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Abstract

We review our recent formulation (with A. Delcroix, S. Pilipović and V. Valmorin) of Colombeau type algebras as Hausdorff sequence spaces with ultranorms, defined by sequences of exponential weights. We extend previous results and give several new perspectives related to echelon type spaces, possible generalisations, asymptotic algebras, concepts of association, and applications thereof.

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1 Introduction

Colombeau's New Generalized Functions [1] are today the most widely used associative differential algebras containing the δ–distribution. Their topology is studied since the late 1990s [3], and investigation in topological duals of such spaces is now emerging as important topic of research in this field.

We define such algebras right from the start as spaces with ultranorms [5, 4], which is natural and especially useful for practical use of the topology, with no need for valuations. Our construction allows for algebras containing ultradistributions and periodic hyperfunctions [6]. Without specializing to a concrete space, we deduce general results about completeness, embedding of duals and functoriality, and generalize known concepts of association, revealing aspects of the underlying structure rather hidden in other approaches. Our approach also shows better the close link with the classical theory of sequence spaces.

2 The basic construction

Consider a sequence $r = (r_n)_n \in (\mathbb{R}_+)^{\mathbb{N}}$ decreasing to zero. For a seminorm p on an $\mathbb{R}$– or $\mathbb{C}$–vector space E, this defines a map $\| \cdot \|_{p,r} : E^{\mathbb{N}} \to \overline{\mathbb{R}}_+ = [0, \infty]$,

$$f = (f_n)_n \quad \longmapsto \quad \| f \| = \| f \|_{p,r} = \limsup_{n\to\infty} \big(p(f_n)\big)^{r_n} \ .$$

Lemma 2.1 *(a) If* $0 < \liminf p(f_n) \leq \limsup p(f_n) < \infty$*, then* $\| f \| = 1$.
(b) For all $f, g \in E^{\mathbb{N}}$, $\lambda \in \mathbb{C}^*$: $\| f + g \| \leq \max(\| f \|, \| g \|)$ *and* $\| \lambda f \| = \| f \|$.
(c) If E is a topological algebra, then $\| f \cdot g \| \leq \| f \| \cdot \| g \|$.

Proof. As $\lim r_n = 0$, we have $\lim k^{r_n} = 1$ for any $k > 0$, thus (a). Writing $p(\lambda f_n) \leq |\lambda|\, p(f_n)$ and $p(f_n + g_n) \leq 2 \max(p(f_n), p(g_n))$, we have (b), and using $\exists C > 0\ \forall x, y \in E : p(x\, y) \leq C\, p(x)\, p(y)$, we get (c) in the same way. $\square$

Definition 2.2 ***The** r**–generalized semi-normed space*** (E,p) *is the factor space* $\mathcal{G}_r(E,p) = \mathcal{F}_r(E,p)\,/\,\mathcal{K}_r(E,p)$*, where*

$$\mathcal{F}_r(E,p) = \left\{ f \in E^{\mathbb{N}} \mid \| f \|_{p,r} < \infty \right\}\ ,\quad \mathcal{K}_r(E,p) = \left\{ f \in E^{\mathbb{N}} \mid \| f \|_{p,r} = 0 \right\}\ .$$

Proposition 2.3 *The map* $\|\cdot\|_{p,r}$ *defines a pseudometric* $d_{p,r}$ *on* $\mathcal{F}_r(E,p)$*, making it a topological ring, if* E *is a topological algebra. As* $\mathcal{K}_r(E,p)$ *is the intersection of neighborhoods of zero,* $\mathcal{G}_r(E,p)$ *is then the associated Hausdorff topological ring, on which* $d_{p,r}$ *is well-defined and an ultrametric.*

Proof. This is a direct consequence of the Definition and preceding Lemma. □

Example 2.4 *For* $E = \mathbb{C}$ *and* $p = |\cdot|$*, we obtain* ***the ring of** r**–generalized complex numbers***, $\mathbb{C}_r = \mathcal{G}_r(\mathbb{C},|\cdot|)$*. For* $r_n = \frac{1}{\log n}$ $(n>1)$*, this are* ***Colombeau's generalized numbers*** $\widetilde{\mathbb{C}}$*, since* $\limsup |x_n|^{1/\log n} < \infty \iff \exists\gamma\in\mathbb{R} : |x_n| = o(n^\gamma)$*, and* $\lim |x_n|^{1/\log n} = 0 \iff \forall\gamma\in\mathbb{R} : |x_n| = o(n^\gamma)$.
The choice $r_n = n^{-1/m}$ *(with* $m>0$*), leads to* ***ultracomplex numbers*** $\overline{\mathbb{C}}^{p!^m}$.

