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Embeddings of ultradistributions and periodic hyperfunctions in Colombeau type algebras through sequence spaces

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Abstract

In a recent paper, we gave a topological description of Colombeau type algebras introducing algebras of sequences with exponential weights. Embeddings of Schwartz spaces into the Colombeau algebra $\mathcal{G}$ are well known, but for ultradistribution and periodic hyperfunction type spaces we give new constructions. We show that the multiplication of regular enough functions (smooth, ultradifferentiable or quasianalytic), embedded into corresponding algebras, is the ordinary multiplication.

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1. *Introduction*

Differential algebras of generalized functions containing embedded distributions are a convenient framework for the analysis of problems with singular coefficients and/or singular data, especially for non linear problems, since multiplication and other non linear operations are in general not defined in classical generalized function spaces. Nowadays, there is a considerable literature concerning such algebras. (For example see [**1**, **2**, **3**, **7**, **12**, **15**, **5**, **6**] and the references therein.)

We have proved in [**4**] that these algebras, here referred to as Colombeau type algebras, can be reconsidered as a class of sequence spaces algebras, and we gave a purely topological description of them.

In analogy to embeddings of Schwarz' distributions, we show in this paper that

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some classes of ultradistributions and periodic hyperfunctions can be embedded into well chosen sequence algebras. Moreover, we show that the product of enough regular elements, ultradifferentiable functions or quasianalytic periodic functions of appropriate classes, is the ordinary multiplication.

The problem of embedding of classical spaces into corresponding sequence spaces algebra is closely related to the choice of sequences of mollifiers, sequences of appropriately smooth functions converging to the delta distribution. While such a problem is easy for embedding of Schwartz distributions defined on an open subset of $\mathbb{R}^n$, it is essential for ultradifferentiable functions and ultradistributions, considered in section 3. The same holds for periodic quasianalytic functions and corresponding periodic hyperfunctions of section 4.

Colombeau ultradistributions corresponding to a general non-quasianalytic sequence were introduced and analyzed in [**14**]. Although we consider here the Gevrey sequence $(p!^m)_p$, $m > 1$, we give sharper estimates and improve results of [**14**]: The construction of appropriate mollifiers enables us to give more precise results concerning embeddings. Colombeau periodic hyperfunctions introduced in this paper are more closely related to the global theory of generalized functions than those of [**16**]. In this sense, we improve results of [**16**].

The novelty of results related to both cited papers and the embedding of both classes of algebras into corresponding sequence space algebras, and so their topological description, are the main results of this paper.

2. *General construction* [**4**]

We use the standard notations $\mathbb{N} = \{0, 1, \dots\}$, $\mathbb{N}^* = \{1, 2, \dots\}$ and $\mathbb{R}_+ = [0, +\infty)$. We recall [**4**] that the algebra of Colombeau complex numbers is given by

$$\overline{\mathbb{C}} = \mathcal{E}_0/\mathcal{N}_0 \equiv \{x \in \mathbb{C}^{\mathbb{N}^*} : \limsup |x_n|^{\frac{1}{\log n}} < \infty\}/\{x \in \mathbb{C}^{\mathbb{N}^*} : \limsup |x_n|^{\frac{1}{\log n}} = 0\} .$$

The passage from this description to Colombeau's original construction is given by the following chain of equivalences, for a complex sequence $(x_n)_n \in \mathbb{C}^{\mathbb{N}^*}$:

$$\begin{aligned} \limsup |x_n|^{1/\log n} < \infty \iff & \exists C \in \mathbb{R}_+ : \limsup |x_n|^{1/\log n} = C \\ \iff & \exists B > 0,\ \exists n_0, \forall n > n_0 : |x_n| \le B^{\log n} = n^{\log B} \\ \iff & \exists \gamma \in \mathbb{R} : |x_n| = o(n^\gamma) . \end{aligned}$$

On the other hand, $\limsup |x_n|^{1/\log n} = 0$ (for the ideal) corresponds to taking $C = 0$ and thus $\forall B > 0$ resp. $\forall \gamma \in \mathbb{R}$ in the last lines.

Let us also recall our construction from [**4**] for the case of $E = \mathbb{C}$, $r_n = n^{-1/m}, n \in \mathbb{N}^*$, where $m > 0$ is fixed. Let

$$\mathcal{E}_0^m = \left\{ c = (c_n)_n \in \mathbb{C}^{\mathbb{N}^*} \mid \| c \|_{|\cdot|, n^{-1/m}} = \limsup_{n\to\infty} |c_n|^{n^{-1/m}} < \infty \right\} ,$$

$$\mathcal{N}_0^m = \left\{ c = (c_n)_n \in \mathbb{C}^{\mathbb{N}^*} \mid \| c \|_{|\cdot|, n^{-1/m}} = 0 \right\} .$$

The factor algebra $\overline{\mathbb{C}}^m = \mathcal{E}_0^m/\mathcal{N}_0^m$, $m > 0$, is called the ring of Colombeau ultracomplex numbers for $m > 1$ and the ring of Colombeau hypercomplex numbers for $m \le 1$.

Now we come to the general construction. Let $(E_\nu^\mu, p_\nu^\mu)_{\mu,\nu\in\mathbb{N}^*}$ be a family of seminormed algebras over $\mathbb{R}$ or $\mathbb{C}$ such that

$$\forall \mu, \nu \in \mathbb{N}^* : \; E_\nu^{\mu+1} \hookrightarrow E_\nu^\mu \quad \text{and} \quad E_{\nu+1}^\mu \hookrightarrow E_\nu^\mu \;\; (\text{resp. } E_\nu^\mu \hookrightarrow E_{\nu+1}^\mu \;) \;,$$

where $\hookrightarrow$ means continuously embedded (i.e., for the ν index we consider inclusions in the two directions). Then let $\overleftarrow{E} = \operatorname*{proj\,lim}_{\mu\to\infty} \operatorname*{proj\,lim}_{\nu\to\infty} E_\nu^\mu = \operatorname*{proj\,lim}_{\nu\to\infty} E_\nu^\nu$, (resp. $\overrightarrow{E} = \operatorname*{proj\,lim}_{\mu\to\infty} \operatorname*{ind\,lim}_{\nu\to\infty} E_\nu^\mu$). Such projective and inductive limits are usually considered with norms instead of seminorms, and with the additional assumption that in the projective case sequences are reduced, while in the inductive case for every $\mu \in \mathbb{N}^*$ the inductive limit is regular, i.e. a set $A \subset \operatorname*{ind\,lim}_{\nu\to\infty} E_\nu^\mu$ is bounded iff it is contained in some E_ν^μ and bounded there.

