

# Extensions of the CQ Algorithm for the Split Feasibility and Split Equality Problems

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## Abstract

The convex feasibility problem (CFP) is to find a member of the intersection of finitely many closed convex sets in Euclidean space. When the intersection is empty, one can minimize a proximity function to obtain an approximate solution to the problem. The split feasibility problem (SFP) and the split equality problem (SEP) are generalizations of the CFP. The approximate SFP (ASFP) and approximate SEP (ASEP) involve finding only approximate solutions to the SFP and SEP, respectively.

We present here the SSEA, a simultaneous iterative algorithm for solving the ASEP. When this algorithm is applied to the ASFP it resembles closely, but is not equivalent to, the CQ algorithm. The SSEA involves orthogonal projection onto the given closed convex sets. The *relaxed* SSEA (RSSEA) is an easily implementable variant of the SSEA that uses orthogonal projection onto half-spaces at each step to solve the SEP.

The perturbed version of the SSEA (PSSEA) is similar to the RSSEA, but uses orthogonal projection onto a sequence of epi-convergent closed convex sets.

**Key Words:** convex feasibility; split feasibility; split equality; iterative algorithms; CQ algorithm.

## 1 Introduction

Recently, the second author presented the ACQA algorithm [15] and the RACQA algorithm [16]. Both algorithms solve the SEP. In the ACQA, the step-length parameters are allowed to vary, while in the RACQA the parameters do not vary, but the projections are onto half-spaces, instead of onto the given closed convex sets. The

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ACQA and the RACQA can be viewed as generalizations of the original CQ algorithm [2, 3]. Because they are sequential, rather than simultaneous, algorithms, they converge only when the SEP has an exact solution; they do not, in general, provide approximate solutions when no exact solutions exist; that is, they solve the SEP, but do not solve the ASEP. In this paper we propose variations of the ACQA and RACQA algorithms, the simultaneous split equality algorithm (SSEA) and relaxed and perturbed variants, the RSSEA and PSSEA, that do provide solutions of the ASEP.

## 2 Preliminaries

Let  $C \subseteq \mathbb{R}^N$  and  $Q \subseteq \mathbb{R}^M$  be closed, non-empty convex sets, and let  $A$  and  $B$  be  $J$  by  $N$  and  $J$  by  $M$  real matrices, respectively. The *split equality* problem (SEP) is to find  $x \in C$  and  $y \in Q$  such that  $Ax = By$ ; the *approximate split equality* problem (ASEP) is to minimize the function

$$f(x, y) = \frac{1}{2} \|Ax - By\|_2^2, \quad (2.1)$$

over  $x \in C$  and  $y \in Q$ . When  $J = M$  and  $B = I$ , the SEP reduces to the *split feasibility problem* (SFP) and the ASEP becomes the *approximate split feasibility problem* (ASFP). Moreover, if we take  $N = M = J$  in the ASEP and let  $C$  and  $Q$  be in  $\mathbb{R}^N$ , the problem becomes that of minimizing the distance between the two sets. The SSEA then is a variant of the Cheney-Goldstein alternating projection algorithm [6].

We present here what we call the *simultaneous split equality algorithm* (SSEA), an iterative algorithm for solving the ASEP. The SSEA is a particular case of the *projected Landweber* (PLW) algorithm. When the SSEA is applied to the ASFP it resembles closely, but is not equivalent to, the CQ algorithm. We also present easily implementable relaxed versions of these algorithms, involving sub-gradient projections.

## 3 The CQ Algorithm

The CQ algorithm is an iterative method for solving the ASFP [2, 3]. It was noted in [8] that the CQ algorithm is a particular case of the forward-backward splitting algorithm.