Proposition 2.5 *The spaces* $\mathcal{G}_r(E,p)$ *(resp.* $\mathcal{F}_r(E,p)$*) are topological algebras over the generalized numbers (resp. over* $\mathcal{F}_r(\mathbb{K},|\cdot|)$*) equipped with* $\|\cdot\|$*–topology, but they aren't topological vector spaces over the field* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*.*

Proof. This is seen by observing that Lemma 2.1-(c) also holds for $f \in \mathbb{C}^{\mathbb{N}}$, while Lemma 2.1-(b) implies that $\|\lambda f\|$ does not go to zero when $\lambda \to 0$. □

Example 2.6 *To obtain* r**–generalized Sobolev algebras** $\mathcal{G}_{W^{s,\infty}(\Omega)} = \mathcal{G}_r\big(W^{s,\infty}(\Omega), p_{s,\infty}\big)$*, we choose* $E = W^{s,\infty}(\Omega)$ *with norm* $p_{s,\infty} = \sum_{|\alpha|<s} \|\partial^\alpha \cdot\|_{L^\infty}$*. This generalizes to any normed algebra.*

Theorem 2.7 ((equivalent scales)) *If* $r = (r_n)_n$*,* $s = (s_n)_n$ *decrease to zero such that* $s = O(r)$*, then* $\mathcal{F}_r(E,p) \subset \mathcal{F}_s(E,p)$*,* $\mathcal{K}_s(E,p) \subset \mathcal{K}_r(E,p)$*. In particular, if* $\lim\limits_{n\to\infty} \frac{s_n}{r_n} = C \in \mathbb{R}_+^*$*, then* $\| f \|_{p,s} = (\| f \|_{p,r})^C$*, and thus* $\mathcal{F}_s(E,p) = \mathcal{F}_r(E,p)$*,* $\mathcal{K}_s(E,p) = \mathcal{K}_r(E,p)$ *and* $\mathcal{G}_s(E,p) = \mathcal{G}_r(E,p)$*.*

Proof. If $s_n = c_n r_n$ with $\limsup c_n = C \in \mathbb{R}_+^*$, we have $\log \| f \|_{p,s} = \limsup(s_n \log p(f_n)) = \limsup(c_n r_n \log p(f_n)) \le C \limsup(r_n \log p(f_n)) = C \log \| f \|_{p,r}$, where we assumed $\lim \log p(f_n) \ge 0$, *i.e.* $\| f \| \ge 1$. Otherwise, $\le$ must be replaced by $\ge$, leading to the inverse inclusion for $\mathcal{K}$. □

Remark 2.8 ((relation to Maddox' sequence spaces)) *Our spaces* $\mathcal{K}_r(\mathbb{C},|\cdot|)$ *and* $\mathcal{F}_r(\mathbb{C},|\cdot|)$ *are the same as* $c_0(r) = \bigcap_{k\in\mathbb{N}} \left\{ x \in \mathbb{C}^{\mathbb{N}} \mid \lim |x_n|\, k^{1/r_n} = 0 \right\}$ *and* $\ell_\infty(r) = \bigcup_{k\in\mathbb{N}} \left\{ x \in \mathbb{C}^{\mathbb{N}} \mid \sup |x_n|\, k^{-1/r_n} < \infty \right\}$*, introduced in [7, 8] and studied extensively by Maddox and his students [9, 10]. To see this, observe that* $\exists k \in \mathbb{N} : \sup |x_n|\, k^{-1/r_n} < \infty \iff \exists k : \limsup |x_n|^{r_n} \le k \iff \| x \|_r < \infty$*, and* $\forall k : \lim |x_n|\, k^{1/r_n} = 0 \iff \forall\varepsilon>0 : |x_n| = o(\varepsilon^{1/r_n}) \iff \| x \|_r = 0$.

These spaces belong to the classes of echelon resp. co-echelon spaces. As we always require $\lim r_n = 0$*, both are Montel and Schwartz spaces. While the cited work on sequence spaces is restricted to* $(\mathbb{C},|\cdot|)$*, our studies concern more general spaces. However, most of the spaces considered in the sequel can be written as intersection and/or union of echelon and co-echelon type spaces. This also allows the generalization of the present construction to any abstract topological module* E*, as will be discussed in a forthcoming publication.*

3 Generalized locally convex spaces

Definition 3.1 *The r–**extension of a locally convex space** $(E,\mathcal{P})$ is the factor space* $\mathcal{G}_r(E,\mathcal{P})=\mathcal{F}_r(E,\mathcal{P})\,/\,\mathcal{K}_r(E,\mathcal{P})=\bigcap\limits_{p\in\mathcal{P}}\mathcal{F}_r(E,p)\,\Big/\bigcap\limits_{p\in\mathcal{P}}\mathcal{K}_r(E,p)$.

Theorem 3.2 *If $(E,\mathcal{P})$ is a topological algebra,* i.e. $\forall p\in\mathcal{P}\ \exists\bar p\in\mathcal{P}\ \exists C>0\,\forall x,y\in E: p(x\,y)\le C\,\bar p(x)\,\bar p(y)$, *then $\mathcal{F}_r(E,\mathcal{P})$ is a subalgebra of $E^{\mathbb{N}}$, $\mathcal{K}_r(E,\mathcal{P})$ is an ideal of $\mathcal{F}_r(E,\mathcal{P})$, and $(d_{p,r})_{p\in\mathcal{P}}$ is a family of pseudo-distances on $\mathcal{G}_r(E,\mathcal{P})$ making it a Hausdoff topological algebra over $\mathbb{C}_r$.*

