Algebras of generalized functions through sequence spaces algebras. Functoriality and associations

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Abstract

Starting from a description of various generalized function algebras based on sequence spaces, we develop the general framework for considering linear problems with singular coefficients or non linear problems. Therefore, we prove functorial properties of those algebras and show how weak equalities, in the sense of various associations, can be described in this setting.

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Introduction

Schwartz [20] proved that differential algebras of generalized functions with the ordinary product of continuous functions and containing the delta distribution, do not exist. But with the ordinary multiplication of smooth functions, such algebras exist as it was proved by Colombeau. Nowadays the theory of such algebras is well-established and it is affirmed through many applications especially in nonlinear problems with strong singularities. We refer to the books [2, 15, 10, 16] and to the numerous papers given in the references.

We have shown in [6] that Colombeau type algebras can be reconsidered as a class of sequence space algebras and have given a purely topological description of Colombeau type algebras. In fact, all these classes of algebras are simply determined by a locally convex algebra E and a sequence $r : \mathbb{N} \to \mathbb{R}_+$ (or sequence of sequences) which serves to construct an ultrametric on subspaces of $E^{\mathbb{N}}$. Such sequences are called weight sequences.

Distribution, ultradistribution and hyperfunction type spaces can be embedded into corresponding algebras of sequences of this class. This is done in [7]. Note, the embeddings of Schwartz' spaces into the Colombeau algebra $\mathcal{G}$ are very well known, but for ultradistribution and hyperfunction type spaces this is quite different problem, especially because of multiplication of regular enough functions (smooth, ultradifferentiable or quasianalytic), embedded into corresponding algebras.

In this paper we continue to develop the foundations of our approach as well as the general framework for considering various linear problems with singular coefficients and nonlinear ones.

We recall in Section 1 our construction of algebras of sequence spaces, defined by a decreasing null sequence, used as exponential weight. Taking $r_n = 1/\log n$, $n \in \mathbb{N}$, we get the simplified and the full Colombeau algebras.

In Section 2 we introduce sequences of scales, construct new algebras and relate them to known algebras as Egorov algebras [9] and asymptotic algebras [5].

This justifies to turn in Section 3 to nowadays classical questions like functorial aspects of Colombeau type algebras [5, 19] in order to apply the following scheme in standard applications: if a classical differential problem for regular data has a unique solution such that the map associating the solution to the initial data verifies convenient growth conditions (with respect to the chosen scale of weights), then this problem can be transferred to corresponding sequence spaces, where it also allows for a unique solution. That way, differential problems with singular data can be solved in such spaces *ad hoc*.

Finally, exact solutions may not exist at all or, even more frequently, may not be needed. For this reason, in spaces of generalized functions the notion of weak solutions has often be used, in the sense of different types of associations. These concepts can nicely be described in our sequential approach, which is done in Section 4. Indeed, we give a generalized and unified scheme of a large number of tools of this kind, including those which can already be found in various places in existing literature [3, 13, 14, 15, and others].

1 The basic construction [6]

1.1 Locally convex vector spaces and algebras

Consider an algebra E which is a locally convex vector space over $\mathbb{C}$, equipped with an arbitrary set of seminorms $p \in \mathcal{P}$ determining its locally convex structure. Assume that

$$\forall p \in \mathcal{P}\ \exists \bar{p} \in \mathcal{P}\ \exists C \in \mathbb{R}_+ : \forall x, y \in E : p(x\,y) \le C\,\bar{p}(x)\,\bar{p}(y)\ .$$

Let $r \in \mathbb{R}_+^{\mathbb{N}}$ be a sequence decreasing to zero. Put, for $f \in E^{\mathbb{N}}$,

$$\|\, f \,\|_{p,r} := \limsup_{n\to\infty} p(f_n)^{r_n}\ .$$

This is well defined for any $f \in E^{\mathbb{N}}$, with values in $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$. With this definition, let

$$\mathcal{F}_{\mathcal{P},r} = \left\{ f \in E^{\mathbb{N}} \mid \forall p \in \mathcal{P} : \|\, f \,\|_{p,r} < \infty \right\}\ ,$$

$$\mathcal{K}_{\mathcal{P},r} = \left\{ f \in E^{\mathbb{N}} \mid \forall p \in \mathcal{P} : \|\, f \,\|_{p,r} = 0 \right\}\ .$$

Remark 1 *The space E is given as domain of the elements $p \in \mathcal{P}$. When we write $\mathcal{F}_{|\cdot|,r}$ in the sequence, it means $E = \mathbb{C}$ and $\mathcal{P} = \{\,|\cdot|\,\}$.*

Proposition 2 *1. $\mathcal{F}_{\mathcal{P},r}$ is a (sub-)algebra of $E^{\mathbb{N}}$, and $\mathcal{K}_{\mathcal{P},r}$ is an ideal of $\mathcal{F}_{\mathcal{P},r}$, thus $\mathcal{G}_{\mathcal{P},r} = \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$ is an algebra.*

2. For every $p \in \mathcal{P}$, $$d_{p,r} : \begin{array}{rcl} E^{\mathbb{N}} \times E^{\mathbb{N}} & \to & \overline{\mathbb{R}}_+ \ , \\ (f,g) & \mapsto & \Vert\!| f - g \Vert\!|_{p,r} \end{array}$$
is an ultrapseudometric on $\mathcal{F}_{\mathcal{P},r}$, and the family $(d_{p,r})_{p\in\mathcal{P}}$ makes $\mathcal{F}_{\mathcal{P},r}$ a topological algebra (over $(\mathcal{F}_{|\cdot|,r}, d_{|\cdot|,r})$).

3. For every $p \in \mathcal{P}$, $$\widetilde{d}_{p,r} : \begin{array}{rcl} \mathcal{G}_{\mathcal{P},r} \times \mathcal{G}_{\mathcal{P},r} & \to & \mathbb{R}_+ \ , \\ ([f],[g]) & \mapsto & d_{p,r}(f,g) \end{array}$$
is an ultrametric on $\mathcal{G}_{\mathcal{P},r}$, where $[f],[g]$ are the classes of $f,g \in \mathcal{F}_{\mathcal{P},r}$. The family of ultrametrics $\{\,\widetilde{d}_{p,r}\,\}_{p\in\mathcal{P}}$ defines a topology, identical to the quotient topology, for which $\mathcal{G}_{\mathcal{P},r} = \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$ is a topological algebra over $\mathbb{C}_r = \mathcal{G}_{|\cdot|,r}$.

Example 3 (Colombeau generalized numbers) *The setting considered here is used to define rings of generalized numbers. For this, E is the underlying field $\mathbb{R}$ or $\mathbb{C}$, and $p = |\cdot|$ the absolute value. The resulting factor algebra $\mathcal{G}_{|\cdot|,r}$, with topology given by $\Vert\!|\cdot\Vert\!|_{|\cdot|,r}$, will be noted $\mathbb{R}_r$ or $\mathbb{C}_r$. As already explained in the introduction, for $r = 1/\log$, we get the ring of Colombeau's numbers $\overline{\mathbb{C}}$. More precisely, let*

$$\forall n \in \mathbb{N}+2 : r_n = \frac{1}{\log n} \ . \tag{1.1}$$

This gives back Colombeau's algebras of elements with polynomial growth modulo elements of more than polynomial decrease, because

$$\begin{aligned} \limsup |x_n|^{1/\log n} < \infty \quad & \iff \quad \exists C : \limsup |x_n|^{1/\log n} = C \\ & \iff \quad \exists B, \exists n_0, \forall n > n_0 : |x_n| \le B^{\log n} = n^{\log B} \\ & \iff \quad \exists \gamma : |x_n| = o(n^\gamma) \ . \end{aligned}$$

On the other hand, $\limsup = 0$ (for the ideal) corresponds to $C = 0$ and thus $\forall B > 0$ and $\forall \gamma$ in the last lines.