**Algorithm 3.1 (CQ)** For arbitrary  $x^0$ , let

$$x^{k+1} = P_C(x^k - \gamma A^T(I - P_Q)Ax^k). \quad (3.1)$$

The operator

$$T = P_C(I - \gamma A^T(I - P_Q)A) \quad (3.2)$$

is averaged whenever  $\gamma$  is in the interval  $(0, 2/L)$ , where  $L$  is the largest eigenvalue of  $A^T A$ , and so the CQ algorithm converges to a fixed point of  $T$ , whenever such fixed points exist. When the SFP has a solution, the CQ algorithm converges to a solution of the SFP; when it does not, the CQ algorithm converges to a minimizer, over  $C$ , of the proximity function  $g(x) = \frac{1}{2}\|P_Q Ax - Ax\|_2^2$ , whenever such minimizers exist, and so solves the ASFP. The function  $g(x)$  is convex and its gradient is

$$\nabla g(x) = A^T(I - P_Q)Ax. \quad (3.3)$$

The convergence of the CQ algorithm then follows from the Krasnosel'skii-Mann-Opial Theorem [10, 12, 17].

## 4 The Projected Landweber Algorithm

Let  $S$  be a closed, nonempty, convex subset of  $\mathbb{R}^I$ ,  $G$  a real  $K$  by  $I$  matrix, and  $L = \rho(G^T G)$ .

**Theorem 4.1 (The Projected Landweber Algorithm)** Let  $\epsilon$  lie in the interval  $(0, \frac{1}{L})$ . The sequence  $\{w^k\}$  generated by the iterative step

$$w^{k+1} = P_S(w^k - \gamma_k G^T(Gw^k - b)) \quad (4.1)$$

converges to a minimizer, over  $w \in S$ , of the function  $f(w) = \frac{1}{2}\|Gw - b\|^2$ , whenever such minimizers exist, for any  $\gamma_k$  in the interval  $[\epsilon, \frac{2}{L} - \epsilon]$ .

**Proof:** Let  $z \in S$  minimize  $f(w)$  over  $w \in S$ . Then we have

$$z = P_S(z - \gamma_k G^T(Gz - b)),$$

for all  $k$ , so that

$$\begin{aligned} \|z - w^{k+1}\|^2 &= \|P_S(z - \gamma_k G^T(Gz - b)) - P_S(w^k - \gamma_k G^T(Gw^k - b))\|^2 \\ &\leq \|z - w^k - \gamma_k G^T(Gz - Gw^k)\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|z - w^k\|^2 - 2\gamma_k \langle z - w^k, G^T(Gz - Gw^k) \rangle + \gamma_k^2 \|G^T(Gz - Gw^k)\|^2 \\
&\leq \|z - w^k\|^2 - (2\gamma_k - \gamma_k^2 L) \|Gz - Gw^k\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|z - w^k\|^2 - \|z - w^{k+1}\|^2 &\geq (2\gamma_k - \gamma_k^2 L) \|Gz - Gw^k\|^2 \\
&\geq L\epsilon^2 \|Gz - Gw^k\|^2.
\end{aligned}$$

It follows that the sequence  $\{\|z - w^k\|^2\}$  is decreasing, the sequence  $\{\|Gz - Gw^k\|^2\}$  converges to zero, the sequence  $\{w^k\}$  is bounded, and a subsequence converges to some  $w^* \in S$  with  $Gw^* = Gz$ . Consequently,  $\{\|w^* - w^k\|^2\}$  is decreasing, and therefore must converge to zero.  $\blacksquare$

## 5 The SSEA

Our simultaneous iterative algorithm for solving the ASEP, the SSEA, is an application to the ASEP of the projected Landweber algorithm.

Let  $I = M + N$ ,  $S = C \times Q$  in  $\mathbb{R}^N \times \mathbb{R}^M = \mathbb{R}^I$ . Define

$$G = [A \quad -B],$$

$$w = \begin{bmatrix} x \\ y \end{bmatrix},$$

so that

$$G^T G = \begin{bmatrix} A^T A & -A^T B \\ -B^T A & B^T B \end{bmatrix}.$$

The original problem can now be reformulated as finding  $w \in S$  with  $Gw = 0$ , or, more generally, minimizing the function  $\|Gw\|$  over  $w \in S$ .