Proof. Lemma 2.1-(b) yields for $f,g\in\mathcal{F}_r$, $\lambda\in\mathbb{C}$, and $p\in\mathcal{P}$: $\|\!|\,\lambda f+g\,|\!\|_p\le\max(\|\!|\,f\,|\!\|_p,\|\!|\,g\,|\!\|_p)$, thus $\mathcal{F}_r$ and $\mathcal{K}_r$ are $\mathbb{C}$–linear subspaces. Continuity of multiplication in $(E,\mathcal{P})$ gives as in 2.1-(c), $\forall p\in\mathcal{P},\ \exists\bar p\in\mathcal{P}:\|\!|\,f\,g\,|\!\|_p\le\|\!|\,f\,|\!\|_{\bar p}\cdot\|\!|\,g\,|\!\|_{\bar p}$. Thus $\mathcal{F}_r$ is a $\mathbb{C}$–subalgebra of $E^{\mathbb{N}}$, and $\mathcal{K}_r$ an ideal of $\mathcal{F}_r$. The inequalities also imply continuity of addition and multiplication, thus $\mathcal{F}_r$ is a topological $\mathcal{F}_{|\cdot|,r}$–algebra, and $\mathcal{G}_{\mathcal{P}}$ is again the associated Hausdorff space. □

Example 3.3 *The classical **simplified Colombeau algebra** [1] is obtained for $r_n=\frac{1}{\log n}$ and $\mathcal{P}=\big\{\,p^\mu_\nu:f\mapsto\sup\limits_{|\alpha|\le\nu,|x|\le\mu}|f^{(\alpha)}(x)|\,\big\}_{\mu,\nu\in\mathbb{N}}$ on $E=\mathcal{C}^\infty(\Omega)$.*

As a last generalization of the base space, consider a family of semi-normed algebras $(E^\mu_\nu,p^\mu_\nu)_{\mu,\nu\in\mathbb{N}}$ with embeddings $\forall\mu,\nu\in\mathbb{N}:E^{\mu+1}_\nu\hookrightarrow E^\mu_\nu$, $E^\mu_\nu\hookrightarrow E^\mu_{\nu+1}$ resp. $E^\mu_{\nu+1}\hookrightarrow E^\mu_\nu$. Let $\overrightarrow{E}=\operatorname{proj}\lim\limits_{\mu\to\infty}\operatorname{ind}\lim\limits_{\nu\to\infty}E^\mu_\nu$, resp. $\overleftarrow{E}=\operatorname{proj}\lim\limits_{\mu\to\infty}\operatorname{proj}\lim\limits_{\nu\to\infty}E^\mu_\nu$, and assume that for all $\mu\in\mathbb{N}$ the inductive limit is regular, *i.e.* a subset is bounded iff it is a bounded subset of $(E^\mu_\nu)^{\mathbb{N}}$ for some $\nu\in\mathbb{N}$. Now let

$\mathcal{F}_r(\overrightarrow{E})=\Big\{\,f\in\overrightarrow{E}^{\mathbb{N}}\ \Big|\ \forall\mu\in\mathbb{N},\exists\nu\in\mathbb{N}:f\in(E^\mu_\nu)^{\mathbb{N}}\wedge\|\!|\,f\,|\!\|_{p^\mu_\nu,r}<\infty\,\Big\}$,

$\mathcal{F}_r(\overleftarrow{E})=\big\{\,f\in\overleftarrow{E}^{\mathbb{N}}\mid\forall\mu,\nu\in\mathbb{N}:\|\!|\,f\,|\!\|_{p^\mu_\nu,r}<\infty\big\}$, and the obvious definition for $\mathcal{K}_r(\overleftrightarrow{E})$, where we write $\overleftrightarrow{\cdot}$ for both, $\overrightarrow{\cdot}$ and $\overleftarrow{\cdot}$. Then again, $\mathcal{G}_r(\overleftrightarrow{E})=\mathcal{F}_r(\overleftrightarrow{E})\,/\,\mathcal{K}_r(\overleftrightarrow{E})$ is a topological algebra for the respective limit topology.

Example 3.4 *In [6] we showed how to embed Gevery class ultradistributions in Colombeau algebras $\mathcal{G}^{(p!^m,p!^{m'})}=\mathcal{G}_{r^{m'}}(\mathcal{E}^{(m)})$, $\mathcal{G}^{\{p!^m,p!^{m'}\}}=\mathcal{G}_{r^{m'}}(\mathcal{E}^{\{m\}})$, where $r^m_n=n^{\frac1m}$ and $\mathcal{E}^{(m)}$, $\mathcal{E}^{\{m\}}$ are the proj–proj resp. proj–ind limit type spaces of Beurling resp. Roumieu ultradifferentiable functions, defined trough spaces on which $p^{m,\mu}_\nu(f)=\sup\limits_{|x|\le\mu,\alpha\in\mathbb{N}^s}\dfrac{\nu^{|\alpha|}}{\alpha!^m}|f^{(\alpha)}(x)|$ resp. $q^{m,\mu}_\nu=p^{m,\mu}_{1/\nu}$ are finite.*