Example 4 *Take $E = \mathcal{C}^\infty(\Omega)$, $\mathcal{P} = \{\,p_\nu\,\}_{\nu\in\mathbb{N}}$, with*

$$p_\nu(f) := \sup_{|\alpha|\le\nu,\,|x|\le\nu} |f^{(\alpha)}(x)| \ ,$$

and $r = \frac{1}{\log}$. Then, $\mathcal{G}_{\mathcal{P},r} = \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$ with

$$\begin{aligned} \mathcal{F}_{\mathcal{P},r} &= \left\{ (f_n)_n \in \mathcal{C}^\infty(\Omega)^{\mathbb{N}} \;\middle|\; \forall \nu \in \mathbb{N} : \limsup_{n\to\infty} p_\nu(f_n)^{1/\log n} < \infty \right\} \ , \\ \mathcal{K}_{\mathcal{P},r} &= \left\{ (f_n)_n \in \mathcal{C}^\infty(\Omega)^{\mathbb{N}} \;\middle|\; \forall \nu \in \mathbb{N} : \limsup_{n\to\infty} p_\nu(f_n)^{1/\log n} = 0 \right\} \ . \end{aligned}$$

is just the simplified Colombeau algebra.

Example 5 *With a slight generalization, we can consider the "full" Colombeau algebra based on these definitions. Following Colombeau, let*

$$\forall q\in\mathbb{N}:\quad \mathcal{A}_q=\left\{\phi\in\mathcal{D}(\mathbb{R}^s)\;\middle|\;\int\phi=1\quad\text{and}\quad\forall n\in\{1,\dots,q\}:\int x^n\phi=0\right\}$$

and p_ν, $\nu\in\mathbb{N}$ *as above. Then, for fixed* $\nu,N\in\mathbb{N}$ *and* $\phi\in\mathcal{A}_N$ *let*

$$\begin{aligned}\mathcal{F}_{\nu,N,\phi} &= \left\{(f_{\phi_n})_n\in E^{\mathbb{N}}\mid \|\!|\,(f_{\phi_n})_n\,|\!\|_{p_\nu,r}<N\right\}\ ,\\ \mathcal{K}_{\nu,N,\phi} &= \left\{(f_{\phi_n})_n\in E^{\mathbb{N}}\mid \|\!|\,(f_{\phi_n})_n\,|\!\|_{p_\nu,r}=0\right\}\ ,\end{aligned}$$

where $\phi_n=n^s\,\phi(n\,\cdot)$.
(Here $(f_{\phi_n})_n$ *are the "extracted sequences" of the elements* $(f_\phi)_\phi\in E^{\mathcal{D}(\mathbb{R})}$*).*
Now define

$$\begin{aligned}\mathcal{F} &= \bigcap_{\nu\in\mathbb{N}}\mathcal{F}_\nu\ ,\quad \mathcal{F}_\nu=\bigcup_{N\in\mathbb{N}}\mathcal{F}_{\nu,N}\ ,\quad \mathcal{F}_{\nu,N}=\bigcap_{\phi\in\mathcal{A}_N}\mathcal{F}_{\nu,N,\phi}\ ,\\ \mathcal{K} &= \bigcap_{\nu\in\mathbb{N}}\mathcal{K}_\nu\ ,\quad \mathcal{K}_\nu=\bigcup_{N\in\mathbb{N}}\mathcal{K}_{\nu,N}\ ,\quad \mathcal{K}_{\nu,N}=\bigcap_{\phi\in\mathcal{A}_N}\mathcal{K}_{\nu,N,\phi}\ .\end{aligned}$$

Then again, $\mathcal{F}$ *is an algebra and* $\mathcal{K}$ *an ideal of* $\mathcal{F}$*, and* $\mathcal{G}=\mathcal{F}\,/\,\mathcal{K}$ *is the "full" Colombeau algebra.*

1.2 Projective and inductive limits

We consider again a positive sequence $r=(r_n)_n\in(\mathbb{R}_+)^{\mathbb{N}}$ decreasing to zero and we use the notations introduced above.

Let $(E^\mu_\nu,p^\mu_\nu)_{\mu,\nu\in\mathbb{N}}$ be a family of semi-normed algebras over $\mathbb{R}$ or $\mathbb{C}$ such that

$$\forall\mu,\nu\in\mathbb{N}:\ E^{\mu+1}_\nu\hookrightarrow E^\mu_\nu\quad\text{and}\quad E^\mu_{\nu+1}\hookrightarrow E^\mu_\nu\quad(\text{resp. } E^\mu_\nu\hookrightarrow E^\mu_{\nu+1}\ )\ ,$$

where $\hookrightarrow$ means continuously embedded. Then let $\overleftarrow{E}=\operatorname{proj\,lim}_{\mu\to\infty}\operatorname{proj\,lim}_{\nu\to\infty}E^\mu_\nu=\operatorname{proj\,lim}_{\nu\to\infty}E^\nu_\nu$, (resp. $\overrightarrow{E}=\operatorname{proj\,lim}_{\mu\to\infty}\operatorname{ind\,lim}_{\nu\to\infty}E^\mu_\nu$). Such projective and inductive limits are usually considered with norms instead of seminorms, and with the additional assumption that in the projective case sequences are reduced, while in the inductive case for every $\mu\in\mathbb{N}$ the inductive limit is regular, i.e. a set $A\subset\operatorname{ind\,lim}_{\nu\to\infty}E^\mu_\nu$ is bounded iff it is contained in some E^μ_ν and bounded there.

Define (with $p\equiv(p^\mu_\nu)_{\nu,\mu}$)

$$\begin{aligned}&\overleftarrow{\mathcal{F}}_{p,r}=\left\{f\in\overleftarrow{E}^{\mathbb{N}}\;\middle|\;\forall\mu,\nu\in\mathbb{N}:\|\!|\,f\,|\!\|_{p^\mu_\nu,r}<\infty\right\}\ ,\\ &\overleftarrow{\mathcal{K}}_{p,r}=\left\{f\in\overleftarrow{E}^{\mathbb{N}}\;\middle|\;\forall\mu,\nu\in\mathbb{N}:\|\!|\,f\,|\!\|_{p^\mu_\nu,r}=0\right\}\\ (\text{resp. }\ &\overrightarrow{\mathcal{F}}_{p,r}=\bigcap_{\mu\in\mathbb{N}}\overrightarrow{\mathcal{F}}^\mu_{p,r}\ ,\quad \overrightarrow{\mathcal{F}}^\mu_{p,r}=\bigcup_{\nu\in\mathbb{N}}\left\{f\in(E^\mu_\nu)^{\mathbb{N}}\;\middle|\;\|\!|\,f\,|\!\|_{p^\mu_\nu,r}<\infty\right\}\ ,\\ &\overrightarrow{\mathcal{K}}_{p,r}=\bigcap_{\mu\in\mathbb{N}}\overrightarrow{\mathcal{K}}^\mu_{p,r}\ ,\quad \overrightarrow{\mathcal{K}}^\mu_{p,r}=\bigcup_{\nu\in\mathbb{N}}\left\{f\in(E^\mu_\nu)^{\mathbb{N}}\;\middle|\;\|\!|\,f\,|\!\|_{p^\mu_\nu,r}=0\right\}\ )\ .\end{aligned}$$

