot \nabla_x f_\varepsilon$ is also supported in K_v, and the elements $|a_i(v)|$ are bounded by

$$a_\infty = \max\{|a_i(v)|,\ v \in K_v,\ i = 1, \ldots, N\}. \qquad (2.14)$$

Remark 2.3. In fact, the support of $f_\varepsilon(t, x, \cdot)$ is mainly contained in $[-u_\infty, +u_\infty]$, because of the decay of the initial data in (2.8).

2.3. The kinetic entropies $H(f_\varepsilon)$. The notion of "kinetic" entropy has the same sense as the classical entropy for conservation laws. It is just a microscopic version of it, and the name is due to the presence of the speed vector $a(v)$.

Definition 2.4. The kinetic entropies are the functions $H(f_\varepsilon)$ such that for any positive test function $\varphi \in \mathcal{D}(\mathbb{R}^t_+ \times \mathbb{R}^N_x)$,

$$-\int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}_v} ([\partial_t + a(v)\nabla_x]\varphi) H(f_\varepsilon)\, dv\, dx\, dt \leq 0 \qquad (2.15)$$

(according to the H-theorem in Boltzmann theory).

Multiplying (2.1) by $\text{sgn}(f_\varepsilon - \chi_\ell)$, $\ell \in \mathbb{R}$, and integrating by parts, we have

$$-\int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}_v} ([\partial_t + a(v)\nabla_x]\varphi)\, |f_\varepsilon - \chi_\ell|\, dv\, dx\, dt$$

$$= -\frac{1}{\varepsilon} \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}_v} (f_\varepsilon - \chi_\ell + \chi_\ell - \chi_{u_\varepsilon})\varphi \, \text{sgn}\,(f_\varepsilon - \chi_\ell)\, dv\, dx\, dt \qquad (2.16)$$

$$\leq -\frac{1}{\varepsilon} \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}_v} |f_\varepsilon - \chi_\ell|\, \varphi\, dv\, dx\, dt + \int_0^{+\infty} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |u_\varepsilon - \ell|\, \varphi\, dx\, dt \leq 0,$$

since $|u_\varepsilon - \ell| = |\int_v f_\varepsilon - \chi_\ell dv|$.

The functions $H(f_\varepsilon) = |f_\varepsilon - \chi_\ell|$ are the kinetic entropies. Notice that ℓ is any given scalar (it may also be any function uniformly bounded in t and x).

3. The approximate entropy solutions

In this section, we show that the Kružkov method can be applied. Let $(s, y) \in (0, T) \times \mathbb{R}^N$ be the new variables, we then define $Q = (0, T)^2 \times \mathbb{R}_x^N \times \mathbb{R}_y^N$ and $\varphi = \varphi(t, s, x, y)$.

PROPOSITION 3.1. *Let $u = u(s, y)$ be the unique entropy solution of (1.1)–(1.3), and $f_\varepsilon \in L^\infty([0, T]; L^1(\mathbb{R}^N \times \mathbb{R}_v))$ the approximate kinetic solution given by (2.1). Then*

$$
- \int_Q |u_\varepsilon(t, x) - u(s, y)| \left((\partial_t + \partial_s)\varphi\right) dQ
$$

$$
- \int_Q \mathrm{sgn}\,(u_\varepsilon - u)\,(A(u_\varepsilon(t, x)) - A(u(s, y)))\,((\nabla_x + \nabla_y)\varphi)\,dQ \tag{3.1}
$$

$$
\leq \int_Q \int_v |\partial_t \varphi|\,|\chi_{u_\varepsilon} - f_\varepsilon|\,dv\,dQ + a_\infty \int_Q \int_v |\nabla_x \varphi|\,|\chi_{u_\varepsilon} - f_\varepsilon|\,dv\,dQ
$$

for any test function $\varphi \in \mathcal{D}(Q)$.

Proof. Recall that $\ell \in \mathbb{R}$ can be chosen as any $\ell = \ell(s, y)$ for example, uniformly bounded with respect to s and y. We write

$$
- \int_Q \int_v \left([\partial_t + a(v)\nabla_x]\varphi\right) |\chi_{u_\varepsilon} - \chi_\ell|\,dv\,dQ
$$

$$
= - \int_Q \int_v \left([\partial_t + a(v)\nabla_x]\varphi\right) \left(|\chi_{u_\varepsilon} - \chi_\ell| - |f_\varepsilon - \chi_\ell|\right) dv\,dQ \tag{3.2}
$$

$$
- \int_Q \int_v \left([\partial_t + a(v)\nabla_x]\varphi\right) |f_\varepsilon - \chi_\ell|\,dv\,dQ.
$$