The iterative step of the PLW algorithm in this case is the following:

$$w^{k+1} = P_S(w^k - \gamma_k G^T(Gw^k)). \quad (5.1)$$

Expressing this in terms of  $x$  and  $y$ , we obtain

$$x^{k+1} = P_C(x^k - \gamma_k A^T(Ax^k - By^k)), \quad (5.2)$$

and

$$y^{k+1} = P_Q(y^k + \gamma_k B^T(Ax^k - By^k)). \quad (5.3)$$

The PLW converges, in this case, to a minimizer of  $\|Gw\|$  over  $w \in S$ , whenever such minimizers exist, for  $\epsilon \leq \gamma_k \leq \frac{2}{\rho(G^T G)} - \epsilon$ .

## 6 Solving the SFP

When  $J = M$  and  $B = I$ , the iterations in Equations in (5.2) and (5.3) become

$$x^{k+1} = P_C(x^k - \gamma_k A^T(Ax^k - y^k)), \quad (6.1)$$

and

$$y^{k+1} = P_Q(y^k + \gamma_k(Ax^k - y^k)). \quad (6.2)$$

This application of the SSEA to the ASFP resembles, but is not equivalent to, the original CQ algorithm, but it does solve the same problem.

## 7 The Relaxed PLW Algorithm

The projected Landweber algorithm requires the orthogonal projection onto  $S$  at each step of the iteration. In this section we modify the PLW algorithm so that, at the  $k$ th step, we project onto a half-space, using sub-gradient projection. In a subsequent section we apply this relaxed PLW (RPLW) algorithm to obtain the relaxed SSEA (RSSEA) that solves the SEP.

We assume now that  $S = \{w | h(w) \leq 0\}$ , where  $h(w)$  is convex and sub-differentiable on  $\mathbb{R}^I$  and the sub-differential is bounded on bounded sets. For each  $k$  we define

$$S_k := \{w | \langle \xi^k, w - w^k \rangle + h(w^k) \leq 0\},$$

where  $\xi^k$  is an arbitrarily chosen member of the sub-differential  $\partial h(w^k)$ . It follows that  $S \subseteq S_k$ , for all  $k$ . We assume that there is  $z \in S$  with  $Gz = 0$ . We prove the following theorem:

**Theorem 7.1** *Let  $\epsilon$  lie in the interval  $(0, \frac{1}{L})$ . For each  $k$ , define*

$$w^{k+1} = P_{S_k}(w^k - \gamma_k G^T G w^k). \quad (7.1)$$

*The sequence  $\{w^k\}$  converges to  $w^* \in S$  with  $Gw^* = 0$ , for any  $\gamma_k$  in the interval  $[\epsilon, \frac{2}{L} - \epsilon]$ .*

**Proof:** A vector  $y^k$  in  $S_k$  minimizes the function  $f(t) = \frac{1}{2} \|Gt\|^2$  over all  $t$  in  $S_k$  if and only if

$$\langle \nabla f(y^k), t - y^k \rangle \geq 0,$$

for all  $t \in S_k$ . This is equivalent to

$$0 \leq \langle y^k - (y^k - \gamma_k \nabla f(y^k)), t - y^k \rangle \geq 0,$$

from which we conclude that

$$y^k = P_{S_k}(y^k - \gamma_k \nabla f(y^k)).$$

Since  $Gz = 0$ ,  $z$  minimizes  $f(t)$  over  $t \in S_k$ , for all  $k$ , so that

$$z = P_{S_k}z = P_{S_k}(z - \gamma_k G^T Gz),$$

for all  $k$ . Then we have

$$\begin{aligned} \|z - w^{k+1}\|^2 &= \|P_{S_k}(z - \gamma_k G^T Gz) - P_{S_k}(w^k - \gamma_k G^T Gw^k)\|^2 \\ &\leq \|z - \gamma_k G^T Gz - w^k + \gamma_k G^T Gw^k\|^2 \\ &= \|z - w^k\|^2 - 2\gamma_k \langle z - w^k, G^T Gz - G^T Gw^k \rangle + \gamma^2 \|G^T Gz - G^T Gw^k\|^2. \end{aligned}$$