Proposition–Definition 6

(i) *Writing* $\overleftrightarrow{\cdot}$ *for both,* $\overleftarrow{\cdot}$ *or* $\overrightarrow{\cdot}$*, we have that* $\overleftrightarrow{\mathcal{F}}_{p,r}$ *is an algebra and* $\overleftrightarrow{\mathcal{K}}_{p,r}$ *is an ideal of* $\overleftrightarrow{\mathcal{F}}_{p,r}$*; thus,* $\overleftrightarrow{\mathcal{G}}_{p,r} = \overleftrightarrow{\mathcal{F}}_{p,r}/\overleftrightarrow{\mathcal{K}}_{p,r}$ *is an algebra.*

(ii) *For every* $\mu, \nu \in \mathbb{N}$, $d_{p^\mu_\nu} : (E^\mu_\nu)^{\mathbb{N}} \times (E^\mu_\nu)^{\mathbb{N}} \to \overline{\mathbb{R}}_+$ *defined by* $d_{p^\mu_\nu}(f,g) = \| f - g \|_{p^\mu_\nu,r}$ *is an ultrapseudometric on* $(E^\mu_\nu)^{\mathbb{N}}$*. Moreover,* $(d_{p^\mu_\nu})_{\mu,\nu}$ *induces a topological algebra*[1] *structure on* $\overleftarrow{\mathcal{F}}_{p,r}$ *such that the intersection of the neighborhoods of zero equals* $\overleftarrow{\mathcal{K}}_{p,r}$*.*

(iii) *From (ii),* $\overleftarrow{\mathcal{G}}_{p,r} = \overleftarrow{\mathcal{F}}_{p,r}/\overleftarrow{\mathcal{K}}_{p,r}$ *becomes a topological algebra (over generalized numbers* $\mathbb{C}_r = \mathcal{G}_{|\cdot|,r}$*) whose topology can be defined by the family of ultrametrics* $(\tilde{d}_{p^\mu_\nu})_{\mu,\nu}$ *where* $\tilde{d}_{p^\mu_\nu}([f],[g]) = d_{p^\mu_\nu}(f,g)$*,* $[f]$ *standing for the class of* f*.*

(iv) *If* τ_μ *denote the inductive limit topology on* $\mathcal{F}^\mu_{p,r} = \bigcup_{\nu\in\mathbb{N}}((\tilde{E^\mu_\nu})^{\mathbb{N}}, d_{\mu,\nu})$*,* $\mu \in \mathbb{N}$*, then* $\overrightarrow{\mathcal{F}}_{p,r}$ *is a topological algebra*[1] *for the projective limit topology of the family* $(\mathcal{F}^\mu_{p,r}, \tau_\mu)_\mu$*.*

We have proved in [6]:

Proposition 7 *(i)* $\overleftarrow{\mathcal{F}}_{p,r}$ *is complete.*

(ii) *If for all* $\mu \in \mathbb{N}$*, a subset of* $\overrightarrow{\mathcal{F}}^\mu_{p,r}$ *is bounded iff it is a bounded subset of* $(E^\mu_\nu)^{\mathbb{N}}$ *for some* $\nu \in \mathbb{N}$*, then* $\overrightarrow{\mathcal{F}}_{p,r}$ *is sequentially complete.*

We showed that various definitions of Colombeau algebras $\bar{\mathbb{C}}$ and $\mathcal{G}$ correspond to the sequence $r_n = 1/\log n$, $n \in \mathbb{N}$. The embedding of Schwartz distributions and of smooth functions into $\mathcal{G}$ is well-known. Also it is well-known that the multiplication of smooth function embedded into $\mathcal{G}$ is the usual multiplication. In [7] we have constructed sequence spaces forming algebras which correspond to Colombeau type algebras of ultradistributions and periodic hyperfunctions. The main objective of [7] was to realise the embeddings of ultradistribution spaces and periodic hyperfunction spaces into such algeberas and realise the multiplication of regular elements embedded into corresponding sequence spaces.

In the definition of our sequence spaces $\overrightarrow{\mathcal{F}}_{p,r}$ (resp. $\overleftarrow{\mathcal{F}}_{p,r}$), we assumed that r_n tends to 0 as n tends to ∞. One could consider more general sequences of weights. But, for example, if r_n is contained in some compact subset of $(0, +\infty)$ then $\overleftrightarrow{E}$ can be embedded in the set-theoretical sense via the canonical map $f \mapsto (f)_n$ $(f_n = f)$.

If $r_n \to \infty$, $\overleftrightarrow{E}$ is no more included in $\overleftrightarrow{\mathcal{F}}_{p,r}$.

In order to have an appropriate topological algebra containing "δ", we need "divergent" sequences; this justifies the choice of $r_n \to 0$. Then, our generalized topological algebra induces the discrete topology on the original algebra $\overleftrightarrow{E}$. In some sense, it is an analogy to Schwartz' impossibility statement for multiplication of distributions [20].

[1] over $(\mathbb{C}^{\mathbb{N}}, \| \cdot \|_{|\cdot|})$, not over $\mathbb{C}$: scalar multiplication is not continuous.

2 Sequences of scales and asymptotic algebras

2.1 Sequences of scales

We can consider a sequence $(r^m)_m$ of positive sequences $(r^m_n)_n$ such that

$$\forall m,n\in\mathbb{N}:\ r^m_{n+1}\le r^m_n\ ;\quad \lim_{n\to\infty} r^m_n = 0\ .$$

In addition to this, we request either of the following conditions :

$$\begin{aligned} &\forall m,n\in\mathbb{N}: && r^{m+1}_n \le r^m_n && (2.1)\\ \text{or}\quad &\forall m,n\in\mathbb{N}: && r^{m+1}_n \ge r^m_n\ . && (2.2)\end{aligned}$$

Then let, in the first (resp. second) case:

$$\begin{aligned} \overleftrightarrow{\mathcal{F}}_{p,r} &= \bigcup_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{F}}_{p,r^m}\ ,\quad \overleftrightarrow{\mathcal{K}}_{p,r} = \bigcap_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{K}}_{p,r^m}\\ (\text{ resp. } \overleftrightarrow{\mathcal{F}}_{p,r} &= \bigcap_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{F}}_{p,r^m}\ ,\quad \overleftrightarrow{\mathcal{K}}_{p,r} = \bigcup_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{K}}_{p,r^m}\ )\ ,\end{aligned}$$

where $p=(p^\mu_\nu)_{\nu,\mu}$.

Proposition 8 *With the previous notations, $\overleftrightarrow{\mathcal{G}}_{p,r} = \overleftrightarrow{\mathcal{F}}_{p,r}/\overleftrightarrow{\mathcal{K}}_{p,r}$ is an algebra.*

Proof. Let us start with the first case (2.1). $r^{m+1}\le r^m \implies \| f \|_{r^{m+1}} \ge \| f \|_{r^m}$ if $p(f_n) < 1$, hence $\mathcal{K}_{m+1}\subset\mathcal{K}_m$. Conversely, $\mathcal{F}_{m+1}\supset\mathcal{F}_m$. Thus, intersection for $\mathcal{K}$ and union for $\mathcal{F}$ makes sense. Moreover, because of this inclusion property, $\mathcal{F}$ is indeed a subalgebra. To prove that $\mathcal{K}$ is an ideal, take $(k,f)\in\mathcal{K}\times\mathcal{F}$, i.e. $\forall m'' : k\in\mathcal{K}_{m''}$, and $\exists m' : f\in\mathcal{F}_{m'}$. We have to show that $\forall m : k\cdot f\in\mathcal{K}_m$. So let m be given.
If $m<m'$, then $\mathcal{K}_{m'}\subset\mathcal{K}_m$, thus $k\cdot f\in\mathcal{K}_{m'}\cdot\mathcal{F}_{m'}\subset\mathcal{K}_{m'}\subset\mathcal{K}_m$.
If $m'<m$, then $\mathcal{F}_{m'}\subset\mathcal{F}_m$, thus $k\cdot f\in\mathcal{K}_m\cdot\mathcal{F}_{m'}\subset\mathcal{K}_m\cdot\mathcal{F}_m\subset\mathcal{K}_m$.