With the definition on the kinetic entropies and the following triangle inequality:

$$
\left|\,|\chi_{u_\varepsilon} - \chi_\ell| - |f_\varepsilon - \chi_\ell|\,\right| \leq |\chi_{u_\varepsilon} - f_\varepsilon|, \tag{3.3}
$$

we obtain

$$
- \int_Q \int_v \left([\partial_t + a(v)\nabla_x]\varphi\right) |\chi_{u_\varepsilon} - \chi_\ell|\,dv\,dQ \leq \int_Q \int_v |[\partial_t + a(v)\nabla_x]\varphi|\,|\chi_{u_\varepsilon} - f_\varepsilon|\,dv\,dQ. \tag{3.4}
$$

Using Remark 2.1, and the finite speed of propagation property (2.14), we deduce

$$
- \int_Q |u_\varepsilon(t, x) - \ell(s, y)|\,\partial_t \varphi\,dQ
$$

$$
- \int_Q \mathrm{sgn}\,(u_\varepsilon - \ell)\,(A(u_\varepsilon(t, x)) - A(\ell(s, y)))\,\nabla_x \varphi\,dQ \tag{3.5}
$$

$$
\leq \int_Q \int_v |\partial_t \varphi|\,|\chi_{u_\varepsilon} - f_\varepsilon|\,dv\,dQ + a_\infty \int_Q \int_v |\nabla_x \varphi|\,|\chi_{u_\varepsilon} - f_\varepsilon|\,dv\,dQ.
$$

From another side, the exact entropy solution $u(s, y)$ of (1.1)–(1.3) verifies for all $\ell = \ell(t, x) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_x^N)$ that

$$
\begin{aligned}
&- \int_Q |u(s, y) - \ell(t, x)| \, \partial_s \varphi \, dQ \\
&- \int_Q \operatorname{sgn}(u(s, y) - \ell(t, x)) \left(A(u(s, y)) - A(\ell(t, x)) \right) \nabla_y \varphi \, dQ \leq 0.
\end{aligned}
\tag{3.6}
$$

If we replace $\ell(s, y)$ by $u(s, y)$ (the exact solution) in (3.5), and $\ell(t, x)$ by $u_\varepsilon(t, x)$ in (3.6), then we sum up to obtain (3.1). $\qquad \square$

3.1. The test functions. We define φ as follows:

$$
\varphi(t, s, x, y) = \psi_\eta \left(\frac{t+s}{2} \right) \theta_h(t - s) \prod_{i=1}^N \theta_h(x_i - y_i),
\tag{3.7}
$$

where ψ_η is the characteristic function defined by

$$
\psi_\eta(t) = \chi^\eta \left(\frac{t - t_0}{\Delta t} \right), \qquad \chi^\eta(\tau) = \int_0^\tau \theta_\eta(s) \, ds + \int_\tau^1 \theta_\eta(s - 1) \, ds,
\tag{3.8}
$$

and where the function θ_h is the classical mollifier. For $h > 0$, we also have $-h \leq \theta_h(\cdot) \leq h$, $\int_{-h}^h \theta_h(s) \, ds = 1$, $\theta_h(s) = (1/h) \theta(s/h)$, where θ is a regularized Dirac function of support $[-1, 1] : \int_{-1}^1 \theta(s) \, ds = 1$. We have the following corollary.

COROLLARY 3.2. *Under the hypothesis of Proposition 3.1, the following entropy inequality holds:*

$$
\begin{aligned}
&- \int_Q |u_\varepsilon(t, x) - u(s, y)| \, [\psi_\eta]' \left(\frac{t+s}{2} \right) \theta_h(t - s) \prod_{i=1}^N \theta_h(x_i - y_i) \, dQ \\
&\leq \int_Q \int_v |\partial_t \varphi| \, |\chi_{u_\varepsilon} - f_\varepsilon| \, dv \, dQ + a_\infty \int_Q \int_v |\nabla_x \varphi| \, |\chi_{u_\varepsilon} - f_\varepsilon| \, dv \, dQ.
\end{aligned}
\tag{3.9}
$$

Proof. To get (3.9), it is convenient to choose φ as in (3.7) (see [15, 18]). With this, we obtain

$$
\partial_{x_i} \varphi + \partial_{y_i} \varphi = 0, \quad \text{for } i = 1, \ldots, N,
$$

$$
\partial_t \varphi + \partial_s \varphi = [\psi_\eta]' \left(\frac{t+s}{2} \right) \theta_h(t - s) \prod_{i=1}^N \theta_h(x_i - y_i),
\tag{3.10}
$$

hence we have the left-hand side of (3.9). $\qquad \square$

3.2. Bounds. Let us begin with some properties of the test function. For t_0 and t_1 in $[0, T]$, we have

$$
\iint_{t_0}^{t_1} \left| \psi_\eta' \left(\frac{t+s}{2} \right) \right| \theta_h(t - s) \, dt \, ds \leq 1.
\tag{3.11}
$$

Indeed, note that

$$\psi_\eta'\left(\frac{t+s}{2}\right) = \frac{1}{\Delta t}\left[\theta_\eta\left(\frac{(t+s)/2 - t_0}{\Delta t}\right) - \theta_\eta\left(\frac{(t+s)/2 - t_1}{\Delta t}\right)\right]. \tag{3.12}$$

Moreover, we easily see that

$$\iint_{t_0}^{t_1} \theta_\eta\left(\frac{(t+s)/2 - t_i}{\Delta t}\right)\theta_h(t - s)dt\, ds$$

$$= 2\Delta t \int_{t_0}^{t_1}\int_{t_0-s}^{t_1-s} \theta_{2\Delta t\eta}(\tau + 2s - 2t_i)\theta_h(\tau)d\tau\, ds = \frac{1}{2}, \quad \text{for } i = 0, 1. \tag{3.13}$$

In particular (3.11) holds, and we have the lemma.