Therefore,

$$\|z - w^k\|^2 - \|z - w^{k+1}\|^2 \geq (2\gamma_k - \gamma_k^2 L) \|Gz - Gw^k\|^2.$$

Continuing as in the previous proof, we find that the sequence  $\{\|z - w^k\|\}$  is decreasing, the sequence  $\{\|Gw^k\|^2\}$  converges to zero, the sequence  $\{w^k\}$  is bounded, and there is a subsequence converging to some  $w^*$  with  $Gw^* = 0$ . We need to show that  $w^*$  is in the set  $S$ .

We show first that the sequence  $\{w^k\}$  is asymptotically regular, that is, the sequence  $\{\|w^k - w^{k+1}\|\}$  converges to zero. From

$$\|w^k - w^{k+1}\|^2 = \|z - w^k\|^2 - \|z - w^{k+1}\|^2 + 2\langle w^{k+1} - w^k, w^{k+1} - z \rangle,$$

$$\langle w^{k+1} - (w^k - \gamma_k G^T Gw^k), z - w^{k+1} \rangle \geq 0,$$

and

$$\langle w^k - w^{k+1}, z - w^{k+1} \rangle \leq \gamma_k \langle G^T Gw^k, z - w^{k+1} \rangle \leq \gamma_k \|G^T Gw^k\| \|z - w^{k+1}\|$$

we have

$$\|w^k - w^{k+1}\|^2 \leq \|z - w^k\|^2 - \|z - w^{k+1}\|^2 + 2\gamma_k \|G^T Gw^k\| \|z - w^{k+1}\|.$$

Since the sequence  $\{\|z - w^{k+1}\|\}$  is bounded, the right hand side converges to zero. Therefore, the left hand side also converges to zero.

Suppose that the subsequence  $\{w^{k_n}\}$  converges to  $w^*$  with  $Gw^* = 0$ . Since  $w^{k_n+1}$  is in  $S_{k_n}$ , we have

$$\langle \xi^{k_n}, w^{k_n+1} - w^{k_n} \rangle + h(w^{k_n}) \leq 0.$$

Therefore,

$$h(w^{k_n}) \leq -\langle \xi^{k_n}, w^{k_n+1} - w^{k_n} \rangle \leq \xi \|w^{k_n+1} - w^{k_n}\|,$$

where  $\|\xi^{k_n}\| \leq \xi$ , for all  $n$ . The lower semi-continuity of  $h(w)$  and the asymptotic regularity of  $\{w^k\}$  lead to

$$h(w^*) \leq \liminf_{n \rightarrow \infty} h(w^{k_n}) \leq 0.$$

Consequently,  $w^* \in S$ . It follows now that the sequence  $\{\|w^* - w^k\|\}$  converges to zero. This completes the proof.  $\blacksquare$

## 8 The Relaxed SSEA

We assume now that  $c : \mathbb{R}^N \rightarrow \mathbb{R}$  and  $q : \mathbb{R}^M \rightarrow \mathbb{R}$  are convex, sub-differentiable functions, and that the sub-differentials are bounded on bounded sets. We define

$$C := \{x | c(x) \leq 0\},$$

and

$$Q := \{y | q(y) \leq 0\}.$$

For each  $k$  we define

$$C_k := \{x | c(x^k) + \langle \xi^k, x - x^k \rangle \leq 0\},$$

and

$$Q_k := \{y | q(y^k) + \langle \eta^k, y - y^k \rangle \leq 0\},$$

where  $\xi^k \in \partial c(x^k)$  and  $\eta^k \in \partial q(y^k)$ . It follows from the definition of sub-differential that  $C \subseteq C_k$  and  $Q \subseteq Q_k$ , for all  $k$ .