Now turn to the the second case (2.2). The same reasoning gives now $\mathcal{K}_{m+1}\supset\mathcal{K}_m$ and $\mathcal{F}_{m+1}\subset\mathcal{F}_m$, justifying definitions of $\mathcal{F}$ and $\mathcal{K}$. $\mathcal{F}$ is obviously a subalgebra. To see that $\mathcal{K}$ is an ideal, take $(k,f)\in\mathcal{K}\times\mathcal{F}$. Then $\exists m : k\in\mathcal{K}_m$, but also $f\in\mathcal{F}_m$, in which $\mathcal{K}_m$ is an ideal. Thus, $k\cdot f\in\mathcal{K}_m\subset\mathcal{K}$. □

Example 9 $r^m_n = \begin{cases} 1 & \textit{if}\ \ n\le m\\ 0 & \textit{if}\ \ n>m\end{cases}$ *(with the convention that $0^0=0$) gives Egorov–type algebras, where the "subalgebra" contains everything and the ideal contains only stationary null sequences.*

Example 10 $r^m_n = 1/|\log a_m(n)|$, *where $(a_m:\mathbb{N}\to\mathbb{R}_+)_{m\in\mathbb{Z}}$ is an asymptotic scale, i.e. $\forall m\in\mathbb{Z} : a_{m+1}=o(a_m)$, $a_{-m}=1/a_m$, $\exists M : a_M = o(a_m^2)$. This gives back the asymptotic algebras of [5], cf. Section 2.2.*

2.2 Asymptotic algebras

Let us recall that $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebras [14] are based on a vector space $\mathcal{E}$ with a filtering family $\mathcal{P}$ of seminorms, and a ring of generalized numbers $\mathcal{C} = A/I$. Here, I is an ideal of A, which is a subring of $\mathbb{K}^\Lambda$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and Λ is some indexing set. Both A and I must be solid as a ring, i.e. $\forall s \in \mathbb{K}^\Lambda : (\exists r \in A : \forall\lambda \in \Lambda : |s_\lambda| \le |r_\lambda|) \Longrightarrow s \in A$, and idem for I. Then, the $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebra is defined as $\mathcal{G}_{\mathcal{C},\mathcal{E},\mathcal{P}} = \mathcal{E}_A/\mathcal{E}_I$, with

$$\mathcal{E}_X = \left\{ f \in \mathcal{E}^\Lambda \mid \forall p \in \mathcal{P} : p \circ f \in X \right\}$$

(where $p \circ f \equiv (\lambda \mapsto p(f_\lambda)) = (p(f_\lambda))_\lambda \in (\mathbb{R}_+)^\Lambda \subset \mathbb{K}^\Lambda$): In other words, the function spaces $\mathcal{E}_A$ and $\mathcal{E}_I$ are determined by $\mathcal{C} = A/I$, by selecting the *functions* with the same respective growth properties than the "*constants*".

It is clear that this is too general to be written in the previously presented setting of sequence spaces, mainly because there are almost no restrictions on I.

So let us consider the interesting subclass of asymptotic algebras [5]. Here, A and I are defined by an asymptotic scale[2] $\mathbf{a} = (a_m : \Lambda \to \mathbb{R}_+)_{m\in\mathbb{Z}}$:

$$\begin{aligned} A_\mathbf{a} &= \left\{ s \in \mathbb{K}^\Lambda \mid \exists m \in \mathbb{Z} : s = o(a_m) \right\} \\ I_\mathbf{a} &= \left\{ s \in \mathbb{K}^\Lambda \mid \forall m \in \mathbb{Z} : s = o(a_m) \right\} \end{aligned}$$

Recall that $\mathbf{a}$ must verify: $\forall m \in \mathbb{N} : a_{m+1} = o(a_m)$, $a_{-m} = 1/a_m$, $\exists M : a_M = o(a_m^2)$. Some examples that have proved to be useful are:

1. $\Lambda = \mathbb{N}$ and $a_m(\lambda) = 1/\lambda^m$: This leads to Colombeau's generalized numbers and algebras.
2. $\Lambda = \mathbb{N}$ and $a_m(\lambda) = 1/\exp^m(\lambda)$ for $m \in \mathbb{N}^*$, where $\exp^m$ is the m-fold iterated exp function: This gives the so-called exponential algebras [5].
3. $r_n^m = 1/n^{\frac{m}{m-1}}$: This is related to ultradistribution spaces, and is discussed in [7].

Proposition 11 *Asymptotic algebras can be recovered in our formulation by choosing the sequence of weights $r^m = 1/|\log a_m|$ (i.e. $r_\lambda^m = 1/|\log a_m(\lambda)|$).*

Proof. We will show that $\mathcal{E}_I = \mathcal{K}_{\mathcal{P},r}$ and $\mathcal{E}_A = \mathcal{F}_{\mathcal{P},r}$, for $r^m = 1/|\log a_m|$. In view of the definitions, this amounts to show the equivalences

$$\forall p, \underset{(\exists)}{\forall} a_m : p \circ f = o(a_m) \iff \forall p, \underset{(\exists)}{\forall} r^m : \| f \|_{p,r^m} \underset{(<\infty)}{=0} .$$

Let us start with $\mathcal{E}_A \subset \mathcal{F}_{\mathcal{P},r}$: Let $f \in \mathcal{E}_A$. Thus, $\forall p \in \mathcal{P}$, $\exists m : p \circ f = o(a_m)$. We can assume $a_m > 1$, such that $r^m = 1/\log a_m \iff a_m = e^{1/r^m}$. Thus $p\circ f = o(e^{1/r^m})$. But $p\circ f < e^{1/r^m} \implies (p\circ f)^{r^m} < e$, thus $\limsup (p\circ f)^{r^m} < \infty$ and $f \in \mathcal{F}_{\mathcal{P},r}$.

[2]The set Λ is supposed to have a base of filters $\mathcal{B}$, to which the $o(\cdot)$ notation refers to. In the preceding paragraph, $\forall\lambda \in \Lambda$ could also be replaced by $\exists\Lambda_0 \in \mathcal{B}$, $\forall\lambda \in \Lambda_0$.

Conversely, $\mathcal{F}_{\mathcal{P},r} \subset \mathcal{E}_A$: We have $\forall p \in \mathcal{P}$, $\exists \bar{m} : \limsup (p \circ f)^{1/|\log a_{\bar{m}}|} < \infty$. With

$$(p \circ f)^{1/|\log a_m|} \leq C \iff p \circ f \leq (a_m)^{\log C} \quad , \quad (a_m, C > 1)$$

we have: $\exists C > 0$, $\exists \Lambda_0 : \forall \lambda \in \Lambda_0 : p(f_\lambda) \leq (a_{\bar{m}}(\lambda))^{|\log C|}$. Thus, using the 3rd property of scales, $\exists m : p \circ f = o(a_m)$.

Now turn to $\mathcal{E}_I \subset \mathcal{K}_{\mathcal{P},r}$: We have $\forall \bar{m} : p \circ f = o(a_{\bar{m}})$.
Take $m \in \mathbb{N}$. Now, for any $q \in \mathbb{N}$, $\exists \hat{m} : a_{\hat{m}} = o({a_m}^q)$. and $p \circ f = o(a_{\hat{m}})$.

Using $a_m = e^{-1/r^m}$, $a_{\hat{m}} = o({a_m}^q) = o((e^{-1/r_m})^q) = o((e^{-q})^{1/r^m})$, i.e., $(p \circ f)^{r^m} \leq e^{-q}$ for λ "large enough". As q was arbitrary, we have $(p \circ f)^{r^m} \to 0$ and thus $f \in \mathcal{K}$.