Definition 3.5 *Consider the spaces $\mathcal{O}_\lambda$ of holomorphic functions on $\Omega_\lambda=\big\{\,z\in\mathbb{C}\mid\frac1\lambda<|z|<\lambda\,\big\}$ with finite norm $q^\lambda=\|\cdot\|_{L^\infty(\Omega_\lambda)}$. Analytic functions on the unit circle are then $\mathcal{A}(\mathbb{T})=\operatorname{ind}\lim\limits_{\lambda\to1}\mathcal{O}_\lambda$. Let $\overrightarrow{E}=\operatorname{ind}\lim\limits_{\lambda\to1}(\mathcal{A}_1(\mathbb{T}),q^\lambda)$, where*

$$\mathcal{A}_1(\mathbb{T})=\operatorname{ind}\lim_{m\to1^-}\operatorname{ind}\lim_{\nu\to\infty}\Big\{\,f\in\mathcal{A}(\mathbb{T})\mid\|f^{(\alpha)}\|_{L^\infty(\mathbb{T})}\underset{\alpha\to\infty}{=}O(\nu^\alpha\,\alpha!^m)\Big\}\,.$$

*Then, r–**generalized hyperfunctions on** $\mathbb{T}$ are defined as $\mathcal{G}_{H,r}(\mathbb{T})=\mathcal{G}_r(\overrightarrow{E})$, quotient space of $\mathcal{F}_r(\overrightarrow{E})=\bigcup\limits_{\lambda>0}\mathcal{F}_r(\mathcal{A}_1(\mathbb{T}),q^\lambda)$ by $\mathcal{K}_r(\overrightarrow{E})=\bigcup\limits_{\lambda>0}\mathcal{K}_r(\mathcal{A}_1(\mathbb{T}),q^\lambda)$.*

The same type of ultranorm can be used to characterize generalized functions f on the unit circle by means of their Fourier coefficients $(\widehat{f}_k)_k\in\mathbb{C}^{\mathbb{Z}}$, for which we define $\|\!|\,(\widehat{f}_k)_{k\in\mathbb{Z}}\,|\!\|^{\pm}_r=\max\big\{\|\!|\,(\widehat{f}_k)_{k\in\mathbb{N}}\,|\!\|_{|\cdot|,r}\,,\ \|\!|\,(\widehat{f}_{-k})_{k\in\mathbb{N}}\,|\!\|_{|\cdot|,r}\big\}$.

Fourier coefficients of analytic functions $f\in\mathcal{A}(\mathbb{T})$, Schwartz distributions $T\in\mathcal{E}'(\mathbb{T})$ and hyperfunctions $H\in\mathcal{B}(\mathbb{T})$ are characterized by $\|\!|\,(\widehat{f}_k)_k\,|\!\|^{\pm}_{(\cdot)^{-1}}<1$, $\|\!|\,(\widehat{T}_k)_k\,|\!\|^{\pm}_{1/\log}<\infty$, resp. $\|\!|\,(\widehat{H}_k)_k\,|\!\|^{\pm}_{(\cdot)^{-1}}\le1$.

Proposition 3.6 ((Fourier characterization)) *The same spaces $\mathcal{F}_r(\overrightarrow{E})$, $\mathcal{K}_r(\overrightarrow{E})$ are obtained in the previous definition if q^λ is replaced by $\widehat{q}^\lambda : f \mapsto \sup_{k\in\mathbb{Z}} \lambda^{|k|}|\widehat{f}_k|$.*

Proof. If $f \in \mathcal{F}_r(\overrightarrow{E})$, $\|\, f \,\|_{q^\lambda,r} < \infty$, there is $C > 0$ such that $q^\lambda(f_n)^{r_n} < C$ for all n. Cauchy's inequalities in Ω_λ then give $|\hat{f}_n(k)| \leq q^\lambda(f_n)\lambda^{-|k|}$, thus $|\hat{f}_n(k)|^{r_n} \leq C\,\lambda^{-|k|r_n}$ for all $k \in \mathbb{Z}$, whence $\|\, f \,\|_{\hat{q}^\lambda,r} < \infty$. Conversely, if $f \in \overrightarrow{E}^{\mathbb{N}}$, $\|\, f \,\|_{\hat{q}^\lambda,r} < \infty$, we have $\hat{q}^\lambda(f_n)^{r_n} < C$ for some $C > 0$ and all n, *i.e.* $|\hat{f}_n(k)| < C^{1/r_n}\lambda^{-|k|}$ for all $k \in \mathbb{Z}$. Consequently, there is $M > 0$ such that $q^\lambda(f_n) \leq M\,C^{1/r_n}$, thus $\|\, f \,\|_{q^\lambda,r} < \infty$. The proof for $\mathcal{K}$ goes the same way. □

Convolution with mollifiers $\phi_n = \sum_{|k|\leq 1/r_n} z^k$ allows to embed hyperfunctions $\mathcal{B}(\mathbb{T})$ into $\mathcal{G}_{H,r}(\mathbb{T})$, preserving the usual product of $\mathcal{A}_1(\mathbb{T})$ [6].