We define  $h : \mathbb{R}^N \times \mathbb{R}^M$  to be  $h(w) = h(x, y) = c(x) + q(y)$ . Then  $C \times Q \subseteq S$ , where  $S = \{w | h(w) \leq 0\}$ . The RPLW algorithm now takes the form

$$x^{k+1} = P_{C_k}(x^k - \gamma_k A^T(Ax^k - By^k)), \tag{8.1}$$

and

$$y^{k+1} = P_{Q_k}(y^k + \gamma_k B^T(Ax^k - By^k)), \tag{8.2}$$

where

$$\beta^k = \begin{bmatrix} \xi^k \\ \eta^k \end{bmatrix}$$

is an arbitrary member of  $\partial h(w^k) = \partial h(x^k, y^k)$ .

It is easily shown that

$$\partial h(w^k) = \partial c(x^k) \times \partial q(y^k).$$

Therefore,  $\xi^k \in \partial c(x^k)$  and  $\eta^k \in \partial q(y^k)$ . Since  $\{w^k\}$  is asymptotically regular, so are the sequences  $\{x^k\}$  and  $\{y^k\}$ . Proceeding as in the final paragraph of the proof of Theorem 7.1, we find that  $\{w^k\}$  converges to a vector

$$w^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

with  $x^* \in C$ ,  $y^* \in Q$ , and  $Ax^* = By^*$ .

## 9 A Relaxed PLW Algorithm for the SFP

When  $J = M$  and  $B = I$ , the iterations in Equations in (8.1) and (8.2) become

$$x^{k+1} = P_{C_k}(x^k - \gamma_k A^T(Ax^k - y^k)), \quad (9.1)$$

and

$$y^{k+1} = P_{Q_k}(y^k + \gamma_k(Ax^k - y^k)). \quad (9.2)$$

This application of the SSEA to the SFP resembles, but is not equivalent to, the original CQ algorithm, but it does solve the same problem. Because our proof of convergence of the RPLW algorithm assumed that  $Gz = 0$ , we have established convergence for the iterative algorithm given by Equations (9.1) and (9.2) only in the consistent case when there is  $z \in C$  with  $Az \in Q$ . This algorithm for the SFP is closely related to the relaxed CQ algorithm of Xu [19], which also applies only in the consistent case.

## 10 A Perturbed PLW Algorithm

Our next algorithm is a modification of the PLW algorithm that we call a perturbed PLW (PPLW) algorithm. We denote by  $\text{NCCS}(\mathbb{R}^I)$  the family of all nonempty, closed and convex subsets of  $\mathbb{R}^I$ . We begin with a definition and a proposition that are also used in [5].



**Definition 10.1** Let  $S$  and  $(S_k)_{k=0}^{\infty}$  be a set and a sequence of sets in  $NCCS(\mathbb{R}^I)$ , respectively. The sequence  $(S_k)_{k=0}^{\infty}$  is said to epi-converge to the set  $S$  if the following two conditions hold:

- (i) for every  $x \in S$ , there exists a sequence  $(S_k)_{k=0}^{\infty}$  such that  $x^k \in S_k$  for all  $k \geq 0$ , and  $\lim_{k \rightarrow \infty} x^k = x$ ; and
- (ii) If  $x^{k_n} \in S_{k_n}$  for all  $n \geq 0$ , and  $\lim_{n \rightarrow \infty} x^{k_n} = x$ , then  $x \in S$ .

Let  $S$  be a closed, nonempty, convex subset of  $\mathbb{R}^I$ ,  $G$  a real  $K$  by  $I$  matrix, and  $L = \rho(G^T G)$ . We assume there is  $z \in S$  with  $Gz = 0$ . Let  $\epsilon$  lie in the interval  $(0, \frac{1}{L})$ .