Consider a positive sequence $r = (r_n)_n \in (\mathbb{R}_+)^{\mathbb{N}^*}$ decreasing to zero and define (with $p \equiv (p_\nu^\mu)_{\nu,\mu}$):

$$\begin{aligned}
\| f \|_{p_\nu^\mu, r} &= \limsup_{n\to\infty} p_\nu^\mu(f_n)^{r_n} \qquad \Big(\text{ for } \; f \in (E_\nu^\mu)^{\mathbb{N}^*} \;\Big) \;, \\
\overleftarrow{\mathcal{F}}_{p,r} &= \Big\{ f \in \overleftarrow{E}^{\mathbb{N}^*} \;\Big|\; \forall \mu, \nu \in \mathbb{N}^* : \| f \|_{p_\nu^\mu, r} < \infty \Big\} \;, \\
\overleftarrow{\mathcal{K}}_{p,r} &= \Big\{ f \in \overleftarrow{E}^{\mathbb{N}^*} \;\Big|\; \forall \mu, \nu \in \mathbb{N}^* : \| f \|_{p_\nu^\mu, r} = 0 \Big\} \\
(\text{resp. } \; \overrightarrow{\mathcal{F}}_{p,r} &= \bigcap_{\mu\in\mathbb{N}^*} \overrightarrow{\mathcal{F}}_{p,r}^\mu \;, \;\; \overrightarrow{\mathcal{F}}_{p,r}^\mu = \bigcup_{\nu\in\mathbb{N}^*} \Big\{ f \in (E_\nu^\mu)^{\mathbb{N}^*} \;\Big|\; \| f \|_{p_\nu^\mu, r} < \infty \Big\} \;, \\
\overrightarrow{\mathcal{K}}_{p,r} &= \bigcap_{\mu\in\mathbb{N}^*} \overrightarrow{\mathcal{K}}_{p,r}^\mu \;, \;\; \overrightarrow{\mathcal{K}}_{p,r}^\mu = \bigcup_{\nu\in\mathbb{N}^*} \Big\{ f \in (E_\nu^\mu)^{\mathbb{N}^*} \;\Big|\; \| f \|_{p_\nu^\mu, r} = 0 \Big\} \;\Big) \;.
\end{aligned}$$

Recall [**4**]:

Writing $\overleftrightarrow{\cdot}$ for both, $\overleftarrow{\cdot}$ or $\overrightarrow{\cdot}$, we have that $\overleftrightarrow{\mathcal{F}}_{p,r}$ is an algebra and $\overleftrightarrow{\mathcal{K}}_{p,r}$ is an ideal of $\overleftrightarrow{\mathcal{F}}_{p,r}$; thus, $\overleftrightarrow{\mathcal{G}}_{p,r} = \overleftrightarrow{\mathcal{F}}_{p,r} / \overleftrightarrow{\mathcal{K}}_{p,r}$ is an algebra.

For every $\mu, \nu \in \mathbb{N}^*$, $d_{p_\nu^\mu} : (E_\nu^\mu)^{\mathbb{N}^*} \times (E_\nu^\mu)^{\mathbb{N}^*} \to \overline{\mathbb{R}}_+$ defined by $d_{p_\nu^\mu}(f,g) = \| f - g \|_{p_\nu^\mu, r}$ is an ultrapseudometric on $(E_\nu^\mu)^{\mathbb{N}^*}$. Moreover, $(d_{p_\nu^\mu})_{\mu,\nu}$ induces a topological algebra[1] structure on $\overleftarrow{\mathcal{F}}_{p,r}$ such that the intersection of the neighborhoods of zero equals $\overleftarrow{\mathcal{K}}_{p,r}$.

From the properties above, the factor space $\overleftarrow{\mathcal{G}}_{p,r} = \overleftarrow{\mathcal{F}}_{p,r} / \overleftarrow{\mathcal{K}}_{p,r}$ is a topological algebra over generalized numbers $\overline{\mathbb{C}}_r = \mathcal{G}_{|\cdot|, r}$ (constructed with the sequence $r = (r_n)$ as above for the Colombeau ultracomplex numbers). The topology of $\overleftarrow{\mathcal{G}}_{p,r}$ is defined by the family of ultrametrics $(\tilde{d}_{p_\nu^\mu})_{\mu,\nu}$ where $\tilde{d}_{p_\nu^\mu}([f],[g]) = d_{p_\nu^\mu}(f,g)$, $[f]$ standing for the class of f.

If τ_μ denotes the inductive limit topology on $\mathcal{F}_{p,r}^\mu = \bigcup_{\nu\in\mathbb{N}^*} ((\tilde{E_\nu^\mu})^{\mathbb{N}^*}, d_{\mu,\nu})$, $\mu \in \mathbb{N}^*$, then $\overrightarrow{\mathcal{F}}_{p,r}$ is a topological algebra[1] for the projective limit topology of the family $(\mathcal{F}_{p,r}^\mu, \tau_\mu)_\mu$.

Remark 1. *The two multiplicative sets $H = [0,1]$ and $I = [0,1)$ verify the relations $H \cdot H = H$, $I \cdot H = I$, $I \cdot I = I$, just like the sets* $[0,\infty)$ *and* $\{ 0 \}$. *Thus, similar constructions can also be made with $\| \cdot \| \leq 1$ and $\| \cdot \| < 1$ instead of $\| \cdot \| < \infty$ and $\| \cdot \| = 0$. This is used in the setting of infra-exponential algebras and also appears in the context of periodic hyperfunctions.*

[1] over $(\mathbb{C}^{\mathbb{N}^*}, \| \cdot \|_{|\cdot|, r})$, not over $\mathbb{C}$: scalar multiplication is not continuous.

3. *Colombeau ultradistributions of Gevrey class*

3·1. *Ultradistributions of Gevrey class*

We refer to [**9**] for definitions of the spaces $\mathcal{E}^{(m)}$, $\mathcal{D}^{(m)}$, $\mathcal{E}^{\{m\}}$, $\mathcal{D}^{\{m\}}$ $(m > 1)$, and their duals, Beurling and Roumieu type ultradistribution spaces. Here we construct Colombeau ultradistribution algebras corresponding to $M_p = p!^m$, $m > 1$. We apply the construction of section 2.

For the function space $E = \mathcal{C}^\infty(\mathbb{R}^s)$, we define for all $\mu, \nu \in \mathbb{R}_+$ and $m > 1$ the seminorms

$$p_\nu^{m,\mu}(f) = \sup_{|x|\leq\mu,\alpha\in\mathbb{N}^s} \frac{\nu^{|\alpha|}}{\alpha!^m}|f^{(\alpha)}(x)| \quad \text{and} \quad q_\nu^{m,\mu} = p_{1/\nu}^{m,\mu} \ .$$

Then let, for $\mu, \nu \in \mathbb{N}^*$, $E_\nu^\mu = E_{p_\nu^{m,\mu}}$ (resp. $E_\nu^\mu = E_{q_\nu^{m,\mu}}$) be the subset of E on which the given seminorm is finite. For the first case, we clearly have $E_\nu^{\mu+1} \hookrightarrow E_\nu^\mu$, $E_{\nu+1}^\mu \hookrightarrow E_\nu^\mu$ and for the second case, we have $E_\nu^{\mu+1} \hookrightarrow E_\nu^\mu$, $E_\nu^\mu \hookrightarrow E_{\nu+1}^\mu$ for any $\mu, \nu \in \mathbb{N}^*$. Let $m > 1$, $m' > 0$ and $r_n = n^{-1/m'}$.

Definition 3·1. *The sets of exponentially growth order ultradistribution nets and null nets of Beurling type are defined, respectively, by*

$$\mathcal{E}_{exp}^{(p!^m,p!^{m'})} = \overleftarrow{\mathcal{F}}_{p^m,r} \ , \quad \mathcal{N}^{(p!^m,p!^{m'})} = \overleftarrow{\mathcal{K}}_{p^m,r} \ .$$

The sets of exponentially growth order ultradistribution nets and null nets of Roumieu type are defined, respectively, by

$$\mathcal{E}_{exp}^{\{p!^m,p!^{m'}\}} = \overrightarrow{\mathcal{F}}_{q^m,r} \ , \quad \mathcal{N}^{\{p!^m,p!^{m'}\}} = \overrightarrow{\mathcal{K}}_{q^m,r} \ .$$

Proposition 3·2.