Finally, $\mathcal{K}_{\mathcal{P},r} \subset \mathcal{E}_I$: We have $\forall \bar{m} : \limsup p(f_\lambda)^{1/|\log a_{\bar{m}}|} = 0$, i.e.,

$$\forall C > 0, \exists \Lambda_0, \forall \lambda \in \Lambda_0 : p(f_\lambda)^{1/|\log a_{\bar{m}}|} < C \ .$$

With $a_m, C < 1$, this gives $p(f_\lambda) \leq C^{|\log a_{\bar{m}}|} = {a_{\bar{m}}}^{|\log C|}$. Now, to show that $f \in \mathcal{E}_I$, take any m. Let $\bar{m} = m+1$ and $C = 1/e$: $\exists \Lambda_0, \forall \lambda \in \Lambda_0 : p(f_\lambda) < a_{\bar{m}}(\lambda)$. But $a_{\bar{m}} = a_{m+1} = o(a_m)$, thus $p \circ f = o(a_m)$. $\square$

2.3 Algebras with infra-exponential growth

A second interesting subclass of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$–algebras are of the form

$$\begin{aligned} A &= \left\{ s \in \mathbb{K}^\Lambda \mid \forall \sigma < 0 : s = o(a_\sigma) \right\} \\ I &= \left\{ s \in \mathbb{K}^\Lambda \mid \exists \sigma > 0 : s = o(a_\sigma) \right\} \end{aligned}$$

where $\mathbf{a} = (a_\sigma)_{\sigma \in \mathbb{R}}$ is again a scale (i.e. $\forall \sigma > \rho$, $a_\sigma = o(a_\rho)$, etc.), but indexed by a real number. (Note that here A is given as intersection and I as union of sets, that's why this case is not covered by the previous one.)

For example (again with $\Lambda = \mathbb{N}$),

$$a_\sigma := \lambda \mapsto 1/\exp(\sigma\,\lambda)$$

gives the so-called algebras with infra–exponential growth [4], pertaining to the embedding of periodic hyperfunctions in $(\mathcal{C}, \mathcal{E}, \mathcal{P})$–algebras.

These algebras can be obtained by taking $\mathcal{F} = \{ f \mid \| f \|_r \leq 1 \}$ and $\mathcal{K} = \{ f \mid \| f \|_r < 1 \}$, with $r_n = \frac{1}{n}$. (As the norm is compared to 1, all scales $r_\sigma = 1/|\log a_\sigma|$ (i.e. $r_\sigma(\lambda) = 1/|\sigma\lambda|$ are equivalent. More details on this "dual" construction, where the conditions $< \infty$ and $= 0$ are replaced by ≤ 1 and < 1, will be given in a forthcoming publication.)

3 Functorial properties

In this section, we want to investigate on conditions sufficient to extend mappings on the topological factor algebras constructed as before. Consider for example

$$\varphi : E \to F$$

where (E, P) and (F, Q) are spaces equipped with families of seminorms P and Q. We shall note in this section $\mathcal{F}_{\Pi,r}(\cdot)$, $\mathcal{K}_{\Pi,r}(\cdot)$ and $\mathcal{G}_{\Pi,r}(\cdot)$ the spaces defined as above, where $\cdot$ stands for E or F and Π stands for P or Q.

Suppose that φ satisfies the following hypotheses:

$$\begin{array}{lrcl} (F_1): & f \in \mathcal{F}_{P,r}(E) & \Longrightarrow & \varphi(f) \in \mathcal{F}_{Q,r}(F) \\ (F_2): & f \in \mathcal{F}_{P,r}(E),\ h \in \mathcal{K}_{P,r}(E) & \Longrightarrow & \varphi(f+h) - \varphi(f) \in \mathcal{K}_{Q,r}(F) \end{array}$$

where we write $\varphi(f) := \left(\varphi\left(f_n\right)\right)_n$. Then we can consider the following

Definition 12 *Under the above hypothesis, we define the r–extension of φ by*

$$\Phi := \mathcal{G}_r(\varphi) := \left(\begin{array}{rcl} \mathcal{G}_{P,r}(E) & \to & \mathcal{G}_{Q,r}(F) \\ {[f]} & \mapsto & \varphi(f) + \mathcal{K}_{Q,r} \end{array} \right)$$

where f is any representative of $[f] = f + \mathcal{K}_{P,r}(E)$.

The above consideration is of course a very general condition for a map to be well defined on a factor space. In fact, it does not depend on details of how the spaces $\mathcal{F}_{P,r}(E)$ and $\mathcal{K}_{P,r}(E)$ are defined. In particular, here r can also be a family of sequences $\left(r^m\right)_m$, and E can be of proj-proj or ind-proj type.

Example 13 *Consider a linear mapping $u \in \mathcal{L}(E, F)$, continuous for (P, Q). Fix $q \in Q$. As u is continuous, there exists $p = p_{(q)}$ such that*

$$\exists c : \forall x \in E : \ q\left(u(x)\right) \le c\, p_{(q)}(x) \ .$$

Thus, $\forall f, h \in E^{\mathbb{N}}$:

$$\begin{array}{rcl} \limsup \left(p_{(q)}\left(f_n\right)\right)^{r_n} < \infty & \Longrightarrow & \limsup \left(q\left(u\left(f_n\right)\right)\right)^{r_n} < \infty \\ \limsup \left(p_{(q)}\left(h_n\right)\right)^{r_n} = 0 & \Longrightarrow & \limsup \left(q\left(u\left(h_n\right)\right)\right)^{r_n} = 0 \end{array}$$

This example shows how we can define moderate or compatible maps with respect to the "scale" r. In fact, the concrete definitions will depend on the monotony properties of the family (r^m) of sequences of weights, according to which $\mathcal{F} = \bigcup \mathcal{F}_m$ and $\mathcal{K} = \bigcap \mathcal{K}_m$ (for $r^{m+1} \le r^m$), or $\mathcal{F} = \bigcap \mathcal{F}_m$ and $\mathcal{K} = \bigcup \mathcal{K}_m$ (for $r^{m+1} \ge r^m$).

The analysis of continuity (in the sense of $\| \cdot \|_{p,r}$) shows that the following definitions are convenient:

Definition 14 (for $r^{m+1} \le r^m$) *The map $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be r–**moderate** iff it is increasing and*

$$\forall m \in \mathbb{N}\ \exists M \in \mathbb{N}\ \forall x \in \mathbb{R}_+ : \sup_{n \in \mathbb{N}} \left(g\left(x^{1/r_n^m}\right)\right)^{r_n^M} < \infty$$

*The map $h : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be r–**compatible** iff it is increasing and*

$$\forall M \in \mathbb{N}\ \exists m \in \mathbb{N} : \left(h\left(x^{1/r_n^m}\right)\right)^{r_n^M} \underset{x \to 0}{\longrightarrow} 0 \ \ \textit{uniformly in } n \ .$$

Proposition 15 *The above definition of an* r*–moderate map* g *is equivalent to*

$$g \text{ increasing, and } \forall m\ \exists M : g(\mathcal{F}^+_{r^m}) \subset \mathcal{F}^+_{r^M}\ .$$

where $\mathcal{F}^+_{r^m} = \mathbb{R}^{\mathbb{N}}_+ \cap \mathcal{F}_{|\cdot|,r^m}$ *are "moderate" sequences of nonnegative numbers. The definition of an* r*–compatible map* h *can be written as*

$$h \text{ increasing, and } \forall M\ \exists m : \|\, h(C)\,\|_M \to 0 \text{ when } \|\, C\,\|_m \to 0\ .$$

or, equivalently

$$h \text{ continuous at 0, increasing, and } \forall M\ \exists m : h(\mathcal{K}^+_{r^m}) \subset \mathcal{K}^+_{r^M}\ .$$