**Theorem 10.1 (The Perturbed PLW Algorithm PPLW)** Assume that  $S \subseteq S_k$  for all  $k \geq 0$ , and that  $(S_k)_{k=0}^{\infty}$  epi-converges to  $S$ . The sequence  $\{w^k\}$  generated by the iterative step

$$w^{k+1} = P_{S_k}(w^k - \gamma_k G^T G w^k) \quad (10.1)$$

converges to  $w^* \in S$  with  $Gw^* = 0$ , for any  $\gamma_k$  in the interval  $[\epsilon, \frac{2}{L} - \epsilon]$ .

**Proof:** Let  $z \in S$  such that  $Gz = 0$ . We have

$$\begin{aligned} \|z - w^{k+1}\|^2 &= \|P_{S_k} z - P_{S_k}(w^k - \gamma_k G^T G w^k)\|^2 \\ &\leq \|z - w^k + \gamma_k G^T G w^k\|^2 \leq \|z - w^k\|^2 - (2\gamma_k - \gamma_k^2 L) \|Gw^k\|^2. \end{aligned}$$

Therefore,

$$\|z - w^k\|^2 - \|z - w^{k+1}\|^2 \geq \gamma_k(2 - \gamma_k L) \|Gw^k\|^2.$$

It follows that the sequence  $\{\|z - w^k\|\}$  is decreasing, and the sequence  $\{Gw^k\}$  converges to zero. Since  $\{w^k\}$  is bounded, there is a subsequence converging to some  $w^*$  with  $Gw^* = 0$ . By the epi-convergence of the sequence  $\{S_k\}$  to  $S$ , we know that  $w^* \in S$ . Replacing  $z$  with  $w^*$  above, we find that  $w^k \rightarrow w^*$ . This completes the proof. ■

We find it necessary to assume that  $Gz = 0$ , rather than simply that  $z$  minimize the function  $f(w) = \frac{1}{2} \|Gw\|^2$  over  $w \in S$ . Since  $S \subseteq S_k$ , the latter assumption would not be sufficient to conclude that  $z$  also minimizes  $f(w)$  over  $w \in S_k$ . A similar problem arises in [5]; the proof given there for Lemma 5.1 is incorrect.

It is well-known that, when  $S \subseteq S_{k+1} \subseteq S_k$ , the sequence  $S_k$  epi-converges to  $S = \bigcap_{k \in \mathbb{N}} S_k$ . Furthermore, the half-space approximations  $(S_k)$  considered previously in connection with the RPLW algorithm may not epi-converge to  $S$ , so Theorem 7.1 cannot be derived from Theorem 10.1.

# 11 Removing the Set-Inclusion Assumption

The following proposition is found in [9, 18]. For completeness, we include a proof of the proposition here.

**Proposition 11.1** *Let  $S$  and  $(S_k)_{k=0}^{\infty}$  be a set and a sequence of sets in  $\text{NCCS}(\mathbb{R}^I)$ , respectively. If  $(S_k)_{k=0}^{\infty}$  epi-converges to  $S$  and  $\lim_{k \rightarrow \infty} y^k = y$ , then*

$$\lim_{k \rightarrow \infty} P_{S_k} y^k = P_S y.$$

**Proof:** Let  $x = P_S y$ . Since  $x \in S$ , there is a sequence  $\{x^k\}$  converging to  $x$ , with  $x^k \in S_k$ , for each  $k$ . Using

$$\|P_{S_k} x - x\| \leq \|x^k - x\|,$$

we conclude that the sequence  $\{P_{S_k} x\}$  also converges to  $x$ . From

$$\|P_{S_k} y - P_{S_k} x\| \leq \|y - x\|,$$

it follows that the sequence  $\{P_{S_k} y\}$  is bounded. From

$$\|P_{S_k} y^k - P_{S_k} y\| \leq \|y^k - y\|,$$

it follows that the sequence  $\{P_{S_k} y^k\}$  is bounded. Then there is a  $z \in S$  and a subsequence  $\{P_{S_{k_n}} y^{k_n}\}$  converging to  $z$ . Because

$$\|P_{S_{k_n}} y^{k_n} - y^{k_n}\| \leq \|P_{S_{k_n}} x - y^{k_n}\|,$$

taking limits, we conclude that

$$\|z - y\| \leq \|x - y\|.$$

Therefore,  $z = P_S y = x$ . ■

With a stronger assumption on the convergence of  $\{P_{S_k} x\}$  to  $P_S x$  we can remove the set-inclusion assumption in Theorem 10.1. We need the following result from [7, 1].