Proposition 3.7 ((Completeness)) *Without assuming completeness of $\overleftrightarrow{E}$, $\mathcal{F}_r(\overleftarrow{E})$ is complete, and $\mathcal{F}_r(\overrightarrow{E})$ is sequentially complete.*

Proof. If $(f^m)_m$ is a Cauchy sequence in $\mathcal{F}_r(\overleftarrow{E})$, there are increasing sequences $(m_\mu), (n_\mu) \in \mathbb{N}^{\mathbb{N}}$ such that $\forall \mu \in \mathbb{N}$, $\forall\, k, \ell \geq m_\mu$, $\limsup_{n\to\infty} p^\mu_\mu\left(f^k_n - f^\ell_n\right)^{r_n} < \frac{1}{2^\mu}$ and more precisely $\forall\, k, \ell \in [m_\mu, m_{\mu+1}]$ $\forall n \geq n_\mu$: $p^\mu_\mu\left(f^k_n - f^\ell_n\right)^{r_n} < \frac{1}{2^\mu}$. Let $\bar{\mu}(n) = \sup\{\, \mu \mid n_\mu \leq n\,\}$, and consider the sequence $\bar{f} = (f^{m_{\bar{\mu}(n)}}_n)_n$. Then we have $f^m \to \bar{f}$ in $\mathcal{F}_r(\overleftarrow{E})$. Indeed, for ε and $p^{\mu_0}_\nu$ given, take $\mu > \mu_0$, ν such that $\frac{1}{2^\mu} < \frac{1}{2}\varepsilon$. As p^μ_ν is increasing in both indices, we have for $m \in [m_{\mu+s}, m_{\mu+s+1}]$:

$$p^{\mu_0}_\nu(f^m_n - \bar{f}_n)^{r_n} \leq p^\mu_\mu(f^m_n - f^{m_{\mu+s+1}}_n)^{r_n} + \textstyle\sum_{\mu'=\mu+s+1}^{\bar{\mu}(n)-1} p^{\mu'}_{\mu'}(f^{m_{\mu'}}_n - f^{m_{\mu'+1}}_n)^{r_n}\ .$$

For $n > n_{\mu+s}$, one has $n \geq n_{\bar{\mu}(n)}$, thus $p^{\mu_0}_\nu(f^m_n - \bar{f}_n)^{r_n} < \sum_{\mu'=\mu+s}^{\bar{\mu}(n)} \frac{1}{2^{\mu'}} < \frac{2}{2^\mu} < \varepsilon$.
As $\mathcal{F}_r(\overleftarrow{E})$ is a metrisable space, this implies completeness.
For Cauchy nets in $\mathcal{F}_r(\overrightarrow{E})$, we use that for all μ there is $\nu(\mu)$ such that $p^\mu_{\nu(\mu)} \leq p^{\mu+1}_{\nu(\mu+1)}$ and $p^\mu_{\nu(\mu)}\left(f^m_n - f^p_n\right)^{r_n} < \varepsilon_\mu$, where $(\varepsilon_\mu)_\mu$ decreases to zero. With this, we can prove the sequential completeness of $\mathcal{F}_r(\overrightarrow{E})$ by the same arguments as above. □

Remark 3.8 ((discreteness of induced topology)) *In [6] we have shown that a net $(\delta_n)_n \in \overleftrightarrow{E}^{\mathbb{N}}$ such that $\forall \psi \in \overleftrightarrow{E}$, $\lim_{n\to\infty} \int_{\mathbb{R}^s} \delta_n \cdot \psi = \psi(0)$, cannot be bounded in $\overleftrightarrow{E}$, under very weak assumptions. From this we deduce that the topology of any algebra containing δ and $\overleftrightarrow{E}$* **must** *induce the discrete topology on $\overleftrightarrow{E}$.*

4 Families of scales and asymptotic algebras

We now generalize the growth conditions. Consider a family $r = (r^m)_m$ of sequences $(r^m_n)_n$ decreasing to zero as $n \to \infty$. Suppose that either

$$\text{(I)}\quad \forall m \in \mathbb{N} : r^m = O(r^{m+1}), \qquad \text{or} \qquad \text{(II)}\quad \forall m \in \mathbb{N} : r^{m+1} = O(r^m).$$

Theorem 4.1 *Define $\mathcal{F}_r(\overleftrightarrow{E}) = \bigcap_{m\in\mathbb{N}} \mathcal{F}_{r^m}(\overleftrightarrow{E})$, $\mathcal{K}_r(\overleftrightarrow{E}) = \bigcup_{m\in\mathbb{N}} \mathcal{K}_{r^m}(\overleftrightarrow{E})$ in case (I), resp. $\mathcal{F}_r(\overleftrightarrow{E}) = \bigcup_{m\in\mathbb{N}} \mathcal{F}_{r^m}(\overleftrightarrow{E})$, $\mathcal{K}_r(\overleftrightarrow{E}) = \bigcap_{m\in\mathbb{N}} \mathcal{K}_{r^m}(\overleftrightarrow{E})$ in case (II). Then again, $\mathcal{G}_r(\overleftrightarrow{E}) = \mathcal{F}_r(\overleftrightarrow{E}) \,/\, \mathcal{K}_r(\overleftrightarrow{E})$ is an algebra.*

Proof. Using $\mathcal{K}_{r^m}(\overleftrightarrow{E}) \cdot \mathcal{F}_{r^m}(\overleftrightarrow{E}) \subset \mathcal{K}_{r^m}(\overleftrightarrow{E})$ and Theorem 2.7, it is easy to verify that in both cases, $\mathcal{F}_r(\overleftrightarrow{E})$ is a subalgebra and $\mathcal{K}_r(\overleftrightarrow{E})$ is an ideal thereof. □

Example 4.2 *For $r^m_n = 1$ if $n \leq m$, 0 elsewhere, we get Egorov-type algebras.*