LEMMA 3.3. *Suppose that $f_\varepsilon(0, x, v) \in L^1(\mathbb{R}; \mathrm{BV}(\mathbb{R}^N))$, which is compactly supported in $\mathbb{R}_v$. Let $u(s, y)$ be the entropy solution of (1.1)–(1.3), and $u_\varepsilon(t, x)$ the approximate solution given by (2.3), then*

$$-\int_Q |u(s, y) - u_\varepsilon(t, x)|\,[\psi_\eta]'\left(\frac{t+s}{2}\right)\theta_h(t - s)dQ$$

$$\leq \left(\frac{1}{2} + \frac{Na_\infty}{h} + \frac{(1+a_\infty)}{h}|\theta'|_{L^1(\mathbb{R})}|\psi_\eta|_{L^1(\mathbb{R})}\right) \sup_{0 \leq t \leq T} \|f_\varepsilon - \chi_{u_\varepsilon}\|_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v)}. \tag{3.14}$$

Proof. Using (3.11), we write

$$\int_Q \int_v |\partial_t \varphi|\,|f_\varepsilon - \chi_{u_\varepsilon}|\,dv\, dQ$$

$$\leq \int_x \left(\int_y \prod_{i=1}^N \theta_h(x_i - y_i)dy\right)\int_t\int_s \left\{\frac{1}{2}\left|\psi_\eta'\left(\frac{t+s}{2}\right)\right|\theta_h(t - s)\right.$$

$$\left. + \frac{1}{h^2}\psi_\eta\left(\frac{t+s}{2}\right)\theta'\left(\frac{t-s}{h}\right)\right\}ds\, dt \int_v |f_\varepsilon - \chi_{u_\varepsilon}|\,dv\, dx$$

$$\leq \sup_{0 \leq t \leq T} \int_x\int_v |f_\varepsilon - \chi_{u_\varepsilon}|\,dv\, dx\left\{\frac{1}{2}\int_t\int_s \left|\psi_\eta'\left(\frac{t+s}{2}\right)\right|\theta_h(t - s)ds\, dt\right.$$

$$\left. + \frac{1}{h^2}\int_t\int_s \psi_\eta\left(\frac{t+s}{2}\right)\theta'\left(\frac{t-s}{h}\right)ds\, dt\right\}$$

$$\leq \sup_{0 \leq t \leq T} \|f_\varepsilon - \chi_{u_\varepsilon}\|_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v)}\left[\frac{1}{2} \times 1 + \frac{1}{h^2} \times h|\theta'|_{L^1(\mathbb{R})}|\psi_\eta|_{L^1(\mathbb{R})}\right], \tag{3.15}$$

and

$$\int_Q \int_v |\nabla_x \varphi| \, |f_\varepsilon - \chi_{u_\varepsilon}| \, dv \, dQ$$

$$\leq \left[\frac{N}{h} + \frac{1}{h} \left(|\psi_\eta|_{L^1(\mathbb{R})} |\theta'|_{L^1(\mathbb{R})} \right) \right] \sup_{0 \leq t \leq T} ||f_\varepsilon - \chi_{u_\varepsilon}||_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v)},$$

(3.16)

with the same arguments. □

We now control the term $\sup_{0 \leq t \leq T} ||f_\varepsilon - \chi_{u_\varepsilon}||_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v)}$ at the right-hand side of (3.14).

PROPOSITION 3.4. *Consider the kinetic initial data* $f_\varepsilon(0, x, v) \in L^1(\mathbb{R}_v; \mathrm{BV}(\mathbb{R}^N))$ *which is compactly supported in* $\mathbb{R}_v$. *Then*

$$||f_\varepsilon(t, x, v) - \chi_{u_\varepsilon(t, x)}(v)||_{L^1(\mathbb{R}^N \times \mathbb{R}_v)}$$

$$\leq 2\varepsilon a_\infty ||f_\varepsilon(0, x, v)||_{\mathrm{BV}(\mathbb{R}^N; L^1(\mathbb{R}_v))} + 2||f_\varepsilon(0, x, v) - \chi_{u_\varepsilon(0, x)}(v)||_{L^1(\mathbb{R}^N \times \mathbb{R}_v)},$$