(i) $\mathcal{E}_{exp}^{(p!^m,p!^{m'})}$ *(resp. $\mathcal{E}_{exp}^{\{p!^m,p!^{m'}\}}$) are algebras under pointwise multiplication, and $\mathcal{N}^{(p!^m,p!^{m'})}$ (resp. $\mathcal{N}^{\{p!^m,p!^{m'}\}}$) are ideals of them.*

(ii) *The pseudodistances induced by $\|\!| \cdot \|\!|_{p_\nu^{m,\mu},m'}$ (resp. $\|\!| \cdot \|\!|_{q_\nu^{m,\mu},m'}$) are ultrapseudometrics on respective domains.*

Proof. With Definition 3·1, this is just a particular case of of the general construction recalled in section 2. $\square$

The Colombeau ultradistribution algebra $\mathcal{G}^{(p!^m,p!^{m'})}$ (resp. $\mathcal{G}^{\{p!^m,p!^{m'}\}}$) is defined by

$$\mathcal{G}^{(p!^m,p!^{m'})} = \overleftarrow{\mathcal{G}}_{p,r} = \mathcal{E}_{exp}^{(p!^m,p!^{m'})}/\mathcal{N}^{(p!^m,p!^{m'})}$$
$$(\text{resp. } \ \mathcal{G}^{\{p!^m,p!^{m'}\}} = \overrightarrow{\mathcal{G}}_{p,r} = \mathcal{E}_{exp}^{\{p!^m,p!^{m'}\}}/\mathcal{N}^{\{p!^m,p!^{m'}\}} \ ) \ .$$

These topological algebras are invariant under the actions of ultradifferential operators of respective classes (m) and $\{m\}$, see e.g. [**9**].

3·2. *Embeddings of ultradifferentiable functions and ultradistributions.*

In what follows, mollifiers will be constructed by elements of spaces Σ^{pow} (resp. Σ_{der}), which consist of smooth functions φ on $\mathbb{R}$ with the property that for some $b > 0$,

$$\sigma^b(\varphi) = \sup_{\beta\in\mathbb{N},x\in\mathbb{R}} \frac{|x^\beta \varphi(x)|}{b^\beta \beta!} < \infty \quad (\text{resp. } \ \sigma_b(\varphi) = \sup_{\alpha\in\mathbb{N},x\in\mathbb{R}} \frac{|\varphi^{(\alpha)}(x)|}{b^\alpha \alpha!} < \infty \ ) \ .$$

Both spaces are endowed with the respective inductive topologies.

DEFINITION 3·3. *Let* $(\phi^n)_{n\in\mathbb{N}^*}$ *be a bounded net in* $\Sigma^{\rm pow}$ *(resp.* $\Sigma_{\rm der}$*) such that* $\forall n\in\mathbb{N}^* : \int_{\mathbb{R}} t^j\,\phi^n(t)\,\mathrm{d}t = \delta_{j,0}$ *for* $j\in\{0,1,2,\dots,[n^{1/m}]+1\}$, $m>1$. *Then* $(\phi_n)_{n\in\mathbb{N}^*}$ *with* $\phi_n = n\,\phi^n(n\,\cdot)$ *is called a net of* $\{m,\mathrm{pow}\}$ *(resp.* $\{m,\mathrm{der}\}$*)–mollifiers generated by* $(\phi^n)_n$.

The following important lemma gives an explicit net of $\{m,\mathrm{pow}\}$– and $\{m,\mathrm{der}\}$–mollifiers:

LEMMA 3·4. *For all* $n\in\mathbb{N}^*$ *and* $x\in\mathbb{R}$, *let*

$$h_n(x) = \exp\left(n^2 - \sqrt[n]{n^{2n}+x^{2n}}\right)\ ,\quad k_n(x) = \exp\left(-x^{2n}\right)\ .$$

Then, for all $n\in\mathbb{N}^*$, $h_n(0)=k_n(0)=1$ *and*

$$\forall\alpha\in\{\,1,\dots,2n-1\,\}:\ h_n^{(\alpha)}(0)=k_n^{(\alpha)}(0)=0\ ,$$

and there exist $r>0$ *and* $C>0$ *such that*

$$\sup_{n\in\mathbb{N}^*}\sigma_r(h_n)<C\ ,\quad \sup_{n\in\mathbb{N}^*}\sigma^r(k_n)<C\ . \tag{3·1}$$

Moreover, for given $m>1$, *the nets*[2]

$$\phi^n = \frac{1}{2\,\pi}\,\mathcal{FT}(h_{g(n)})\quad and\quad \phi^n = \frac{1}{2\,\pi}\,\mathcal{FT}(k_{g(n)})\ ,$$

where $g(n)=\frac12[n^{1/(m-1)}]+1$ *for* $n\in\mathbb{N}^*$, *generate a net of* $\{m,\mathrm{pow}\}$*–mollifiers and a net of* $\{m,\mathrm{der}\}$*–mollifiers, respectively.*

Proof. The first claims, $h_n^{(\alpha)}(0)=k_n^{(\alpha)}(0)=\delta_{\alpha,0}$ are easily verified, and imply obviously $\int x^p\,\mathcal{FT}(h_n)=\int x^p\,\mathcal{FT}(k_n)=2\pi\,\delta_{p,0}\quad\forall p\in\{\,0,\dots,2n-1\,\}$, which gives the second condition on $\{\,m,\mathrm{der}\,\}$ resp. $\{\,m,\mathrm{pow}\,\}$–mollifiers for ϕ^n.

So let us show (3·1), i.e. $h_n\in\Sigma_{\rm der}$, $k_n\in\Sigma^{\rm pow}$ with constants independent of n. Consider first h_n.

The function $\mathbb{C}\ni z\mapsto\sqrt[n]{n^{2n}+z^{2n}}$ has singularities at $z=n\,e^{i\,\pi\,(2k+1)/(2n)}$. The nearest one to the real axis has the imaginary part $n\sin\frac{\pi}{2n}\geq 1\quad\forall n\in\mathbb{N}^*$. So for every $x\in\mathbb{R}$, the open disc $\{\,|z-x|<1\,\}$ lies in the domain of analyticity of h_n. Applying Cauchy's integral formula, we have

$$\begin{aligned}\forall x\in\mathbb{R},\forall n\in\mathbb{N}^*:\ |h_n^{(\alpha)}(x)| &= \left|\frac{\alpha!}{2\pi i}\int_{|\zeta-x|=\frac12}\frac{h_n(\zeta)\,\mathrm{d}\zeta}{(\zeta-x)^{\alpha+1}}\right|\\ &\leq 2^\alpha\,\alpha!\,\max_{\theta\in[0,2\pi]}\left|h_n(x+\tfrac12 e^{i\theta})\right|\ .\end{aligned}$$

Thus we have $\sigma_2(h_n)\leq C$ and therefore (3·1), if $\max|h_n(x+\frac12 e^{i\theta})|<C$. So let us show that there exists $C>0$ such that

$$\forall n\in\mathbb{N}^*,x\in\mathbb{R}:\Re\left(n^2-\sqrt[n]{n^{2n}+(x+\tfrac12\,e^{i\theta})^{2n}}\right)<\ln C\ . \tag{3·2}$$

Let $x+\frac12\,e^{i\,\theta}=\rho\,e^{i\,\phi}$ with $\rho\in\mathbb{R}$, $|\phi|<\frac{\pi}{2}$. Consider first $|\rho|\geq\frac34\,n$. Then, $\sin\phi\leq\frac{2}{3n}$, thus $2n\,\phi\leq 2n\arcsin\frac{2}{3n}<\frac{\pi}{2}\quad\forall n\geq 1$. Therefore $\Re\left(1+\left(\frac1n\,\rho\,e^{i\phi}\right)^{2n}\right)>1$ and (3·2)

