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Research Article

A Penalization-Gradient Algorithm for Variational Inequalities

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This paper is concerned with the study of a penalization-gradient algorithm for solving variational inequalities, namely, find $\bar{x} \in C$ such that $\langle A\bar{x}, y - \bar{x} \rangle \geq 0$ for all $y \in C$, where $A : H \rightarrow H$ is a single-valued operator, C is a closed convex set of a real Hilbert space H . Given $\Psi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ which acts as a penalization function with respect to the constraint $\bar{x} \in C$, and a penalization parameter β_k , we consider an algorithm which alternates a proximal step with respect to $\partial\Psi$ and a gradient step with respect to A and reads as $x_k = (I + \lambda_k \beta_k \partial\Psi)^{-1}(x_{k-1} - \lambda_k A x_{k-1})$. Under mild hypotheses, we obtain weak convergence for an inverse strongly monotone operator and strong convergence for a Lipschitz continuous and strongly monotone operator. Applications to hierarchical minimization and fixed-point problems are also given and the multivalued case is reached by replacing the multivalued operator by its Yosida approximate which is always Lipschitz continuous.

1. Introduction

Let H be a real Hilbert space, $A : H \rightarrow H$ a monotone operator, and let C be a closed convex set in H , we are interested in the study of a gradient-penalization algorithm for solving the problem of finding $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x} \rangle \geq 0 \quad \forall y \in C, \quad (1.1)$$

or equivalently

$$A\bar{x} + N_C(\bar{x}) \ni 0, \quad (1.2)$$

where N_C is the normal cone to a closed convex set C . The above problem is a variational inequality, initiated by Stampacchia [1], and this field is now a well-known branch of pure and applied mathematics, and many important problems can be cast in this framework.

In [2], Attouch et al., based on seminal work by Passty [3], solve this problem with a multivalued operator by using splitting proximal methods. A drawback is the fact that the convergence in general is only ergodic. Motivated by [2, 4] and by [5] where penalty methods for variational inequalities with single-valued monotone maps are given, we will prove that our proposed forward-backward penalization-gradient method (1.9) enjoys good asymptotic convergence properties. We will provide some applications to hierarchical fixed-point and optimization problems and also propose an idea to reach monotone variational inclusions.

To begin with, see, for instance [6], let us recall that an operator with domain $D(T)$ and range $R(T)$ is said to be monotone if

$$\langle u - v, x - y \rangle \geq 0 \quad \text{whenever } u \in T(x), v \in T(y). \quad (1.3)$$

It is said to be maximal monotone if, in addition, its graph, $\text{gph } T := \{(x, y) \in H \times H : y \in T(x)\}$, is not properly contained in the graph of any other monotone operator. An operator sequence T_k is said to be graph convergent to T if $(\text{gph}(T_k))$ converges to $\text{gph}(T)$ in the Kuratowski-Painlevé's sense, that is, $\limsup_k \text{gph}(T_k) \subset \text{gph}(T) \subset \liminf_k \text{gph}(T_k)$. It is well-known that for each $x \in H$ and $\lambda > 0$ there is a unique $z \in H$ such that $x \in (I + \lambda T)z$. The single-valued operator $J_\lambda^T := (I + \lambda T)^{-1}$ is called the resolvent of T of parameter λ . It is a nonexpansive mapping which is everywhere defined and is related to its Yosida approximate, namely $T_\lambda(x) := (x - J_\lambda^T(x))/\lambda$, by the relation $T_\lambda(x) \in T(J_\lambda^T(x))$. The latter is $1/\lambda$ -Lipschitz continuous and satisfies $(T_\lambda)_\mu = T_{\lambda+\mu}$. Recall that the inverse T^{-1} of T is the operator defined by $x \in T^{-1}(y) \Leftrightarrow y \in T(x)$ and that, for all $x, y \in H$, we have the following key inequality

$$\|J_\lambda^T(x) - J_\lambda^T(y)\|^2 \leq \|x - y\|^2 + \|(I - J_\lambda^T)(x) - (I - J_\lambda^T)(y)\|^2. \quad (1.4)$$

Observe that the relation $(T_\lambda)_\mu(x) = T_{\lambda+\mu}(x)$ leads to

$$J_\mu^{T_\lambda}(x) = \frac{\lambda}{\lambda + \mu}x + \left(1 - \frac{\lambda}{\lambda + \mu}\right)J_{\lambda+\mu}^T(x). \quad (1.5)$$

Now, given a proper lower semicontinuous convex function $f : H \rightarrow \mathbb{R} \cup \{+\infty\}$, the subdifferential of f at x is the set

$$\partial f(x) = \{u \in H : f(y) \geq f(x) + \langle u, y - x \rangle \forall y \in H\}. \quad (1.6)$$

Its Moreau-Yosida approximate and proximal mapping f_λ and $\text{prox}_{\lambda f}$ are given, respectively, by

$$f_\lambda(x) = \inf_{y \in H} \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}, \quad \text{prox}_{\lambda f}(x) = \operatorname{argmin}_{y \in H} \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}. \quad (1.7)$$

We have the following interesting relation $(\partial f)_\lambda = \nabla f_\lambda$. Finally, given a nonempty closed convex set $C \subset H$, its indicator function is defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise. The projection onto C at a point u is $P_C(u) = \inf_{c \in C} \|u - c\|$. The normal cone to C at x is

$$N_C(x) = \{u \in H : \langle u, c - x \rangle \leq 0 \ \forall c \in C\} \quad (1.8)$$

if $x \in C$ and \emptyset otherwise. Observe that $\partial \delta_C = N_C$, $\text{prox}_{\lambda f} = J_\lambda^{\partial f}$, and $J_\lambda^{N_C} = P_C$.