(3.17)

for every $t > 0$, *where*

$$||f_\varepsilon(0, x, v)||_{\mathrm{BV}(\mathbb{R}^N; L^1(\mathbb{R}_v))} = \sup_{|\Delta x| \neq 0} \int_x \int_v \frac{|f_\varepsilon(0, x + \Delta x, v) - f_\varepsilon(0, x, v)|}{|\Delta x|} dx \, dv.$$

(3.18)

Proof. Suppose that $||f_\varepsilon(0, x, v)||_{\mathrm{BV}(\mathbb{R}^N; L^1(\mathbb{R}_v))}$ is uniformly bounded in ε, then using the contraction inequality (2.11), the corresponding solution $f_\varepsilon(t, x, v)$ satisfies that for every $0 \leq t \leq T$,

$$||f_\varepsilon(t, x, v)||_{\mathrm{BV}(\mathbb{R}^N; L^1(\mathbb{R}_v))} \leq ||f_\varepsilon(0, x, v)||_{\mathrm{BV}(\mathbb{R}^N; L^1(\mathbb{R}_v))}.$$

(3.19)

Moreover,

$$\chi_{u_\varepsilon(t, x)(v)} - f_\varepsilon(t, x, v) = \varepsilon \left\{ \frac{\partial}{\partial t} f_\varepsilon(t, x, v) + a(v) \cdot \nabla_x f_\varepsilon(t, x, v) \right\} \in L^1(\mathbb{R}_x^N \times \mathbb{R}_v),$$

(3.20)

that is for every $t \geq 0$ and every $\varepsilon > 0$,

$$||\chi_{u_\varepsilon(t, x)}(v) - f_\varepsilon(t, x, v)||_{L^1(\mathbb{R}^N \times \mathbb{R}_v)} \leq \varepsilon \left\| \frac{\partial f_\varepsilon}{\partial t}(t, x, v) + a(v) \nabla_x f_\varepsilon(t, x, v) \right\|_{L^1(\mathbb{R}^N \times \mathbb{R}_v)}.$$

(3.21)

It is important to notice that we do not necessarily recover the L^1 regularity of the term $\partial f_\varepsilon / \partial t$ since

$$\frac{\partial f_\varepsilon}{\partial t} = \frac{1}{\varepsilon} [\chi_{u_\varepsilon} - f_\varepsilon] - a(v) \cdot \nabla_x f_\varepsilon,$$

(3.22)

and not for $a(v) \cdot \nabla_x f_\varepsilon$ either. The quantities $|\partial f_\varepsilon / \partial t|$ and $|\partial f_\varepsilon / \partial x_i|$ for $1 \leq i \leq N$ are only measures that we can bound by the BV norm. Indeed, for $\varphi \in \mathcal{D}$, denote by $||\cdot||_{\mathcal{M}}$ the norm

$$||f_\varepsilon||_{\mathcal{M}} := \sup_{||\varphi|| \equiv 1} \left| \int_v \int_z \frac{\partial f_\varepsilon}{\partial z} \varphi \, dv \, dz \right|,$$

(3.23)

where $z = t, x_i$, for $i = 1, \ldots, N$. Then we can write

$$||f_\varepsilon||_{\mathcal{M}} = \sup_{||\varphi|| \equiv 1} \left| \int_v \left(\int_z f_\varepsilon \varphi'(z) dz \right) dv \right| \leq ||f_\varepsilon||_{L^1(\mathbb{R}_v; BV(z))}. \tag{3.24}$$

Using this definition together with the finite speed of propagation property (2.14), we obtain

$$\int_x \int_v |a(v) \nabla_x f_\varepsilon(t, x, v)| \leq a_\infty ||f_\varepsilon||_{\mathcal{M}} \leq a_\infty ||f_\varepsilon(t, x, v)||_{L^1(\mathbb{R}_v; BV(\mathbb{R}^N))}. \tag{3.25}$$

By (3.22) and (3.24), we also deduce

$$\int_\Omega \int_v \left| \frac{\partial f_\varepsilon}{\partial t}(t) \right| \leq ||f_\varepsilon(t)||_{BV(t)} \leq ||f_\varepsilon(0, x, v)||_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v; BV(0, T))}$$

$$\leq a_\infty ||f_\varepsilon(0, x, v)||_{L^1(\mathbb{R}_v; BV(\mathbb{R}^N))} + ||\chi_{u_\varepsilon(0, x)}(v) - f_\varepsilon(0, x, v)||_{L^1}. \tag{3.26}$$

That is for all $T > 0$ and every $0 \leq t \leq T$, we have

$$||f_\varepsilon(t, x, v) - \chi_{u_\varepsilon}(v)||_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v)} \leq 2\varepsilon a_\infty ||f_\varepsilon(0, x, v)||_{L^1(\mathbb{R}_v; BV(\mathbb{R}^N))}$$

$$+ ||f_\varepsilon(0, x, v) - \chi_{u_\varepsilon(0, x)}(v)||_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v)}. \tag{3.27}$$

$\square$

Remark 3.5. Denote $u_\varepsilon(0, x) \equiv u_{\varepsilon 0}(x) := u_{\varepsilon 0}$. Then, it is not restrictive to replace $\lim_{\varepsilon \to 0}$ $||f_\varepsilon(0, x, v) - \chi_{u_{\varepsilon 0}}(v)||_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v)} = 0$ by $f_\varepsilon(0, x, v) = \chi_{u_{\varepsilon 0}}(v)$ in the above proposition. This is useful for the applications.