[2] we denote the Fourier transform by $\mathcal{FT}(\cdot)$ to avoid confusion with spaces $\mathcal{F}_{p,r}$ etc.

with $\ln C = 0$. Next, if $|\rho| < \frac{3}{4}\, n$, then

$$\Re\mathrm{e}\left(n^2 - \sqrt[n]{n^{2n} + (\rho\, e^{i\phi})^{2n}} \right) < n^2 - n^2 \sqrt[n]{1 - \left(\tfrac{3}{4}\right)^{2n}} < 1\ .$$

(The second function is decreasing for $n \geq 2$.) Again, this implies (3·2), with $\ln C = 1$. So we have shown that $\forall n \in \mathbb{N}^*,\ \sigma_2(h_n) < 3$, which proves (3·1) for h_n. With all that precedes, it is easy to see that the given ϕ^n generate a net of $\{m, \mathrm{pow}\}$–mollifiers.

Now turn to $k_n \in \Sigma^{\mathrm{pow}}$. Estimating $x^\beta k_n(x)$ separately for $|x| \leq 2$ and $|x| > 2$ one can easily prove (3·1). Once again, this allows to conclude that the given ϕ^n generate a net of $\{m, \mathrm{der}\}$–mollifiers. $\square$

The embedding of ultradistributions into the corresponding weighted algebra of sequences is realized through the first part of the next theorem. Its second part deals with the representatives of ultradifferentiable functions implying that the multiplication of regular enough elements within corresponding algebras is the ordinary multiplication.

Theorem 3·5. *Assume $m > 1$.*

(i) *Let $\psi \in \mathcal{D}^{(m)}$ (resp. $\psi \in \mathcal{D}^{\{m-\rho\}}$) with $\rho > 0$ such that $m-\rho > 1$) be compactly supported, and $(\phi^n)_n$ generate a net of $\{m, \mathrm{pow}\}$–mollifiers. Then*

$$\psi * \phi_n - \psi \in \mathcal{N}^{(p!^m, p!^m)}\ , \qquad (\phi_n = n\, \phi^n(n\cdot))$$
$$(\ \textit{resp.}\ \ \psi * \phi_n - \psi \in \mathcal{N}^{(p!^m, p!^m)}\ \subset\ \mathcal{N}^{\{p!^m, p!^m\}}\ )\ .$$

(ii) *Let $f \in \mathcal{E}'^{(m)}$ (resp. $f \in \mathcal{E}'^{\{m\}}$) with compact support; and $(\phi^n)_n$ generate a net of $\{m, \mathrm{der}\}$–mollifiers. Then $f * \phi_n \in \mathcal{E}_{exp}^{(p!^m, p!^{m-1})}$, (resp. $f * \phi_n \in \mathcal{E}_{exp}^{(p!^m, p!^{m-1})} \subset \mathcal{E}_{exp}^{\{p!^m, p!^{m-1}\}}$).*
If $(\phi^n)_n$ and $(\phi'^n)_n$ generate nets of $\{m, \mathrm{pow}\}$–mollifiers, then

$$\forall \psi \in \mathcal{D}^{(m)} : \langle f * \phi_n - f * \phi'_n, \psi \rangle \in \mathcal{N}_0^m\ ,$$
$$(\ \textit{resp.}\ \ \forall \psi \in \mathcal{D}^{\{m-\rho\}} : \langle f * \phi_n - f * \phi'_n, \psi \rangle \in \mathcal{N}_0^m\ )\ .$$

Remark 2. *If $\psi \in \mathcal{D}^{(m)}$, $m > 1$, then $(\psi)_n \in \mathcal{E}^{(p!^m, p!^{m'})}$ for all $m' > 0$. Fix a net of $\{m, \mathrm{pow}\}$–mollifiers $(\phi_n)_n$. The embedding $\mathcal{D}^{(m)} \to \mathcal{E}^{(p!^m, p!^m)}$ can be realized through $\psi \mapsto (\psi * \phi_n)_n$ as well as through $\psi \to (\psi)_n$. This is a consequence of assertion (i). The similar conclusion follows for $\mathcal{D}^{\{m-\rho\}}$. Thus, the product of $\varphi, \psi \in \mathcal{D}^{(m)}$ (resp. $\varphi, \psi \in \mathcal{D}^{\{m-\rho\}}$) is the usual one in $\mathcal{E}^{(p!^m, p!^m)}$ (resp. in $\mathcal{E}^{\{p!^m, p!^m\}}$).*
Assertion (ii) characterizes the embedding of elements in $\mathcal{E}'^{(m)}$ (resp. $\mathcal{E}'^{\{m\}}$) into the corresponding algebra by regularizations by $\{m, \mathrm{der}\}$–mollifiers. Moreover, we have that the regularization of elements in $\mathcal{E}'^{(m)}$ (resp. $\mathcal{E}'^{\{m\}}$) with $\{m, \mathrm{pow}\}$–mollifiers are weakly equal in the sense of ultracomplex numbers.
Note that $\mathcal{D}^{(m_1)} \hookrightarrow D^{\{m_1\}} \hookrightarrow D^{(m_2)}$, $m_2 > m_1 > 1$, where the left space is dense in the right one. This implies $\mathcal{D}'^{(m_2)} \hookrightarrow D'^{\{m_1\}} \hookrightarrow D'^{(m_1)}$. With these relations theorem 3·5 implies various embedding results depending on the parameter $m > 1$.

Proof. **(i)** Assume $\mathrm{supp}\ \psi \subset [-\mu, \mu]$. Since $\psi * \phi_n - \psi = 0$ for $|x| > \mu$, $n > n_0$, we assume in this proof $x \in [-\mu, \mu]$, $n > n_0$. First, we prove the assertion for the Beurling case; the Roumieu case is treated in a similar way.

Let $s \in \mathbb{N}$. We have

$$(\psi * \phi_n - \psi)^{(s)}(x) \;=\; \int_{\mathbb{R}} \Big(\psi^{(s)}(x+t/n) - \psi^{(s)}(x)\Big)\,\phi^n(t)\,\mathrm{d}t$$

$$= \int_{\mathbb{R}} \left(\sum_{p=0}^{N-1} \frac{t^p}{n^p\,p!}\psi^{(p+s)}(x) + \frac{t^N}{n^N N!}\psi^{(N+s)}(\xi) - \psi^{(s)}(x)\right)\phi^n(t)\,\mathrm{d}t\ ,$$

where $x \le \xi \le x+t/n$. For $N = [n^{1/m}]+1$ as in the definition of $\{m, \mathrm{pow}\}$–mollifiers,

$$(\psi * \phi_n - \psi)^{(s)}(x) = \int_{\mathbb{R}} \frac{t^N}{n^N\,N!}\psi^{(N+s)}(\xi)\,\phi^n(t)\,\mathrm{d}t\ .$$

Let $b > 1$ such that $\sigma^b(\phi^n) < \infty$. Then

$$\left|\frac{\nu^s}{s!^m}(\psi * \phi_n - \psi)^{(s)}(x)\right|$$
$$\le \int_{\mathbb{R}} \frac{1}{(N+s)!^m}\left|\psi^{(N+s)}(\xi)\right| \frac{\nu^s(N+s)!^m}{n^N s!^m N!} t^N |\phi^n(t)|\,\mathrm{d}t\ .$$