Proof. (writing $\mathcal{F}_m$ for $\mathcal{F}^+_{r^m}$): We have $g(\mathcal{F}_m) \subset \mathcal{F}_M$

$$\begin{aligned}
\iff\quad & \forall C \in \mathbb{R}^{\mathbb{N}}_+ : C \in \mathcal{F}_m \implies g(C) \in \mathcal{F}_M \\
\iff\quad & \forall C \in \mathbb{R}^{\mathbb{N}}_+ : \|\, C\,\|_m < \infty \implies \|\, g(C)\,\|_M < \infty \\
\iff\quad & (\exists x > 0, \forall n : C_n < x^{1/r^m_n}) \implies \sup_n g(C_n)^{r^M_n} < \infty \\
\iff\quad & \forall x \in \mathbb{R}_+ : \sup_n g(x^{1/r^m_n})^{r^M_n} < \infty\ .
\end{aligned}$$

For h, again take $C_n = x^{1/r^m_n}$, such that $x \to 0 \iff \|\, C\,\|_m \to 0$.
Clearly, the first form implies that h is continuous at 0, thus "$\to 0$" can be replaced by "$= 0$". Thus we have $\forall M\ \exists m : C \in \mathcal{K}_m \implies h(C) \in \mathcal{K}_M$, i.e. $h(\mathcal{K}_m) \subset \mathcal{K}_M$. □

Definition 16 (for $r^{m+1} \geq r^m$) *The map* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *is said to be* r*–****moderate*** *iff it is increasing and*

$$\forall M \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall x \in \mathbb{R}_+ : \sup_{n\in\mathbb{N}} \left(g\left(x^{1/r^m_n} \right) \right)^{r^M_n} < \infty$$

The map $h : \mathbb{R}_+ \to \mathbb{R}_+$ *is said to be* r*–****compatible*** *iff it is increasing and*

$$\forall m \in \mathbb{N}\ \exists M \in \mathbb{N} : \left(h\left(x^{1/r^m_n} \right) \right)^{r^M_n} \underset{x\to 0}{\longrightarrow} 0 \text{ uniformly in } n\ .$$

Proposition 17 *The condition of* r*–moderateness can be written*

$$g \text{ increasing and } \forall M, \exists m : g(\mathcal{F}^+_{r^m}) \subset \mathcal{F}^+_{r^M}\ .$$

The condition of r*–compatibility can be written*

$$h \text{ increasing and } \forall m, \exists M : \|\, h(C)\,\|_M \to 0 \text{ for } \|\, C\,\|_m \to 0$$

or equivalently

$$h \text{ continuous at 0, increasing, and } \forall m, \exists M : h(\mathcal{K}^+_{r^m}) \subset \mathcal{K}^+_{r^M}\ .$$

Proof. The previous proof applies, *mutatis mutandis*: supremum is to be replaced by the corresponding ultranorm, and uniform convergence by convergence of the ultranorm. The definition implies continuity at $h(0) = 0$, so "$\to 0$" can be replaced by "$= 0$", and thus $h \in \mathcal{K}_M$ (resp. $\mathcal{K}_m$). □

Lemma 18 *If g is r–moderate, then $g(\mathcal{F}_r^+) \subset \mathcal{F}_r^+$; if h is r–compatible, then $h(\mathcal{K}_r^+) \subset \mathcal{K}_r^+$.*

Proof. Start with the first case, $\mathcal{F} = \cup\mathcal{F}_m$ and $\mathcal{K} = \cap\mathcal{K}_m$: We have $\forall m\ \exists M : g(\mathcal{F}_m) \subset \mathcal{F}_M \implies \forall m : g(\mathcal{F}_m) \subset \bigcup_M \mathcal{F}_M = \mathcal{F} \iff \bigcup_m g(\mathcal{F}_m) = g(\mathcal{F}) \subset \mathcal{F}$. Similarly, $\forall M\ \exists m : h(\mathcal{K}_m) \subset \mathcal{K}_M \implies \forall M : \bigcap_m h(\mathcal{K}_m) = h(\mathcal{K}) \subset \mathcal{K}_M \iff h(\mathcal{K}) \subset \bigcap_M \mathcal{K}_M = \mathcal{K}$.
Turn to the second case, $\mathcal{F} = \cap\mathcal{F}_m$ and $\mathcal{K} = \cup\mathcal{K}_m$: The proofs for $g(\mathcal{F})$, $h(\mathcal{K})$ are identical to the proofs for $h(\mathcal{K})$, $g(\mathcal{F})$ in the first case. $\square$

Now we give the definition (valid for both of the above cases) characterizing maps that extend canonically to $\mathcal{G}_r$:

Definition 19 *The map $\varphi : (E, P) \to (F, Q)$ is said to be **continuously** r–**temperate** iff*

$$
\begin{aligned}
(\alpha) \quad & \exists\, r\text{–moderate } g, \forall q \in Q, \exists p \in P, \forall f \in E : q(\varphi(f)) \le g(p(f)) \\
(\beta) \quad & \exists\, r\text{–moderate } g, \exists\, r\text{–compatible } h : \forall q \in Q, \exists p \in P, \\
& \qquad \forall f \in E, \forall k \in E : q(\varphi(f + k) - \varphi(f)) \le g(p(f))\, h(p(k))
\end{aligned}
$$

Proposition 20 *Any continuously r–temperate map φ extends canonically to*

$$\Phi = \mathcal{G}_r(\varphi) : \mathcal{G}_{P,r}(E) \to \mathcal{G}_{Q,r}(F)\ .$$

Furthermore, this canonical extension is continuous for the topologies $\Big(\mathcal{G}_{P,r}(E), (\|\!| \cdot |\!\|_{p,r})_{p\in P}\Big)$ and $\Big(\mathcal{G}_{Q,r}(F), (\|\!| \cdot |\!\|_{q,r})_{q\in Q}\Big)$.

Proof. The proof has two parts: first, the well-definedness of the extension; secondly, the continuity of Φ. As a preliminary remark, observe that $\mathcal{F}_{P,r^m} = \big\{ f \mid \forall p \in P : p(f) \in \mathcal{F}_{r^m}^+ \big\}$, and idem for $\mathcal{K}$. This, and the fact that $\mathcal{K}_{r^m}$ is an ideal in $\mathcal{F}_{r^m}$ (and $\mathcal{F}_{r^m}^+ \cdot \mathcal{K}_{r^m}^+ \subset \mathcal{K}_{r^m}^+$) helps us to write the proof using the preceding two characterizations of moderate and compatible maps.
First part of the proof: We will show that (α) implies (F_1) and (β) gives (F_2). Using respective definitions of moderateness and compatibility, the proof will be different for the two cases $r^{m+1} \le r^m$ and $r^{m+1} \ge r^m$.
First case, $r^{m+1} \le r^m$, where $\mathcal{F} = \cup\mathcal{F}_m$ et $\mathcal{K} = \cap\mathcal{K}_m$:
ad (F_1) : Take $f \in \mathcal{F}_{P,r}(E)$, i.e. $\exists m\ \forall p : p(f) \in \mathcal{F}_m^+$. By (α), there is g such that $\exists M : g(\mathcal{F}_m^+) \subset \mathcal{F}_M^+$, and $\forall q : q(\varphi(f)) \le g(p(f)) \in g(\mathcal{F}_m^+)$, thus $\exists M\ \forall q : q(\varphi(f)) \in \mathcal{F}_M^+$, i.e. $\varphi(f) \in \mathcal{F}_{Q,r}(F)$.