Definition 4.3 *Let* $\mathbf{a} = (a_m : \mathbb{N} \to \mathbb{R}_+)_{m\in\mathbb{Z}}$ *be an asymptotic scale,* i.e. $\forall m \in \mathbb{Z}$, $a_{m+1} = o(a_m)$, $a_{-m} = 1/a_m$ *and* $\exists M \in \mathbb{Z} : a_M = o(a_m^2)$. *The asymptotic algebra defined by* $\mathbf{a}$ *and a locally convex algebra* $(E, \mathcal{P})$ *is the factor space* $\mathcal{A}_{(\mathbf{a})}(E,\mathcal{P}) = \dfrac{\{ f \in E^{\mathbb{N}} \mid \exists m \in \mathbb{Z} \;\; \forall p \in \mathcal{P} : \;\; p(f) = O(a_m) \}}{\{ f \in E^{\mathbb{N}} \mid \forall m \in \mathbb{Z} \;\; \forall p \in \mathcal{P} : \;\; p(f) = o(a_m) \}}$.

Example 4.4 *(i)* $a_m(n) = n^{-m}$ *leads to Colombeau type generalized algebras.*
(ii) $a_m = 1/\exp^m$ *(m-fold iterated* exp *function) gives exponential algebras [2].*

Theorem 4.5 *For* $r_n^m = |\log a_m(n)|^{-1}$, *we have* $\mathcal{G}_r(E,\mathcal{P}) = \mathcal{A}_{(\mathbf{a})}(E,\mathcal{P})$.

Proof. If $p(f_n) < C\, a_m(n) = C\, e^{1/r^m}$ (for $a_m > 1$), then $\limsup (p \circ f)^{r^m} < \infty$ and $f \in \mathcal{F}_r(E,\mathcal{P})$. Conversely, if $\limsup (p\circ f)^{1/|\log a_{\bar m}|} < C$ then $p \circ f \le (a_m)^{\log C}$, $(a_m, C > 1)$. Using the third property of scales, $\exists M : p\circ f = o(a_M)$.
Now consider $\forall \bar m : p \circ f = o(a_{\bar m})$. Take $m \in \mathbb{N}$. Then, for any $q \in \mathbb{N}$, there is $\hat m$ such that $a_{\hat m} = o({a_m}^q)$ and $p\circ f = o(a_{\hat m}) = o({a_m}^q) = o((e^{-1/r_m})^q) = o((e^{-q})^{1/r^m})$. Therefore $\limsup (p\circ f)^{r^m} \le e^{-q}$, and as q was arbitrary, we have $\| f \|_{p,r^m} = 0$, thus $f \in \mathcal{K}_r(E,\mathcal{P})$.
Finally assume $\forall \bar m : \limsup p(f_n)^{r^{\bar m}} = 0$, *i.e.* $\forall C > 0 : (p\circ f)^{1/|\log a_{\bar m}|} = o(C)$, thus $p(f) = O(C^{|\log a_{\bar m}|} = O({a_{\bar m}}^{|\log C|})$. Now for any m, let $\bar m = m+1$ and $C = 1/e$. Then $p(f) = O(a_{\bar m}) = o(a_m)$, as required. □

A second kind of "asymptotic" algebras is of the form

$$\mathcal{A}^{(\mathbf{a})}(E,\mathcal{P}) = \frac{\{ f \in E^{\mathbb{N}} \mid \forall \sigma < 0 \;\; \forall p \in \mathcal{P} : \;\; p(f) = o(a_\sigma) \}}{\{ f \in E^{\mathbb{N}} \mid \exists \sigma > 0 \;\; \forall p \in \mathcal{P} : \;\; p(f) = o(a_\sigma) \}},$$

where $\mathbf{a} = (a_\sigma)_{\sigma\in\mathbb{R}}$ is a scale (*i.e.* $\forall \sigma > \rho$, $a_\sigma = o(a_\rho)$, etc.), indexed by a real number. As the subalgebra is here given as intersection and the ideal as union of sets, this case is not covered by the previous one.

Proposition 4.6 *For* $r^m = \frac{1}{|\log a_{1/m}|}$, *we have* $\mathcal{A}^{(\mathbf{a})}(E,\mathcal{P}) = \mathcal{F}'_r(\mathcal{P})/\mathcal{K}'_r(\mathcal{P})$, *with*
$\mathcal{F}'_r(\mathcal{P}) = \mathcal{F}'_r(E,\mathcal{P}) = \left\{ f \in E^{\mathbb{N}} \mid \forall m \in \mathbb{N}\; \forall p \in \mathcal{P} : \| f \|_{p,r^m} \le 1 \right\}$ *and*
$\mathcal{K}'_r(\mathcal{P}) = \mathcal{K}'_r(E,\mathcal{P}) = \left\{ f \in E^{\mathbb{N}} \mid \exists m \in \mathbb{N}\; \forall p \in \mathcal{P} : \| f \|_{p,r^m} < 1 \right\}$.