We use $N!^m \le (N^N)^m$, $(N+s)! \le e^{N+s}\,N!\,s!$ and $\frac{1}{n^N} \le \frac{2^N}{N^{Nm}}$, to get

$$\left|\frac{\nu^s}{s!^m}(\psi * \phi_n - \psi)^{(s)}(x)\right|$$
$$\le \int_{\mathbb{R}} \frac{(2e\,(\nu+b))^{N+s}}{(N+s)!^m}\left|\psi^{(N+s)}(\xi)\right| \frac{N!^m}{N^{mN}}\frac{|t|^N}{b^N N!}|\phi^n(t)|\,\mathrm{d}t\ .$$

Let $\ell > 1$. Inserting $e^{-\ell N}e^{\ell N}$, with $\nu_0 = 2\,\ell\,e\,(\nu+b)$, we have

$$\left|\frac{r^s}{s!^m}\left(\psi * \phi_n - \psi\right)^{(s)}(x)\right| \le 2^{-\ell N} p^{m,\mu}_{\nu_0}(\psi)\,\sigma^b(\phi^n)\ .$$

Now we use $e^{-\ell\,N} \sim e^{-\ell\,n^{1/m}}$ as $n \to \infty$. This implies for every $\nu > 0$ and $\ell > 0$ there exist $C > 0$ so that

$$\left|\frac{\nu^s}{s!^m}(\psi * \phi_n - \psi)^{(s)}(x)\right| \le C\,e^{-\ell\,n^{1/m}}\ .$$

Taking the supremum over all s and x, we obtain that

$$\|\,\psi * \phi_n - \psi\,\|_{p^{m,\mu}_{\nu}, n^{-1/m}} = 0\ .$$

Roumieu case: Let $d > 1$ such that $\sigma^d(\phi^n) < \infty$ and $h > 0$ such that $p^{m-\rho,\mu}_{e^{m-\rho}h}(\psi) < \infty$. With arbitrary $\nu > 0$, as above, we have

$$\left|\frac{\nu^s}{s!^m}(\psi * \phi_n - \psi)^{(s)}(x)\right|$$
$$\le \int_{\mathbb{R}} \frac{|\psi^{(N+s)}(\xi)|}{(N+s)!^{m-\rho}}\frac{\nu^s(N+s)!^{m-\rho}}{n^N s!^m N!} t^N|\phi^n(t)|\,\mathrm{d}t\ .$$
$$\le \int_{\mathbb{R}} \frac{(he^{m-\rho})^{N+s}|\psi^{(N+s)}(\xi)|}{(N+s)!^{m-\rho}}\frac{N!^m}{N^{Nm}}\frac{(h\nu)^s s!^{m-\rho}(2dh)^N}{s!^m N!^\rho}\frac{|t|^N}{d^N N!}|\phi^n(t)|\,\mathrm{d}t\ .$$

Let $\ell > 1$. Note

$$\sup\{\frac{(h\nu)^s s!^{m-\rho}}{s!^m}, s \in \mathbb{N}\} < \infty, \quad \sup\{\frac{(2dhe^{\ell})^N}{N!^\rho}, N \in \mathbb{N}\} < \infty.$$

As above we have, with suitable $C > 0$, (inserting $e^{-\ell N}e^{\ell N}$),

$$\left|\frac{\nu^s}{s!^m}\left(\psi * \phi_n - \psi\right)^{(s)}(x)\right| \le C\,e^{-\ell N}\,p^{m-\rho,\mu}_{e^{m-\rho}h}(\psi)\,\sigma^d(\phi^n)\ .$$

Again as above we finish the proof.
(ii) We will give the proof in the Beurling case. The proof in the Roumieu case is similar. Recall [**9**], if $f \in \mathcal{E}'^{(m)}$, then there exists an ultradifferential operator of class (m), $P(D) = \sum_{k\in\mathbb{N}} a_k\, D^k$, $\mu_0 > 0$ and continuous functions $(F_k)_{k\in\mathbb{N}}$, with the property $\operatorname{supp} F_k \subset [-\mu_0, \mu_0]$, $\sup\limits_{k\in\mathbb{N},x\in\mathbb{R}} |F_k(x)| \le M$, such that $f = \sum_{k\in\mathbb{N}} a_k\, D^k F_k$. This implies

$$\forall x \in \mathbb{R} : f * \phi_n(x) = \sum_{k=0}^{\infty} (-1)^k a_k\, n^k \int_{\mathbb{R}} F_k(x + t/n)\, D^k \phi^n(t)\, \mathrm{d}t\ ,$$

where $(\phi_n)_n$ is a net of $\{m, \mathrm{der}\}$–mollifiers such that $\sigma_b(\phi^n) < \infty$ and a_k, $k \in \mathbb{N}$ satisfy

$$\exists h, B > 0 : \forall k \in \mathbb{N} : |a_k| < Bh^k / k!^m\ .$$

As in the part (i), we take $x \in [-\mu, \mu]$, $\mu > \mu_0$ and $n > n_0$. Let $\nu > 1$ be given and $s \in \mathbb{N}$. We have

$$\begin{aligned}
\frac{\nu^s}{s!^m} \left| f^{(s)} * \phi_n(x) \right| &= \left| \sum_{k=0}^{\infty} (-1)^k a_k n^{k+s} \frac{\nu^s}{s!^m} \int\limits_{\mathbb{R}} F_k(x + t/n)\, D^{k+s} \phi^n(t)\, \mathrm{d}t \right| \\
&\le \sum_{k=0}^{\infty} B \frac{\nu^s h^k n^{k+s}}{k!^m s!^m} \int\limits_{\mathbb{R}} |F_k(x + t/n)|\, |D^{k+s} \phi^n(t)|\, \mathrm{d}t \\
&\le \sum_{k=0}^{\infty} B \frac{(\nu h e)^{s+k} n^{k+s}}{(k+s)!^m} \int\limits_{\mathbb{R}} |F_k(x + t/n)|\, |D^{k+s} \phi^n(t)|\, \mathrm{d}t \\
&\le \sum_{k=0}^{\infty} \frac{B}{2^k} \frac{(2eb\nu h)^{s+k} n^{k+s}}{(k+s)!^{m-1}} \int\limits_{\mathbb{R}} \frac{|F_k(x + \frac{t}{n})|}{b^{k+s}\,(k+s)!} \left| D^{k+s} \phi^n(t) \right| \mathrm{d}t \\
&\le C\, e^{(2eb\nu h n)^{1/(m-1)}} \sigma_b(\phi^n)\ .
\end{aligned}$$

This proves that $f * \phi_n \in \mathcal{E}_{exp}^{(p!^m, p!^{m-1})}$.