ad (F_2) : Take $f \in \mathcal{F}$ and $k \in \mathcal{K}$, i.e. $\exists m, \forall p : p(f) \in \mathcal{F}_m^+$ and $\forall m', \forall p : p(k) \in \mathcal{K}_{m'}^+$. Now fix M and q. With (β), $\exists g\ \forall m\ \exists M' : g(\mathcal{F}_m^+) \subset \mathcal{F}_{M'}^+$, and $\exists h\ \forall M''\ \exists m' : h(\mathcal{K}_{m'}^+) \subset \mathcal{K}_{M''}^+$. We use this for $M'' = \max(M, M')$, such that $\mathcal{K}_{M''}^+ \subset \mathcal{K}_{M'}^+$ and $\mathcal{K}_{M''}^+ \subset \mathcal{K}_M^+$. Finally, $\exists p : q\,(\varphi(f + k) - \varphi(f)) \le g(p(f))\, h(p(k)) \in g(\mathcal{F}_m^+) \cdot h(\mathcal{K}_{m'}^+) \subset \mathcal{F}_{M'}^+ \cdot \mathcal{K}_{M''}^+$. Now distingush two cases: if $M' \le M$, this is in $\mathcal{F}_{M'}^+ \cdot \mathcal{K}_M^+ \subset \mathcal{F}_M^+ \cdot \mathcal{K}_M^+ \subset \mathcal{K}_M^+$. Conversely, if $M < M'$, then this is subset of $\mathcal{F}_{M'}^+ \cdot \mathcal{K}_{M'}^+ \subset \mathcal{K}_{M'}^+ \subset \mathcal{K}_M^+$, because the $\mathcal{K}_m^+$ form a decreasing sequence. Thus, $\varphi(f+k) - \varphi(f) \in \mathcal{K}_{Q,r}(F)$.

Second case, $r^{m+1} \ge r^m$, where $\mathcal{F} = \cap\mathcal{F}_m$ and $\mathcal{K} = \cup\mathcal{K}_m$:
ad (F_1) : $f \in \mathcal{F}_{P,r}(E) \iff \forall m\ \forall p : p(f) \in \mathcal{F}_m^+$. By (α),

$$\exists g\ \forall M\ \exists m : g(\mathcal{F}_m^+) \subset \mathcal{F}_M^+, \text{ and } \forall q\ \exists p : q(\varphi(f)) \le g(p(f)) \in g(\mathcal{F}_m^+).$$

Thus, $\forall M\ \forall q : q(\varphi(f)) \in \mathcal{F}_M^+$, i.e. $\varphi(f) \in \mathcal{F}_{Q,r}(F)$.

ad (F_2) : Take $f \in \mathcal{F}$ and $k \in \mathcal{K}$, i.e. $\forall m, \forall p : p(f) \in \mathcal{F}_m^+$ and $\exists m', \forall p : p(k) \in \mathcal{K}_{m'}^+$. Now fix q. With (β),

$$\exists h\ \forall m'\ \exists M : h(\mathcal{K}_{m'}^+) \subset \mathcal{K}_M^+ \text{ and } \exists g\ \forall M\ \exists m : g(\mathcal{F}_m^+) \subset \mathcal{F}_M^+,$$

and there exists p such that

$$q\left(\varphi(f+k) - \varphi(f)\right) \le g(p(f))\, h(p(k)) \in g(\mathcal{F}_m^+)\, h(\mathcal{K}_{m'}^+) \subset \mathcal{F}_M^+ \cdot \mathcal{K}_M^+ \subset \mathcal{K}_M^+,$$

thus $\varphi(f+k) - \varphi(f) \in \mathcal{K}_{Q,r}(F)$.

Second part of the proof : continuity of Φ. We must show that

$$\forall q \in Q : \| \varphi(f+k) - \varphi(f) \|_{q,r^M} \to 0 \text{ when } \forall p \in P : \| k \|_{p,r^m} \to 0$$

for all M (resp. for some M), in respective cases. The proof goes analogous to the above proof of (F_2), by replacing $p(f) \in \mathcal{F}_m^+$ with $\| f \|_{p,m} \le K$, and $p(k) \in \mathcal{F}_m^+$ with $\| k \|_{p,m} \le \varepsilon$, etc. □

4 Association in $\mathcal{G}$

We will introduce different types of association, according to what has already been considered in the literature on generalized function spaces. Generally speaking, we will adopt the following terminology: *strong association* is expressed directly on the level of the factor algebra, while *weak association* will be defined in terms of a duality product, and thus with respect to a certain test function space.

Association in Colombeau type generalized numbers. To start with, recall that Colombeau generalized numbers $[x]$ and $[y]$ are said to be associated, $[x] \approx [y]$, iff

$$x_n - y_n \underset{n\to\infty}{\longrightarrow} 0 \quad (\text{ in } \mathbb{C}\ ) .$$

This can also be expressed by considering the subset of null sequences, $N = \left\{ x \in \mathbb{C}^{\mathbb{N}} \mid \lim x_n = 0 \right\}$, and by defining $[x] \approx [y] \iff x - y \in N$

As any element j of the ideal verifies $j_n \to 0$, this is clearly independant of the representative. In other words, it is well defined because $I \subset N$.

4.1 The general concept of $\mathcal{J}, X$–association

The following general concept of association allows to recover all known notions of association, as well as the the types we shall consider below.

Definition 21 ($\mathcal{J}, X$–association) *Let $\mathcal{J}$ be an additive subgroup of $\mathcal{F}$ containing the ideal $\mathcal{K}$, and X a set of generalized numbers. Then, two elements $F, G \in \mathcal{G} = \mathcal{F}/\mathcal{K}$ are called $\mathcal{J}, X$–associated,*

$$F \underset{\mathcal{J},X}{\approx} G \quad \text{iff} \quad \forall x \in X : x \cdot (F - G) \in \mathcal{J}/\mathcal{K} .$$

For $X = \{ 1 \}$, we simply write

$$F \underset{\mathcal{J}}{\approx} G \iff F - G \in \mathcal{J}/\mathcal{K} \ .$$

Remark 22 *As $\mathcal{J}$ is not an ideal, association is not compatible with multiplication in $\mathcal{F}$ (not even by generalized numbers, only by elements of E). However, in the case of differential algebras, $\mathcal{J}$ is usually chosen such that $\underset{\mathcal{J},X}{\approx}$ is stable under differentiation.*

Example 23 *Usual association of generalized numbers, as recalled above, is obtained for $\mathcal{J} = N$, the set of null sequences:*

$$[x] \approx [y] \iff [x] \underset{N}{\approx} [y] \ .$$

As already mentioned, all elements of the ideal $\mathcal{K}$ tend to zero, i.e. $\mathcal{K} \subset N$, as needed for well-definedness at the level of the factor algebra.

4.2 Strong association

As mentioned, strong association is defined directly in terms of the ultranorm (or ultrametric) of elements of the factor space.

Definition 24 *For $s \in \mathbb{R}_+$,* **strong s–association** *is defined by*

$$F \overset{s}{\simeq} G \iff F \underset{\mathcal{J}^{(s)}_{\mathcal{P},r}}{\approx} G$$

with

$$\mathcal{J}^{(s)}_{\mathcal{P},r} = \left\{ f \in \mathcal{F} \mid \forall p \in \mathcal{P} : |\!|\!| f |\!|\!|_{p,r} < e^{-s} \right\} \ , \tag{4.1}$$

which is equivalent to say

$$F \overset{s}{\simeq} G \iff \widetilde{d}_{p,r}(F,G) < e^{-s} \ .$$

For $s = 0$, we write $F \simeq G$ and simply call them **strongly associated**.

If one has $F \overset{s}{\simeq} G$ for all $s \geq 0$, then $F = G$. Indeed, this means that $F - G$ is in the intersection of all balls of positive radius, which is equal to $\mathcal{K} = 0_{\mathcal{G}}$.