**Lemma 11.1** *Let  $\{a_k\}$  and  $\{\epsilon_k\}$  be positive sequences, with  $a_{k+1} \leq a_k + \epsilon_k$ , and  $\sum_{k=1}^{\infty} \epsilon_k < +\infty$ . Then the sequence  $\{a_k\}$  is convergent.*

To remove the set inclusion assumption  $S \subseteq S_k$  we propose the following.

**Theorem 11.1** *Assume that there is  $z \in S$  with  $Gz = 0$ , that the sequence  $\{S_k\}_{k=0}^{\infty}$  epi-converges to  $S$ , and that  $\sum_{k=0}^{\infty} \|P_{S_k}x - x\| < +\infty$  for all  $x \in S$ . Then the sequence  $\{w^k\}$  given by Equation (10.1) converges to  $w^* \in S$  with  $Gw^* = 0$ .*

**Proof:** We can write

$$\begin{aligned} \|z - w^{k+1}\| &= \|z - P_{S_k}(w^k - \gamma_k G^T G w^k)\| \\ &\leq \|z - w^k + \gamma_k G^T G w^k\| + \|P_{S_k}z - z\|. \end{aligned}$$

Therefore

$$\|z - w^{k+1}\| \leq \sqrt{\|z - w^k\|^2 - \gamma_k(2 - \gamma_k L)\|Gw^k\|^2} + \|P_{S_k}z - z\|. \quad (11.1)$$

Applying Lemma 11.1 to the inequality (11.1), combined with Proposition 11.1 and the same argument as in the proof of Theorem 10.1, allows us to conclude that the sequence  $\{w^k\}$  converges. ■

## 12 A Perturbed SSEA Algorithm

Now, let  $I = M + N$ ,  $S = C \times Q$  in  $\mathbb{R}^N \times \mathbb{R}^M = \mathbb{R}^I$  and remember that, by defining

$$G = [A \quad -B],$$

$$w = \begin{bmatrix} x \\ y \end{bmatrix},$$

the SEP can be reformulated as finding  $w \in S$  with  $Gw = 0$ . Let  $S_k = C_k \times Q_k$  in  $\mathbb{R}^N \times \mathbb{R}^M = \mathbb{R}^I$ , where  $C_k$  and  $Q_k$  are sequences of sets satisfying  $C \subseteq C_k$  and  $Q \subseteq Q_k$  for all  $k \geq 0$ , that  $(C_k)_{k=0}^{\infty}$  epi-converges to  $C$  and  $(Q_k)_{k=0}^{\infty}$  epi-converges to  $Q$ . In this case the perturbed PLW algorithm PPLW gives a perturbed SSEA that we call the PSSEA:

$$x^{k+1} = P_{C_k}(x^k - \gamma_k A^T(Ax^k - By^k)), \quad (12.1)$$

and

$$y^{k+1} = P_{Q_k}(y^k + \gamma_k B^T(Ax^k - By^k)). \quad (12.2)$$

This PPLW iteration converges to  $w^* \in S$  with  $Gw^* = 0$ ; in other words,  $\{w^k\}$  converges to a vector

$$w^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

with  $x^* \in C$ ,  $y^* \in Q$ , and  $Ax^* = By^*$ .

## 13 Related Algorithms

In a previous paper [15] Moudafi presented the ACQA algorithm. The ACQA algorithm has the following iterative step:

$$x^{k+1} = P_C(x^k - \gamma_k A^T(Ax^k - By^k)), \quad (13.1)$$

and

$$y^{k+1} = P_Q(y^k + \gamma_k B^T(Ax^{k+1} - By^k)). \quad (13.2)$$

Here the parameters  $\gamma_k$  are allowed to vary with each iterative step.