Let us prove (for the Beurling case) that

$$\left\langle f, (\check{\phi}_n - \check{\phi}'_n) * \psi \right\rangle \in \mathcal{N}_0^m\ .$$

By continuity, we know that there exist $\mu \in \mathbb{N}^*$, $\nu > 0$ and $C > 0$ such that

$$\begin{aligned}
|\langle f, (\check{\phi}_n - \check{\phi}'_n) * \psi \rangle| &\le C\, p_\nu^{\mu,m}((\check{\phi}_n - \check{\phi}'_n) * \psi) \\
&\le C \left[p_\nu^{\mu,m}(\check{\phi}_n * \psi - \psi) + p_\nu^{\mu,m}(\check{\phi}'_n * \psi - \psi) \right]\ . \qquad (3{\cdot}3)
\end{aligned}$$

By the first part of the theorem we have that

$$\psi * \phi_n - \psi,\ \psi * \phi'_n - \psi \in \mathcal{N}^{(p!^m, p!^m)}\ .$$

This implies that for every $k > 0$ there exists $C > 0$ such that for every $n \in \mathbb{N}^*$ both addends in (3·3) are $\le C\, e^{-k\, n^{1/m}}$. $\square$

4. *Generalized hyperfunctions on the circle*

For $\lambda > 1$, let $\Omega_\lambda = \left\{ z \in \mathbb{C} \mid \frac{1}{\lambda} < |z| < \lambda \right\}$ and $\mathcal{O}_\lambda$ the Banach space of bounded holomorphic functions on Ω_λ. We denote by $\mathcal{E}(\mathbb{T})$ (resp. $\mathcal{A}(\mathbb{T}) := \operatorname{ind\,lim}_{\lambda\to 1} \mathcal{O}_\lambda$) the space of smooth (resp. analytic) functions on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and by $\mathcal{E}'(\mathbb{T})$ (resp. $\mathcal{B}(\mathbb{T})$) the corresponding space of distributions (resp.

hyperfunctions), cf. [**11**]. For $f \in \mathcal{A}(\mathbb{T})$, the coefficient $\widehat{T}(k)$ of $e_k(z) = z^k$ in the Laurent expansion of f is its k-th Fourier coefficient. Complex numbers c_k, $k \in \mathbb{Z}$, are the Fourier coefficients of some analytic function (resp. some hyperfunction) if and only if $\|\!| (c_{\pm k})_{k\in\mathbb{N}^*} |\!\|_{|\cdot|,1/k} < 1$ (resp. ≤ 1).

Let $m \in [0,1)$ and $\nu > 0$. We denote by $\mathcal{A}_{m,\nu}(\mathbb{T})$ the set of functions $f \in \mathcal{A}(\mathbb{T})$ such that $q_\nu^{m,\infty}(f) := \sup_{t\in\mathbb{R},\alpha\in\mathbb{N}} \frac{|\tilde{f}^{(\alpha)}(t)|}{\nu^\alpha \alpha!^m} < \infty$ where $\tilde{f}(t) = f(e^{it}),\ t \in \mathbb{R}$. We set

$$\mathcal{A}_m(\mathbb{T}) = \operatorname*{ind\,lim}_{\nu\to\infty} \mathcal{A}_{m,\nu}(\mathbb{T}) \ \text{ and } \ \mathcal{A}_1(\mathbb{T}) = \operatorname*{ind\,lim}_{m\to 1^-} \mathcal{A}_m(\mathbb{T}).$$

Clearly $\mathcal{A}_1(\mathbb{T})$ a subalgebra of $\mathcal{A}(\mathbb{T})$ whose elements are holomorphic in $\mathbb{C}^*$.

To prove the following theorem, we establish

Lemma 4·1. *Let $m \in (0,1)$ and $\rho > e/2$. Then the function*

$$\varphi : t \mapsto \rho^{-t}\, t^{m(t+\frac{1}{2})}\, e^{-mt}\ , \quad t \in \left[\tfrac{1}{2},\infty\right)$$

reaches its minimum in a unique point t_ρ such that $\frac{1}{2} < t_\rho < \rho^{\frac{1}{m}} - \frac{1}{2}$. Moreover, φ is strictly increasing on $[t_\rho,\infty)$ and $\varphi(\rho^{1/m}+\frac{1}{2}) < \sqrt{e\,\rho}\, e^{-m\rho^{1/m}}$.

Proof. The derivative of $\psi = \ln\varphi$, given by $\psi'(t) = -\ln\rho + m\left(\ln t + \frac{1}{2t}\right)$, is strictly increasing for $t \in (\frac{1}{2},\infty)$ and verifies

$$\psi'(t_\rho) = 0 \iff t_\rho\, e^{1/2\,t_\rho} = \rho^{1/m}\ .$$

This yields $\rho^{1/m} - t_\rho = t_\rho(e^{1/2\,t_\rho} - 1)$, and, using $x < e^x - 1 < x\,e^x$ for $x \neq 0$, the claimed inequalities on t_ρ. Writing $\ln(\rho^{1/m}+\frac{1}{2}) = \frac{1}{m}\ln\rho + \ln(1+\frac{1}{2\,\rho^{1/m}})$ gives $\psi(\rho^{1/m}+\frac{1}{2}) = \frac{1}{2}\ln\rho + m(\rho^{1/m}+1)\ln(1+\frac{1}{2\rho^{1/m}}) - m(\rho^{1/m}+\frac{1}{2})$. Since $\ln(1+\frac{1}{2\rho^{1/m}}) \leq \frac{1}{2\rho^{1/m}}$ it follows that $\psi(\rho^{1/m}+\frac{1}{2}) \leq \frac{1}{2}\ln\rho + m(\frac{1}{2\rho^{1/m}} - \rho^{1/m})$. Using $\rho > e/2$, and $m \in (0,1)$, we find $\frac{m}{2\rho^{1/m}} < \frac{1}{2}$ and thus $\varphi(\rho^{1/m}+\frac{1}{2}) \leq \sqrt{e\,\rho}\, e^{-m\rho^{1/m}}$. $\square$

Previously, we defined the $\|\!| \cdot |\!\|$ norm only for sequences indexed by $\mathbb{N}$. Here it is convenient to use it also for the nets $\hat{f}(k)$ indexed by $\mathbb{Z}$, for which we take it to be the maximum of the norms of $(\hat{f}(k))_{k\in\mathbb{N}}$ and $(\hat{f}(-k))_{k\in\mathbb{N}}$, or equivalently

$$\|\!| (\hat{f}(k))_k |\!\|^{\pm}_{(\cdot)^{-1/m}} = \limsup_{k\to\infty} \Big(\max(|\hat{f}(k)|, |\hat{f}(-k)|)\Big)^{k^{-1/m}}\ .$$

Theorem 4·2. *Let $f \in \mathcal{A}(\mathbb{T})$ and $m \in (0,1)$.*
(i) If $f \in \mathcal{A}_{m,\nu}(\mathbb{T})$ then:

$$\|\!| (\hat{f}(k))_k |\!\|^{\pm}_{(\cdot)^{-1/m}} \leq e^{-m/\nu^{1/m}}.$$

Conversely if the above condition holds, then $f \in \mathcal{A}_{m,\nu'}(\mathbb{T})$ for all $\nu' > \nu$.
(ii) $f \in \mathcal{A}_m(\mathbb{T})$ if and only if

$$\|\!| (\hat{f}(k))_k |\!\|^{\pm}_{(\cdot)^{-1/m}} < 1.$$

(iii) $f \in \mathcal{A}_{0,\nu}(\mathbb{T})$ if and only if $\hat{f}(k) = 0$ for $|k| > \nu$.
(iv) $f \in \mathcal{A}_0(\mathbb{T})$ if and only if $(\hat{f}(k))_{k\in\mathbb{Z}}$ has finite support.
(v) For all $f \in \mathcal{A}_1(\mathbb{T})$ there exists $g \in \mathcal{O}(\mathbb{C}^*)$ such that $g|_{\mathbb{T}} = f$.