We then have the proposition on the error bound.

PROPOSITION 3.6. *Suppose that $f_\varepsilon(0, x, v) \in L^1(\mathbb{R}; BV(\mathbb{R}^N))$, which is compactly supported in $\mathbb{R}_v$, and that initially $f_\varepsilon(0, x, v) = \chi_{u_{\varepsilon 0}}(v)$. Let $u(s, y)$ be the entropy solution of (1.1)–(1.3), and $u_\varepsilon(t, x)$ the approximate solution given by (2.3). Then*

$$-\int_Q |u(s, y) - u_\varepsilon(t, x)| [\psi_\eta]' \left(\frac{t+s}{2} \right) \theta_h(t-s) dQ \leq M \frac{\varepsilon}{h} ||u_{\varepsilon 0}||_{BV(\mathbb{R}^N)}, \tag{3.28}$$

where

$$M = a_\infty \left(h + 2N a_\infty + 2(1 + a_\infty) |\theta'|_{L^1(\mathbb{R})} |\psi_\eta|_{L^1(\mathbb{R})} \right). \tag{3.29}$$

Proof. If we take the simplified initial data form (Maxwellian data) of Remark 3.5, then Proposition 3.4 gives

$$||f_\varepsilon(t, x, v) - \chi_{u_\varepsilon(t, x)}(v)||_{L^1(\mathbb{R}^N \times \mathbb{R}_v)} \leq 2\varepsilon a_\infty ||f_\varepsilon(0, x, v)||_{L^1(\mathbb{R}_v; BV(\mathbb{R}^N))}. \tag{3.30}$$

Moreover, noticing that

$$||f_\varepsilon(0, x, v)||_{L^1(\mathbb{R}_v; BV(\mathbb{R}^N))} = ||u_\varepsilon(0, x)||_{BV(\mathbb{R}^N)} = ||u_{\varepsilon 0}(x)||_{BV(\mathbb{R}^N)}, \tag{3.31}$$

we immediately deduce the result from Lemma 3.3.

$\square$

4. Generalization to the Kuznetsov technique

In this section, we will estimate the difference

$$\|u(t_1,\cdot) - u_\varepsilon(t_1,\cdot)\|_{L^1(\mathbb{R}^N)} - \|u(t_0,\cdot) - u_\varepsilon(t_0,\cdot)\|_{L^1(\mathbb{R}^N)}, \tag{4.1}$$

where t_0 and t_1 are in $[0,T]$, $T > 0$. The techniques we will use are not new (see, e.g., Bouchut and Perthame [3]), and are known as the Kuznetsov techniques (see Kuznetsov [11], and Kuznetsov and Volshin [12]). The new thing here is the generalization of these techniques to the kinetic approach of Perthame and Tadmor, as the approximate solution is obtained from the kinetic model. We prepare by the lemma.

LEMMA 4.1. *Suppose that $f_\varepsilon(0,x,v) \in L^1(\mathbb{R}; BV(\mathbb{R}^N))$ is compactly supported in v, and that $u_{\varepsilon 0}(x) = u_0(x)$. Then*

$$\|u(t,\cdot)\|_{BV(\mathbb{R}^N)} \le \|u_0\|_{BV(\mathbb{R}^N)}, \qquad \|u_\varepsilon(t,\cdot)\|_{BV(\mathbb{R}^N)} \le \|u_0\|_{BV(\mathbb{R}^N)}, \tag{4.2}$$

for every $t \in [0,T]$. Moreover, u and u_ε belong to the $BV([0,T]; L^1(\mathbb{R}^N))$ for every $T > 0$.

Proof. The proof is classical and relies on the translation property in x and the contraction (2.11). $\qquad\square$

PROPOSITION 4.2. *Suppose that the hypothesis of Proposition 3.6 is satisfied. Then*

$$I = -\int_Q |u(s,y) - u_\varepsilon(t,x)| \, [\psi_\eta]' \left(\frac{t+s}{2}\right) \theta_h(t-s) dQ$$

$$\ge \|u(t_1,x) - u_\varepsilon(t_1,x)\|_{L^1(\mathbb{R}^N)} - \|u(t_0,x) - u_\varepsilon(t_0,x)\|_{L^1(\mathbb{R}^N)} \tag{4.3}$$

$$- \left(4\|u_0\|_{BV(\mathbb{R};L^1(\mathbb{R}^N))} + 2\|u_0\|_{BV(\mathbb{R}^N)}\right) h.$$