In [16] the same author proposed a second iterative algorithm, the RACQA, to solve the SEP. The RACQA algorithm has the following iterative step:

$$x^{k+1} = P_{C_k}(x^k - \gamma A^T(Ax^k - By^k)), \quad (13.3)$$

and

$$y^{k+1} = P_{Q_k}(y^k + \gamma B^T(Ax^{k+1} - By^k)). \quad (13.4)$$

The  $\{C_k\}$  and  $\{Q_k\}$  are the sequences of half-spaces defined previously that contain  $C$  and  $Q$ , respectively. Now the parameters do not vary, but the projections are onto half-spaces, instead of onto the sets  $C$  and  $Q$ .

Both the ACQA and the RACQA can be viewed as generalizations of the original CQ algorithm. Because they are sequential, rather than simultaneous, algorithms, they converge only when the SEP has an exact solution; they do not, in general, provide a minimizer of the function  $\|Ax - By\|$  over  $x \in C$  and  $y \in Q$ .

## 14 Conclusions

The approximate split equality problem (ASEP) is to minimize the function  $f(x) = \frac{1}{2}\|Ax - By\|^2$ , over  $x \in C$  and  $y \in Q$ . The simultaneous split equality algorithm (SSEA), obtained from the projected Landweber algorithm (PLW), is an iterative procedure that solves the ASEP, and, therefore, the approximate split feasibility problem (ASFP). As applied to the ASFP, the SSEA is similar, but not equivalent, to the CQ algorithm.

The relaxed PLW (RPLW) algorithm replaces orthogonal projection onto  $S$  with orthogonal projection onto half-spaces  $S_k$  containing  $S$ ; the relaxed SSEA (RSSEA)

is a particular case of the RPLW algorithm. We are able to prove convergence of the RSSEA for the case in which there are  $x \in C$  and  $y \in Q$  with  $Ax = By$ .

The perturbed PLW algorithm PPLW is similar to the RPLW algorithm and use orthogonal projection onto sets  $S_k$  epi-converging to  $S$ . The PSSEA algorithm is the application of PPLW, to the SEA problem. As is the case for the RSSEA, we are able to prove that the PSSEA algorithm generates sequences that converge to solutions of the SEA problem.

Finally, note that our results are still valid in an infinite dimensional Hilbert space with weak convergence. To reach strong convergence, a first approach is the introduction of a quadratic term for  $\varepsilon > 0$ , namely to consider the function

$$f_\varepsilon(w) = \frac{1}{2}\|Gw - b\|^2 + \frac{\varepsilon}{2}\|w\|^2.$$

The minimizer  $w^\varepsilon$  of this function can be computed by the following regularized PLW algorithm

$$w^{k+1} = \frac{1}{1 + \gamma\varepsilon} P_S(w^k - \gamma G^T(Gw^k - b)). \quad (14.1)$$

For  $\gamma \in (0, \frac{2}{L})$ , the function

$$w \mapsto \frac{1}{1 + \gamma\varepsilon} P_S(w - \gamma G^T(Gw - b))$$

is a contraction, hence the iteration converges linearly to the unique minimizer of  $f_\varepsilon$ . One can easily show that there exist sequences of minimizers  $w^\varepsilon$  which converge to a minimizer  $w^*$  (the minimum-norm solution) for  $f$ . A further alternative is to consider, for instance, the following diagonal version

$$w^{k+1} = \frac{1}{1 + \gamma_k \varepsilon_k} P_S(w^k - \gamma_k G^T(Gw^k - b)). \quad (14.2)$$

Roughly speaking, if the sequence of the regularized parameters  $(\varepsilon_k)$  converges slowly to zero, one can show that the sequence  $(w^k)$  generated by (14.2) strongly converges to a particular minimizer (the minimum-norm solution) of  $f$ . For more details about Tikhonov (or viscosity) approximation, see, for example, [14] or [19] and the references therein.

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