Example 4.7 $a_\sigma(n) = e^{-n\,\sigma}$ *gives algebras with infra–exponential growth [3], of particular interest for embeddings of periodic hyperfunctions.*

5 Functorial properties

A map $\varphi : \overleftrightarrow{E} \to \overleftrightarrow{F}$ obviously extends canonically to $\mathcal{G}_r(\varphi) : \mathcal{G}_r(\overleftrightarrow{E}) \to \mathcal{G}_r(\overleftrightarrow{F})$ if for all $f \in \mathcal{F}_r(\overleftrightarrow{E})$ and $k \in \mathcal{K}_r(\overleftrightarrow{E})$, we have

$$(F_1): \quad \varphi(f) = (\varphi(f_n))_n \in \mathcal{F}_r(\overleftrightarrow{F}) \;, \qquad \text{and} \quad (F_2): \quad \varphi(f+k) - \varphi(f) \in \mathcal{K}_r(\overleftrightarrow{F}) \;.$$

Definition 5.1 *The* $r-$***extension of a map*** $\varphi : \overleftrightarrow{E} \to \overleftrightarrow{F}$ *satisfying the above conditions* $(F_1), (F_2)$, *is defined as the map* $\mathcal{G}_r(\varphi) : \mathcal{G}_r(\overleftrightarrow{E}) \to \mathcal{G}_r(\overleftrightarrow{F})$ *such that* $[f] \mapsto \varphi(f) + \mathcal{K}_r(\overleftrightarrow{F})$, *where* f *is any representative of* $[f] = f + \mathcal{K}_r(\overleftrightarrow{E})$.

Example 5.2 *Linear mappings* φ *of locally convex vector spaces* $(E,\mathcal{P}) \to (F,\mathcal{Q})$ *are continuous iff* $\forall q \in \mathcal{Q} \;\; \exists p \in P \;\; \exists c > 0 \;\; \forall x \in E : \;\; q(\varphi(x)) \le c\, p(x)$.
Then, $\forall f \in E^{\mathbb{N}} : \| \varphi(f) \|_{q,r} \le \| f \|_{p,r}$, *whence* (F_1) *and* (F_2), *using linearity.*

We consider again a sequence of scales (r^m) such that $r^{m+1} \le r^m$. Let us denote $\mathcal{F}^+_{r^m} = \mathcal{F}_{r^m}(\mathbb{R}_+, |\cdot|)$ and $\mathcal{K}^+_{r^m} = \mathcal{K}_{r^m}(\mathbb{R}_+, |\cdot|)$. $(\ge)$

Definition 5.3 *In case $r^{m+1} \le r^m$, an increasing map $g : \mathbb{R}_+ \to \mathbb{R}_+$ is called r–**moderate** iff $\forall m \in \mathbb{N}\ \exists M \in \mathbb{N} : g(\mathcal{F}^+_{r^m}) \subset \mathcal{F}^+_{r^M}$, and r–**compatible** iff it is continuous at 0 and $\forall M \in \mathbb{N}\ \exists m \in \mathbb{N} : h(\mathcal{K}^+_{r^m}) \subset \mathcal{K}^+_{r^M}$.*
In case $r^{m+1} \ge r^m$, the definitions hold with $\forall m\ \exists M \leftrightarrow \forall M\ \exists m$ exchanged.

These notions allow to characterize maps that extend canonically to $\mathcal{G}_r$:

Definition 5.4 *A map $\varphi : (E, \mathcal{P}) \to (F, \mathcal{Q})$ is **continuously** r–**temperate** iff : (a) there is an r–moderate function g such that*

$$\forall q \in \mathcal{Q}\quad \exists p \in \mathcal{P}\quad \forall f \in E : q(\varphi(f)) \le g(p(f)),$$

and (b) there is an r–moderate function g and an r–compatible function h such that $\forall q \in \mathcal{Q}\quad \exists p \in \mathcal{P}\quad \forall f, k \in E : q(\varphi(f + k) - \varphi(f)) \le g(p(f))\, h(p(k)).$

Theorem 5.5 *Any continuously r–temperate map φ extends canonically to $\mathcal{G}_r(\varphi) : \mathcal{G}_r(E, \mathcal{P}) \to \mathcal{G}_r(F, \mathcal{Q})$, and this canonical extension is continuous for the topologies induced by $(\|\cdot\|_{p,r^m})_{p\in\mathcal{P},m\in\mathbb{N}}$ resp. $(\|\cdot\|_{q,r^m})_{q\in\mathcal{Q},m\in\mathbb{N}}$.*

Proof. Condition (a) of Definition 5.4 implies (F_1), and (b) gives (F_2). We omit the straightforward calculations, a bit lengthy in view of the four cases to be treated [4]. Continuity of $\mathcal{G}_r(\varphi)$ is obtained in the same way as (F_2), replacing $p(f) \in \mathcal{F}^+_{r^m}$ by $\| f \|_{p,m} \le K$, and $p(k) \in \mathcal{K}^+_{r^m}$ by $\| k \|_{p,m} \le \varepsilon$. □

6 Association in r–generalized algebras

In several situations, e.g. when solving PDE, strong equality is impossible to obtain or not needed, and approximation expressed by *association* is sufficient.