Proof. We split the integral I using (3.12) into $I = I_0 + I_1$ relatively to t_0 and t_1, where

$$I_0 = -\int_Q |u(s,y) - u_\varepsilon(t,x)| \frac{1}{\Delta t} \theta_\eta \left(\frac{(t+s)/2 - t_0}{\Delta t}\right) \theta_h(t-s) \prod_{i=1}^N \theta_h(x_i - y_i) dQ,$$

$$\tag{4.4}$$

$$I_1 = +\int_Q |u(s,y) - u_\varepsilon(t,x)| \frac{1}{\Delta t} \theta_\eta \left(\frac{(t+s)/2 - t_1}{\Delta t}\right) \theta_h(t-s) \prod_{i=1}^N \theta_h(x_i - y_i) dQ.$$

For $t = t_0$, we use the triangle inequality

$$|u(s,y) - u_\varepsilon(t,x)| \le |u(s,y) - u(t_0,y)| + |u(t_0,y) - u(t_0,x)|$$

$$+ |u(t_0,x) - u_\varepsilon(t_0,x)| + |u_\varepsilon(t_0,x) - u_\varepsilon(t,x)|, \tag{4.5}$$

which also may be written as

$$-|u(t_0,x) - u_\varepsilon(t_0,x)| \le |u(s,y) - u(t_0,y)| + |u(t_0,y) - u(t_0,x)|$$

$$+ |u_\varepsilon(t_0,x) - u_\varepsilon(t,x)| - |u(s,y) - u_\varepsilon(t,x)|. \tag{4.6}$$

We then integrate as follows:

$$-\int_Q |u(t_0,x) - u_\varepsilon(t_0,x)| \, \theta_\eta\left(\frac{(t+s)/2 - t_0}{\Delta t}\right) \theta_h(t-s) \prod_{i=1}^N \theta_h(x_i - y_i) dQ$$

$$\leq \int_Q |u(s,y) - u(t_0,y)| \, \theta_\eta\left(\frac{(t+s)/2 - t_0}{\Delta t}\right) \theta_h(t-s) \prod_{i=1}^N \theta_h(x_i - y_i) dQ$$

$$+ \int_Q |u(t_0,y) - u(t_0,x)| \, \theta_\eta\left(\frac{(t+s)/2 - t_0}{\Delta t}\right) \theta_h(t-s) \prod_{i=1}^N \theta_h(x_i - y_i) \, dx \, dy \, dt \, ds$$

$$+ \int_Q |u_\varepsilon(t_0,x) - u_\varepsilon(t,x)| \, \theta_\eta\left(\frac{(t+s)/2 - t_0}{\Delta t}\right) \theta_h(t-s) \prod_{i=1}^N \theta_h(x_i - y_i) dQ$$

$$- \int_Q |u(s,y) - u_\varepsilon(t,x)| \, \theta_\eta\left(\frac{(t+s)/2 - t_0}{\Delta t}\right) \theta_h(t-s) \prod_{i=1}^N \theta_h(x_i - y_i) dQ$$

$$\equiv J + K + L^\varepsilon + I_0.$$

(4.7)

At the left-hand side, we have

$$-\int_Q |u(t_0,x) - u_\varepsilon(t_0,x)| \, \theta_\eta\left(\frac{(t+s)/2 - t_0}{\Delta t}\right) \theta_h(t-s) \prod_{i=1}^N \theta_h(x_i - y_i) dQ$$

$$= -\int_x |u(t_0,x) - u_\varepsilon(t_0,x)| \int_y \prod_{i=1}^N \theta_h(x_i - y_i) \int_t \int_s \theta_\eta\left(\frac{(t+s)/2 - t_0}{\Delta t}\right) \theta_h(t-s) dQ$$

$$= -\frac{1}{2} \|u(t_0,\cdot) - u_\varepsilon(t_0,\cdot)\|_{L^1(\mathbb{R}_x^N)}.$$

(4.8)

At the right-hand side, we have the following:

$$J = 1 \cdot \iint_{t_0}^{t_1} \|u(s,\cdot) - u(t_0,\cdot)\|_{L^1(\mathbb{R}_y^N)} \theta_\eta\left(\frac{(t+s)/2 - t_0}{\Delta t}\right) \theta_h(t-s) ds \, dt$$

$$\leq \iint_{t_0}^{t_1} |t_0 - s| \, \|u(\xi,\cdot)\|_{BV(\mathbb{R}_+; L^1(\mathbb{R}_y^N))} \theta_\eta\left(\frac{(t+s)/2 - t_0}{\Delta t}\right) \theta_h(t-s) ds \, dt, \quad s \leq \xi \leq t_0,$$

$$\leq h \|u_0(\cdot)\|_{BV(\mathbb{R}_t^+; L^1(\mathbb{R}^N))} \iint_{t_0}^{t_1} \theta_\eta\left(\frac{(t+s)/2 - t_0}{\Delta t}\right) \theta_h(t-s) ds \, dt$$