4.3 Weak association in $\overleftrightarrow{\mathcal{G}}_{p,r}$

In contrast to the above, weak association is defined by comparing sequences of *numbers* (not *functions*), obtained by means of a duality product

$$\langle \cdot, \cdot \rangle : \overleftrightarrow{E} \times \boldsymbol{D} \to \mathbb{C} \ ,$$

where $\boldsymbol{D}$ is a test function space such that $E \hookrightarrow \boldsymbol{D}'$ (as for example $\boldsymbol{D} = \mathcal{D}$ for $E = \mathcal{C}^\infty$). The subset $\mathcal{J}$ defining the association will then be of the form

$$\mathcal{J} = \mathcal{J}_M = \left\{ f \in \overleftrightarrow{E}^{\mathbb{N}} \mid \forall \psi \in \boldsymbol{D} : (\langle f_n, \psi \rangle)_n \in M \right\} \ , \tag{4.2}$$

where M is some additive subgroup of $\mathbb{C}^{\mathbb{N}}$ (like e.g. $M = N$, the null sequences).

Example 25 *For the choices given above,* $\boldsymbol{D} = \mathcal{D}$, $E = \mathcal{C}^\infty$ *and* $M = N$, *in the case of Colombeau's algebra, we get the usual, so-called weak association* $[f] \approx [g] \iff f_n - g_n \to 0$ *in* $\mathcal{D}'$.

Again, this is independent of the representatives, because $\mathcal{J} \supset \mathcal{K}_{r,p}$. *To see this, consider* $j \in \mathcal{K}_{r,p}$. *Then for any* $\varepsilon > 0$ *there is* n_0 *such that for* $n > n_0$,

$$|\langle j_n, \psi\rangle| \le \varepsilon^{1/r_n} \int |\phi| \underset{n\to\infty}{\longrightarrow} 0 \ .$$

Thus, $\langle f_n, \psi\rangle \underset{n\to\infty}{\longrightarrow} 0 \iff \langle f_n + j_n, \psi\rangle \underset{n\to\infty}{\longrightarrow} 0.$

This is a special case of the following definition.

Definition 26 $s - \boldsymbol{D}'$**–association** *is defined by*

$$F \overset{s}{\underset{\boldsymbol{D}}{\approx}} G \iff F \underset{\mathcal{J}_N, X_s}{\approx} G$$

with $X_s = \left\{ \left[\left(e^{s/r_n}\right)_n\right] \right\}$ *for* $s \in \mathbb{R}$.

Note that this generalized number is always of the same form, but depends in each case on the sequence $(r_n)_n$ defining the topology.

Example 27 *In Colombeau's case,* $r = 1/\log$, *we have* $X_s = \{ [(n^s)_n] \}$. *For* $s = 0$ $(X_0 = \{1\})$, *we get the already mentioned weak association.*

For $s \neq 0$, $[f] \overset{s}{\underset{\mathcal{D}}{\approx}} [g] \iff n^s(f_n - g_n) \to 0$ *in* $\mathcal{D}'$. *This also has already been considered (with* $\boldsymbol{D} = \mathcal{D}$*), for example in [14] (where it had been denoted by* $\underset{s}{\approx}$*). This association is of course stronger than the simple weak association (again, because association is not compatible with multiplication even only by generalized numbers).*

As an extension of this example, consider $\mathcal{J}$ *as above, and*
$X = \{ [(n^s)_n] \}_{s\in\mathbb{N}}$. *This means that*

$$[f] \approx [g] \iff \forall s \in \mathbb{N} : \lim n^s(f_n - g_n) = 0 \ \text{in}\ \mathcal{D}'$$

In Colombeau's algebra, this amounts in fact to strict equality.

Definition 28 ***Weak*** s**–*association*** *is defined for any* $s \in \mathbb{R}$ *by*

$$F \overset{(s)}{\simeq} G \iff F \underset{\mathcal{J}_{(s)}}{\approx} G$$

where

$$\begin{aligned} \mathcal{J}_{(s)} &= \left\{ f \in E^{\mathbb{N}} \mid \forall \psi \in \boldsymbol{D} : \| \left(\langle f_n - g_n, \psi\rangle \right)_n \|_{|\cdot|,r} < e^{-s} \right\} \\ &= \left\{ f \in E^{\mathbb{N}} \mid \forall \psi \in \boldsymbol{D} : \limsup_{n\to\infty} |\langle f_n - g_n, \psi\rangle|^{r_n} < e^{-s} \right\} \ . \end{aligned}$$

It is obtained from the general setting (4.2) by observing that $\mathcal{J}_{(s)} = J_M$ *with (cf. eq. (4.1))*

$$M = \mathcal{J}^{(s)}_{|\cdot|,r} = \left\{ c \in \mathbb{C}^{\mathbb{N}} \mid \| c \|_{|\cdot|,r} < e^{-s} \right\} \ .$$

For $s = 0$, *we write* $F \overset{\text{sw}}{\approx} G$ *and call* F *and* G **strong–weak associated**.

Weak s–association implies $s - \boldsymbol{D}'$–association, but conversely $s - \boldsymbol{D}'$–association only implies weak s'–association for all $s' < s$.

Let us consider some details concerning the structure of strong-weak association. In the following we will note $|\cdot|_r = \|\!|\cdot\|\!|_{|\cdot|,r}$, i.e.

$$|c|_r = \limsup_{n\to\infty} |c_n|^{r_n} \ .$$

To start with, let us remark that $\mathcal{I}_{|\cdot|,r} = \left\{ c \in \mathbb{C}^{\mathbb{N}} \mid |c|_r < 1 \right\}$ is an ideal in the subalgebra $\mathcal{H}_{|\cdot|,r} = \left\{ c \in \mathbb{C}^{\mathbb{N}} \mid |c|_r \le 1 \right\}$ of $\mathbb{C}^{\mathbb{N}}$.

Let us now consider the topology on $\mathbb{C}^{\mathbb{N}}$ induced by the $|\cdot|_r$–norm. We have

$$|c|_r \le a \iff \forall b > a, \exists n_0, \forall n > n_0 : |c_n| \le b^{1/r_n} \ .$$

But now observe that for $b > 1$, $b = 1$ and $b < 1$, the limit of the last expression is respectively ∞, 1 and 0. This means that

1. $|c|_r < 1 \implies \lim c_n = 0$
2. $\lim c_n = 0 \implies \forall b > 1\ \exists n_0\ \forall n \ge n_0 : |c_n| \le b^{1/r_n} \to \infty, \implies |c|_r \le 1$
3. $|c|_r = 1$: any value in $\mathbb{R}_+ \cup \{ \infty \}$ (or none at all) is possible as limit for c_n. Indeed, whatever be the null sequence (r_n), the sequences $c_n = r_n$ (resp. $c_n = 1/r_n$) have limits 0 (resp. ∞), but

$$|c_n|^{r_n} = \exp(\pm r_n \log r_n) \underset{n\to\infty}{\to} 1 \quad (\text{ because } x \log x \underset{x\to 0}{\to} 0\ )$$

   i.e. $|c|_r = 1$.

Thus, all elements of the open unit ball are weakly associated to zero. This is very similar to classical results related to ultrametric spaces and weak topology.

Proposition 29 *Weak s–association implies $s - \boldsymbol{D}'$–association, but the converse is not true.*

Proof. This follows from $\|\!| c \|\!|_{|\cdot|,r} < 1 \implies \lim_{n\to\infty} c_n = 0 \implies \|\!| c \|\!|_{|\cdot|,r} \le 1$, with $c_n = \langle f_n, \psi\rangle\, e^{s/r_n}$. (As already seen, for $\|\!| c \|\!|_{|\cdot|,r} = 1$, nothing can be concluded about the limit of (c_n)). $\square$

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