Definition 6.1 *Generalized numbers $[x], [y] \in \mathbb{C}_r$ are **associated** iff $x - y$ is a null sequence, $[x] \approx [y] \iff x - y \in N = \{ x \in \mathbb{C}^{\mathbb{N}} \mid \lim x = 0 \}$. For $s \in \mathbb{R}$, they are s–associated, $[x] \overset{s}{\approx} [y]$, iff $x - y \in N^{(s)} = \{ x \in \mathbb{C}^{\mathbb{N}} \mid x_n = o(e^{-s/r_n}) \}$.*

Remark 6.2 *(i) The definition is well-posed since $\mathcal{K}_r(\mathbb{C}, |\cdot|) \subset N$.*
(ii) We have $N^{(s)} = e_r^{-s}\, N$, where $e_r = (e^{\frac{1}{r_n}})_n$ represents a positive unit of $\mathbb{C}_r$.
(iii) All elements of the open unit ball are associated to zero, $\| x \|_{|\cdot|,r} < 1 \implies x \in N$, and $x \in N \implies \| x \|_{|\cdot|,r} \le 1$, but $\frac{1}{r_n} \underset{n\to\infty}{\to} \infty$ also verifies $\| \frac{1}{r} \|_{|\cdot|,r} = 1$.

Definition 6.3 *If $\mathcal{J}$ is an additive subset of $\mathcal{F}_r(\overleftrightarrow{E})$ containing $\mathcal{K}_r(\overleftrightarrow{E})$, two elements $F, G \in \mathcal{G}_r(\overleftrightarrow{E})$ are $\mathcal{J}$–**associated**, $F \underset{\mathcal{J}}{\approx} G$, iff $F - G \in \mathcal{J}/\mathcal{K}_r(\overleftrightarrow{E})$.*

Proposition 6.4 *If $\mathcal{J}$ is absolutely convex, the relation $\underset{\mathcal{J}}{\approx}$ is stable under multiplication with elements of the closed unit ball.*

Clearly, $\underset{\mathcal{J}}{\approx}$ is compatible with derivation iff $\mathcal{J}$ is stable under differentiation.

Definition 6.5 *We call $F, G \in \mathcal{G}_r(E, \mathcal{P})$ **strongly associated**, $F \simeq G$, iff $\forall p \in \mathcal{P} : d_{p,r}(F, G) < 1$. For any $s \in \mathbb{R}$, **strong** s–**association** is defined as*

$$F \overset{s}{\simeq} G \iff \forall p \in \mathcal{P} : d_{p,r}(F, G) < e^{-s} \iff F \underset{\mathcal{J}^{(s)}}{\approx} G$$

where $\mathcal{J}^{(s)} = \{ f \in E^{\mathbb{N}} \mid \forall p \in \mathcal{P} : \| f \|_{p,r} < e^{-s} \} =: B^{(\mathcal{P})}_{e^{-s}}$.

Remark 6.6 *If $F \overset{s}{\simeq} G$ for all $s > 0$, then $F = G$, because $\bigcap_{s>0} \mathcal{J}^{(s)} = o_{\mathcal{G}_r(E,\mathcal{P})}$.*

In order to define **weak association**, notice that a continuous bilinear form $\langle\cdot,\cdot\rangle : \overleftrightarrow{E} \times \boldsymbol{D} \to \mathbb{C}$ canonically extends to $\mathcal{G}_r(\overleftrightarrow{E}) \times \boldsymbol{D} \to \mathbb{C}_r$ (cf. Example 5.2). This allows to define, for any convex subset M of $\mathcal{F}_r(\mathbb{C}, |\cdot|)$ containing $0_{\mathbb{C}_r}$, subspaces $\mathcal{J}$ of the form $J_M = \{ f \in \overleftrightarrow{E}^{\mathbb{N}} \mid \forall \psi \in \boldsymbol{D} : \langle f, \psi\rangle \in M \}$.

Definition 6.7 *For $M = N^{(s)}$ resp. $M = B^{(|\cdot|)}_{e^{-s}}$, association with respect to J_M is called* ***weak*** $(s, \boldsymbol{D}')$*– resp.* ***strong–weak*** $(s, \boldsymbol{D})$***–association*** *and is written $F \overset{s}{\underset{\boldsymbol{D}}{\approx}} G$ ($\iff \forall\psi \in \boldsymbol{D} : \langle F - G, \psi\rangle \overset{s}{\approx} 0$), resp. $F \overset{s}{\underset{\boldsymbol{D}}{\sim}} G$ ($\iff \forall\psi \in \boldsymbol{D} : |\langle F - G, \psi\rangle|_r < e^{-s}$). If $s = 0$, it is omitted from notation.*

Example 6.8 *In Colombeau's case, $[f]$, $[g]$ are weakly $(s, \mathcal{D}')$–associated iff $n^s(f_n - g_n) \to 0$ in $\mathcal{D}'(\Omega)$. For ultradistributions and for periodic hyperfunctions, with $\boldsymbol{D} = \mathcal{D}^{(m)}$, $\boldsymbol{D} = \mathcal{D}^{\{m\}}$ resp. $\boldsymbol{D} = \mathcal{A}(\mathbb{T})$, this is a new construction.*

Proposition 6.9 *Strong–weak $(s, \boldsymbol{D})$–association implies $(s, \boldsymbol{D}')$–association, which conversely implies strong–weak $(s', \boldsymbol{D})$–association only for all $s' < s$.*

Proof. This follows from $\| x \|_r < e^{-s} \implies x \in N^{(s)} \implies \| x \|_r < e^{-s'}$ for $s' < s$, while a counter-example for $s' = s$ can be built as in Remark 6.2. □

In a forthcoming paper, we explain in detail how these concepts of association are useful in the context of regularity theory and microlocal analysis.

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