$$\leq \frac{h}{2} \|u_0\|_{BV(\mathbb{R}_t; L^1(\mathbb{R}^N))},$$

(4.9)

using Lemma 4.1. We also have

$$
K \leq \int_z \int_y |u(t_0,y) - u(t_0, y+z)| \prod_{i=1}^N \theta_h(z_i) \, dz \, dy \iint_{t_0}^{t_1} \theta_\eta \left(\frac{(t+s)/2 - t_0}{\Delta t} \right) \theta_h(t-s) \, dt \, ds
$$

$$
\leq h \int_y \frac{|u(t_0,y) - u(t_0, y+z)|}{|z|} \, dy \int_z \prod_{i=1}^N \theta_h(z_i) \, dz \times \frac{1}{2}
$$

$$
\leq \frac{h}{2} \|u(\cdot)\|_{BV(\mathbb{R}^N)} \leq \frac{h}{2} \|u_0\|_{BV(\mathbb{R}^N)}
$$

(4.10)

using the change of variables

$$
x_i - y_i = z_i \longleftrightarrow dx_i = dz_i.
$$

(4.11)

Finally,

$$
L^\varepsilon \leq 1 \cdot h \|u(\xi, \cdot)\|_{BV(\mathbb{R}_t^+; L^1(\mathbb{R}_y^N))} \iint_{t_0}^{t_1} \theta_\eta \left(\frac{(t+s)/2 - t_0}{\Delta t} \right) \theta_h(t-s) \, ds \, dt, \quad t_0 \leq \xi \leq s,
$$

$$
\leq \frac{h}{2} \|u_\varepsilon\|_{BV(\mathbb{R}_t; L^1(\mathbb{R}^N))} \leq \frac{h}{2} \|u_{\varepsilon 0}\|_{BV(\mathbb{R}_t; L^1(\mathbb{R}^N))}.
$$

(4.12)

Hence

$$
-\frac{1}{2} \|u(t_0, \cdot) - u_\varepsilon(t_0, \cdot)\|_{L^1(\mathbb{R}^N)} \leq h \|u_0\|_{BV(\mathbb{R}; L^1(\mathbb{R}^N))} + \frac{h}{2} \|u_0\|_{BV(\mathbb{R}^N)} + I_0.
$$

(4.13)

For $t = t_1$, we consider the following triangle inequality:

$$
|u(t_1, x) - u_\varepsilon(t_1, x)| \leq |u(t_1, x) - u(t_1, y)| + |u(t_1, y) - u(s, y)| + |u(s, y) - u_\varepsilon(t, x)|.
$$

(4.14)

With the same techniques, we obtain

$$
\int_Q |u(t_1, x) - u_\varepsilon(t_1, x)| \, \theta_\eta \left(\frac{(t+s)/2 - t_1}{\Delta t} \right) \theta_h(t-s) \prod_{i=1}^N \theta_h(x_i - y_i) \, dQ
$$

$$
= \frac{1}{2} \|u(t_1, x) - u_\varepsilon(t_1, x)\|_{L^1(\mathbb{R}^N)} \leq h \|u_0\|_{BV(\mathbb{R}; L^1(\mathbb{R}^N))} + \frac{h}{2} \|u_0\|_{BV(\mathbb{R}^N)} + I_1.
$$

(4.15)

We sum up by

$$
\|u(t_1, x) - u_\varepsilon(t_1, x)\|_{L^1(\mathbb{R}^N)} - \|u(t_0, x) - u_\varepsilon(t_0, x)\|_{L^1(\mathbb{R}^N)}
$$

$$
\leq 4h \|u_0\|_{BV(\mathbb{R}; L^1(\mathbb{R}^N))} + 2h \|u_0\|_{BV(\mathbb{R}^N)} + 2(I_0 + I_1).
$$

(4.16)

□

THEOREM 4.3. *Suppose that initially $f_\varepsilon(0,x,v) \in L^1(\mathbb{R}_v; \mathrm{BV}(\mathbb{R}^N))$, which is compactly supported in v, and that $f_\varepsilon(0,x,v) \equiv f_0(x,v) = \chi_{u_0(x)}(v)$. Let $u(s,y)$ be the entropy solution of (1.1)–(1.3), and $u_\varepsilon(t,x)$ the approximate solution given by (2.3). Then for any $T > 0$, and t_0, t_1 in $[0,T]$, the following L^1_x error bound*

$$\|u(t_1, \cdot) - u_\varepsilon(t_1, \cdot)\|_{L^1(\mathbb{R}^N)} \le \|u(t_0, \cdot) - u_\varepsilon(t_0, \cdot)\|_{L^1(\mathbb{R}^N)} + C\sqrt{\varepsilon} \tag{4.17}$$

holds true, with C being a positive constant independent on ε.