Proof. Let $f \in \mathcal{A}_{m,\nu}(\mathbb{T})$ with $0 < m < 1$. For all $\alpha \in \mathbb{N}$, $\tilde{f}^{(\alpha)}(t) =$

$\sum_{p\in\mathbb{Z}}(ip)^\alpha \hat{f}(p)\, e^{ipt}$. It follows that $\int_{-\pi}^{\pi} \tilde{f}^\alpha(t)e^{-ikt}\,\mathrm{d}t = 2\pi(ik)^\alpha \hat{f}(k)$, and then consequently there is a positive constant C_1 such that $|k|^\alpha|\hat{f}(k)| \leq C_1\nu^\alpha\alpha!^m$.
Using Stirling's formula: $\alpha! = \alpha^{\alpha+1/2}\, e^{-\alpha}\sqrt{2\pi}\,(1+\varepsilon_\alpha)$, $\varepsilon_\alpha \searrow 0$, we find a positive constant C_2 such that:

$$\forall\alpha\in\mathbb{N}^*,\ \forall k\in\mathbb{Z},\ |k|^\alpha|\hat{f}(k)| \leq C_2\nu^\alpha\alpha^{m(\alpha+1/2)}e^{-m\alpha}.$$

It follows that:

$$\forall\alpha\in\mathbb{N}^*,\ \forall k\in\mathbb{Z}^*,\ |\hat{f}(k)| \leq C_2\left(\frac{\nu}{|k|}\right)^\alpha \alpha^{m(\alpha+1/2)}\, e^{-m\alpha}.$$

Using the notations of Lemma 4·1 by taking $\rho = \frac{|k|}{\nu}$ with $|k| > e\,\nu$, yields $|\hat{f}(k)| \leq C_2\,\varphi(t)$ for all $t\in\mathbb{N}^*$. Following the Lemma, we have $\varphi(\rho^{1/m}+\frac{1}{2}) \leq \sqrt{e\,\rho}\, e^{-m\rho^{1/m}}$. Since φ increases on $[\rho^{1/m}-\frac{1}{2}, \rho^{1/m}+\frac{1}{2}]$ which contains a positive integer α_ρ, then $|\hat{f}(k)| \leq C_2\,\varphi(\alpha_\rho) \leq C_2\sqrt{e\,\rho}\, e^{-m\rho^{1/m}}$ for $|k| > e\nu$, that is $|\hat{f}(k)| \leq C_2\,(\frac{e\,|k|}{\nu})^{1/2}\, e^{-m/\nu^{1/m}}$ for $|k| > e\nu$ from which inequality of *(i)* follows.
Conversely assume that f satisfies the condition of *(i)*. Let $\nu' > \nu$. Choose ν'' such that $\nu' > \nu'' > \nu$ and set $\beta' = m/(\nu')^{1/m}$, $\beta'' = m/(\nu'')^{1/m}$. Since $e^{-\beta''} > e^{-m/\nu^{1/m}}$, there exists a positive constant $C > |\hat{f}(0)|$ such that $|\hat{f}(k)| \leq Ce^{-\beta''|k|^{1/m}}$ for every $k\in\mathbb{Z}$. For every $\alpha\in\mathbb{N}$, we have $\tilde{f}^{(\alpha)}(t) = \sum_{k\in\mathbb{Z}}(ik)^\alpha\hat{f}(k)\, e^{ikt}$. Then, using the last inequality we find

$$\|\tilde{f}^{(\alpha)}\|_\infty \leq C\left(\sum_{k\in\mathbb{Z}} e^{-(\beta''-\beta')|k|^{1/m}}\right)\sup_{k\in\mathbb{Z}}|k|^\alpha\, e^{-\beta'|k|^{1/m}}.$$

Let $\phi(t) = t^\alpha e^{-\beta' t^{1/m}}$; $t\geq 0$. A simple study of ϕ shows that $\sup_{t\geq 0}\phi(t) = \phi(\nu'\alpha^m) = (\nu')^\alpha\alpha^{m\alpha}\, e^{-m\alpha}$. Using Stirling's formula, we get a positive constant C_1 such that for all $\alpha\in\mathbb{N}$, $\left\|\tilde{f}^{(\alpha)}\right\|_\infty \leq C_1\,\nu'^\alpha\alpha!^m$, showing that $f\in\mathcal{A}_{m,\nu'}(\mathbb{T})$ and proving *(i)*.

Let $f\in\mathcal{A}_m(\mathbb{T})$. Then $f\in\mathcal{A}_{m,\nu}(\mathbb{T})$ for some $\nu>0$ and the inequality follows from *(i)* and $e^{-m/\nu^{1/m}} < 1$. Conversely if $\|\!|\,(\hat{f}(k))_k\,\|\!|^{\pm}_{(\cdot)^{-1/m}} < 1$, there exists $\nu > 0$ such that $\|\!|\,(\hat{f}(k))_k\,\|\!|^{\pm}_{(\cdot)^{-1/m}} \leq e^{-m/\nu^{1/m}}$. From *(i)*, it follows that $f\in\mathcal{A}_{m,\nu'}(\mathbb{T})$ for $\nu' > \nu$. Hence $f\in\mathcal{A}_m(\mathbb{T})$ proving *(ii)*.

Let $f \in \mathcal{A}_{0,\nu}(\mathbb{T})$. The previous shows that there exists $C_1 > 0$ such that $|k|^\alpha|\hat{f}(k)| \leq C_1\nu^\alpha$. It follows that $|\hat{f}(k)| \leq C_1\left(\frac{\nu}{|k|}\right)^\alpha$ for all $k\in\mathbb{Z}^*$ and all $\alpha\in\mathbb{N}$. If $|k| > \nu$, then $\frac{\nu}{|k|} < 1$, and making $\alpha\to\infty$ yields $\hat{f}(k) = 0$.
Conversely, assume that $\hat{f}(k) = 0$ for $|k| > \nu$. Then we have $\forall z\in\mathbb{C}^*$, $f(z) = \sum_{|k|\leq\nu}\hat{f}(k)\, z^k$. It follows that for all $\alpha\in\mathbb{N}$, $\left\|\tilde{f}^{(\alpha)}\right\|_\infty \leq \left(\sum_{|k|\leq\nu}\left|\hat{f}(k)\right|\right)\nu^\alpha$, that is $f\in\mathcal{A}_{0,\nu}(\mathbb{T})$ proving *(iii)*.
Claim *(iv)* follows from *(iii)* straightforwardly.
Claims *(ii)* and *(iv)* show that for $f\in\mathcal{A}_1(\mathbb{T})$, the series $\sum_{k\in\mathbb{Z}}\hat{f}(k)\, z^k$ converges absolutely for any $z\in\mathbb{C}$, proving *(v)*. $\square$

Now let $r = (r_n)_n$ with $r_n > 0$ and $r_n \searrow 0$. For $n\in\mathbb{N}$, we set $\psi_n(z) = \sum_{|k|\leq 1/r_n} z^k$. We have $\psi_n * \psi_n = \psi_n$ and $\lim_{n\to\infty}\psi_n = \delta$ in $\mathcal{E}'(\mathbb{T})$. If $H\in\mathcal{B}(\mathbb{T})$, $H*\psi_n = \sum_{|k|\leq 1/r_n}\hat{H}(k)z^k$ (where $S*T = z\mapsto\sum_{k\in\mathbb{Z}}\widehat{S}(k)\,\widehat{T}(k)\, z^k$) and consequently $\lim_{n\to\infty} H*\psi_n = H$ in $\mathcal{B}(\mathbb{T})$.
For $f\in\mathcal{O}_\lambda$ we set $q^\lambda(f) = \|\, f\,\|_{L^\infty(\Omega_\lambda)}$ and $\hat{q}^\lambda(f) = \sup\limits_{k\in\mathbb{Z}}\lambda^{|k|}|\hat{f}(k)|$. Recall that, for

$f = (f_n)_n \in (\mathcal{O}_\lambda)^{\mathbb{N}}$, we note $\| f \|_{q^\lambda, r} = \limsup_{n\to\infty} q^\lambda(f_n)^{r_n}$ and

$$\overrightarrow{\mathcal{F}}_{q,r} = \left\{ f \in \mathcal{A}_1(\mathbb{T})^{\mathbb{N}} \mid \exists \lambda > 1 : \| f \|_{q^\lambda, r} < \infty \right\} ,$$
$$\overrightarrow{\mathcal{K}}_{q,r} = \left\{ f \in \mathcal{A}_1(\mathbb{T})^{\mathbb{N}} \mid \exists \lambda > 1 : \| f \|_{q^\lambda, r} = 0 \right\} .$$

and exactly the same for $\hat{q}$ instead of q.