Proof. From Proposition 3.6, we have

$$I_0 + I_1 = I \le M \frac{\varepsilon}{h} \|u_0\|_{\mathrm{BV}(\mathbb{R}^N_x)}, \tag{4.18}$$

where M is given by (3.29). Hence

$$\|u(t_1, \cdot) - u_\varepsilon(t_1, \cdot)\|_{L^1(\mathbb{R}^N)} - \|u(t_0, \cdot) - u_\varepsilon(t_0, \cdot)\|_{L^1(\mathbb{R}^N)}$$
$$\le \left(4\|u_0\|_{\mathrm{BV}(\mathbb{R}; L^1(\mathbb{R}^N))} + 2\|u_0\|_{\mathrm{BV}(\mathbb{R}^N)} \right) h + M\frac{\varepsilon}{h}\|u_0\|_{\mathrm{BV}(\mathbb{R}^N_x)}. \tag{4.19}$$

The optimal choice of h is $h = \sqrt{\varepsilon}$. Then

$$\left(4\|u_0\|_{\mathrm{BV}(\mathbb{R}; L^1(\mathbb{R}^N))} + 2\|u_0\|_{\mathrm{BV}(\mathbb{R}^N)} \right) h + M\frac{\varepsilon}{h}\|u_0\|_{\mathrm{BV}(\mathbb{R}^N_x)} = C\sqrt{\varepsilon}, \tag{4.20}$$

where

$$C = \left\{ 2 + a_\infty \left(\sqrt{\varepsilon} + 2Na_\infty + 2(1+a_\infty)|\theta'|_{L^1(\mathbb{R})}|\psi_\eta|_{L^1(\mathbb{R})} \right) \right\} \|u_0\|_{\mathrm{BV}(\mathbb{R}^N_x)} + 4\|u_0\|_{\mathrm{BV}(\mathbb{R}; L^1(\mathbb{R}^N))}. \tag{4.21}$$

□

Remark 4.4. Notice that the error is global in time, and optimal in the sense that we cannot have more than $\sqrt{\varepsilon}$, since we use the Kuznetsov techniques.

5. The parabolic case

In this short section, we discuss the kinetic approach for the following parabolic case:

$$\partial_t u + \mathrm{div}_x A(u) = \Delta u, \tag{5.1}$$

where $\Delta = \sum_{i=1}^N$ is the Laplacian operator. The corresponding kinetic model is

$$\left[\partial_t + \sum_{i=1}^N a_i(v)\partial_{x_i} \right] f_\varepsilon = -\frac{1}{\varepsilon}[\chi_{u_\varepsilon} - f_\varepsilon] + \sum_{i=1}^N \partial_{x_i^2} f_\varepsilon, \tag{5.2}$$

which reads as

$$\left[\partial_t + \sum_{i=1}^N a_i(v)\partial_{x_i} - \sum_{i=1}^N \partial_{x_i^2} \right] f_\varepsilon = -\frac{1}{\varepsilon}[\chi_{u_\varepsilon} - f_\varepsilon]. \tag{5.3}$$

We multiply by sgn$(f_\varepsilon - \chi_\ell)$ and integrate over v, we then easily obtain in the weak sense that

$$\int_v \left[\partial_t + \sum_{i=1}^N a_i(v)\partial_{x_i} - \sum_{i=1}^N \partial_{x_i^2} \right] |f_\varepsilon - \chi_\ell|\, dv = -\frac{1}{\varepsilon}\left[\int_v |f_\varepsilon - \chi_\ell| - |u_\varepsilon - \ell| \right] \le 0. \quad (5.4)$$

We consider here $|f_\varepsilon - \chi_\ell|$ as the "kinetic entropies" for (5.2) in the sense that

$$-\int_Q \int_v \left(\left[\partial_t + \sum_{i=1}^N a_i(v)\partial_{x_i} - \sum_{i=1}^N \partial_{x_i^2} \right]\varphi \right) |f_\varepsilon - \chi_\ell|\, dv\, dQ \le 0. \quad (5.5)$$

We do the same analysis as in Sections 3 and 4. The error found is the same. Notice the following remainder:

$$\left| \varepsilon \int_v \int_Q \sum_{i=1}^N \partial_{x_i} |f_\varepsilon - \chi_\ell| \, \partial_{x_i}(\varphi) dv\, dQ \right| \le \left(\varepsilon \sum_{i=1}^N ||\partial_{x_{i_0}} \cdot f_\varepsilon||_{L^\infty} \right) \left(\sum_{i=1}^N |\varphi_{x_i}|_{L^1(t,x)} \right)$$
$$\le \left(\varepsilon N ||\partial_{x_{i_0}} f_\varepsilon||_{L^\infty} \right) \left(||\varphi||_{L^1(\mathbb{R};W^{1,1}(\mathbb{R}^N))} \right), \quad (5.6)$$

where x_{i_0}, $i_0 \in 1,\dots,N$, is the space variable so that $|\partial_{x_i} f| \le |\partial_{x_{i_0}} f|$.

Remark 5.1. It would be interesting to study the kinetic approach to the case of parabolic-hyperbolic problems in the sense of Chen and Karlsen in [7] and Chen and Perthame in [8].

Acknowledgment

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