Proposition 4·3. *Let $\lambda > 1$ and $f = (f_n)_n \in \mathcal{O}_\lambda^{\mathbb{N}}$. Then we have for any $\mu \in (1, \lambda)$*

$$\| f \|_{q^\mu, r} \leq \| f \|_{\hat{q}^\lambda, r} \leq \| f \|_{q^\lambda, r} .$$

Consequently, $\overrightarrow{\mathcal{F}}_{q,r} = \overrightarrow{\mathcal{F}}_{\hat{q},r}$ and $\overrightarrow{\mathcal{K}}_{q,r} = \overrightarrow{\mathcal{K}}_{\hat{q},r}$

Proof. Let $\lambda > \mu > 1$ and $f = (f_n)_n \in \mathcal{O}_\lambda^{\mathbb{N}}$. For every $k \in \mathbb{Z}$, $\lambda^{|k|} |\hat{f}_n(k)| \leq \hat{q}^\lambda(f_n)$ and then $|\hat{f}_n(k)| \leq \hat{q}^\lambda(f_n) \lambda^{-|k|}$. Using this inequality, we find from $|f_n(z)| \leq \sum_{k\in\mathbb{Z}} |\hat{f}_n(k)|\, |z|^k$ that $|f_n(z)| \leq \hat{q}^\lambda(f_n) \sum_{k\in\mathbb{Z}} \left(\frac{\mu}{\lambda}\right)^k$. It follows that there exists a positive constant $C(\lambda,\mu)$ such that $C(\lambda,\mu) q^\mu(f_n) \leq \hat{q}^\lambda(f_n)$.
From Cauchy's formula $\lambda^{|k|} |\hat{f}_n(k)| \leq q^\lambda(f_n)$ and then $\hat{q}^\lambda(f_n) \leq q^\lambda(f_n)$. Finally we obtain

$$\| f \|_{q^\mu, r} \leq \| f \|_{\hat{q}^\lambda, r} \leq \| f \|_{q^\lambda, r},$$

and then it follows straightforwardly that $\overrightarrow{\mathcal{F}}_{q,r} = \overrightarrow{\mathcal{F}}_{\hat{q},r}$ and $\overrightarrow{\mathcal{K}}_{q,r} = \overrightarrow{\mathcal{K}}_{\hat{q},r}$. □

Definition 4·4. *Let $\mathcal{X}_r(\mathbb{T}) = \overrightarrow{\mathcal{F}}_{q,r}$ and $\mathcal{N}_r(\mathbb{T}) = \overrightarrow{\mathcal{K}}_{q,r}$. The algebra of generalized hyperfunctions on $\mathbb{T}$ is $\mathcal{G}_{H,r} = \mathcal{X}_r(\mathbb{T}) / \mathcal{N}_r(\mathbb{T})$.*

We have an embedding of $\mathcal{B}(\mathbb{T})$ in $\mathcal{G}_{H,r}(\mathbb{T})$ which preserves the usual multiplication of elements in $\mathcal{A}_1(\mathbb{T})$:

Theorem 4·5. *Let*

$$\begin{array}{rccc} \overline{\mathbf{i}} : & \mathcal{B}(\mathbb{T}) & \to & \mathcal{G}_{H,r}(\mathbb{T}) \\ & H & \mapsto & [(H * \psi_n)_n] \end{array} \quad \text{and} \quad \begin{array}{rccc} \overline{\mathbf{i}}_{\mathbf{0}} : & \mathcal{A}_1(\mathbb{T}) & \to & \mathcal{G}_{H,r}(\mathbb{T}) \\ & f & \mapsto & [(f)_n] \end{array} .$$

Then, $\overline{\mathbf{i}}$ is a linear embedding and $\overline{\mathbf{i}}_{\mathbf{0}}$ is a one to one morphism of algebras such that $\overline{\mathbf{i}}|_{\mathcal{A}_1(\mathbb{T})} = \overline{\mathbf{i}}_{\mathbf{0}}$.

Proof. The claim on $\overline{\mathbf{i}}_{\mathbf{0}}$ is easy to prove. Let us focus on the properties of the first part related to $\overline{\mathbf{i}}$. The linearity of $\overline{\mathbf{i}}$ is quite obvious. Let $H \in \mathcal{B}(\mathbb{T})$ and set $h = (h_n)_n$ with $h_n = H * \psi_n$. From Theorem 4·2,(v), we have $h \in \mathcal{X}(\mathbb{T})$.

Now take $\lambda > 1$. From the property of the Fourier coefficients of H, there exists $C > 0$ such that $|\hat{H}(k)| \leq C\, \lambda^{|k|}$ for all $k \in \mathbb{Z}$. It follows that $\lambda^{|k|} |\hat{h}_n(k)| \leq C \lambda^{2/r_n}$ showing that $\| h \|_{\hat{q}^\lambda, r} \leq \lambda^2$. By Proposition 4·3, $h \in \mathcal{X}_r(\mathbb{T})$. It is sufficient to consider restrictions to the spaces $\mathcal{A}_m(\mathbb{T})$ with $0 < m < 1$. Let $f \in \mathcal{A}_m(\mathbb{T})$ with $0 < m < 1$. There is $\lambda > 1$ such that $f(z) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\, z^k$ for $1/\lambda \leq |z| \leq \lambda$. Then we have $\overline{\mathbf{i}}_{\mathbf{0}}(f) - \overline{\mathbf{i}}(f) = [f_n]$ where $f_n = f - f * \psi_n$, that is $f_n(z) = \sum_{|k| > 1/r_n} \hat{f}(k)\, z^k$. Then we have $(f_n)_n \in \mathcal{O}_\lambda$.
We claim that $(f_n)_n \in \mathcal{N}_r(\mathbb{T})$. From Theorem 4·2, there exist $p \in (0,1)$ and $C > 0$ such that every $k \in \mathbb{Z}$, $|\hat{f}(k)| \leq C p^{|k|^{1/m}}$. For $|k| > r_n^{-1}$, writing $p^{|k|^{1/m}} \leq p^{\frac{1}{2}|k|^{1/m}} p^{\frac{1}{2} r_n^{-1/m}}$, we find $\left(\lambda^{|k|} |\hat{f}_n(k)|\right)^{r_n} \leq \left(C \lambda^{|k|} p^{\frac{1}{2}|k|^{1/m}}\right)^{r_n} p^{\frac{1}{2} r_n^{(m-1)/m}}$.

Since $C\lambda^{|k|}p^{\frac{1}{2}|k|^{1/m}}$ is bounded with respect to k, because of $1/m > 1$ and $p \in (0,1)$, it follows that $\| f \|_{\hat{q}^\lambda, r} = 0$, proving our claim. $\square$

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