Given some $x_{k-1} \in H$, the current approximation to a solution of (1.2), we study the penalization-gradient iteration which will generate, for parameters $\lambda_k > 0, \beta_k \rightarrow +\infty, x_k$ as the solution of the regularized subproblem

$$\frac{1}{\lambda_k}(x_k - x_{k-1}) + Ax_{k-1} + \beta_k \partial \Psi(x_k) \ni 0, \quad (1.9)$$

which can be rewritten as

$$x_k = (I + \lambda_k \beta_k \partial \Psi)^{-1}(x_{k-1} - \lambda_k Ax_{k-1}). \quad (1.10)$$

Having in view a large range of applications, we shall not assume any particular structure or regularity on the penalization function Ψ . Instead, we just suppose that Ψ is convex, lower semicontinuous and $C = \text{argmin} \Psi \neq \emptyset$. We will denote by $\text{VI}(A, C)$ the solution set of (1.2).

The following lemmas will be needed in our analysis, see for example [6, 7], respectively.

Lemma 1.1. *Let T be a maximal monotone operator, then $(\beta_k T)$ graph converges to $N_{T^{-1}(0)}$ as $\beta_k \rightarrow +\infty$ provided that $T^{-1}(0) \neq \emptyset$.*

Lemma 1.2. *Assume that α_k and δ_k are two sequences of nonnegative real numbers such that*

$$\alpha_{k+1} \leq \alpha_k + \delta_k. \quad (1.11)$$

If $\lim_{k \rightarrow +\infty} \delta_k = 0$, then there exists a subsequence of (α_k) which converges. Furthermore, if $\sum_{k=0}^{\infty} \delta_k < +\infty$, then $\lim_{k \rightarrow +\infty} \alpha_k$ exists.

2. Main Results

2.1. Weak Convergence

Theorem 2.1. *Assume that $\text{VI}(A, C) \neq \emptyset$, A is inverse strongly monotone, namely*

$$\langle Ax - Ay, x - y \rangle \geq \frac{1}{L} \|Ax - Ay\|^2 \quad \forall x, y \in H, \text{ for some } L > 0. \quad (2.1)$$

If

$$\sum_{k=0}^{\infty} \left\| \bar{x} - J_{\lambda_k}^{\beta_k \partial \Psi} (\bar{x} - \lambda_k A \bar{x}) \right\| < +\infty \quad \forall \bar{x} \in \text{VI}(A, C), \quad (2.2)$$

and $\lambda_k \in]\varepsilon, 2/L - \varepsilon[$ (where $\varepsilon > 0$ is a small enough constant), then the sequence $(x_k)_{k \in \mathbb{N}}$ generated by algorithm (1.9) converges weakly to a solution of Problem (1.2).

Proof. Let \bar{x} be a solution of (1.2), observe that \bar{x} solves (1.2) if and only if $\bar{x} = (I + \lambda_k N_C)^{-1} (\bar{x} - \lambda_k A \bar{x}) = P_C(\bar{x} - \lambda_k A \bar{x})$. Set $\bar{x}_k = (I + \lambda_k \beta_k \partial \Psi)^{-1} (\bar{x} - \lambda_k A \bar{x})$, by the triangular inequality, we can write

$$\|x_k - \bar{x}\| \leq \|x_k - \bar{x}_k\| + \|\bar{x}_k - \bar{x}\|. \quad (2.3)$$

On the other hand, by virtue of (1.4) and (2.1), we successively have

$$\begin{aligned} \|x_k - \bar{x}_k\|^2 &\leq \|x_{k-1} - \bar{x} - \lambda_k (Ax_{k-1} - A\bar{x})\|^2 - \|x_{k-1} - x_k - \lambda_k (Ax_{k-1} - A\bar{x}) + \bar{x}_k - \bar{x}\|^2 \\ &\leq \|x_{k-1} - \bar{x}\|^2 - \lambda_k \left(\frac{2}{L} - \lambda_k \right) \|Ax_{k-1} - A\bar{x}\|^2 \\ &\quad - \|x_{k-1} - x_k - \lambda_k (Ax_{k-1} - A\bar{x}) + \bar{x}_k - \bar{x}\|^2. \end{aligned} \quad (2.4)$$

Hence

$$\begin{aligned} \|x_k - \bar{x}\| &< \sqrt{\|x_{k-1} - \bar{x}\|^2 - \varepsilon^2 \|Ax_{k-1} - A\bar{x}\|^2 - \|x_{k-1} - x_k - \lambda_k (Ax_{k-1} - A\bar{x}) + \bar{x}_k - \bar{x}\|^2} \\ &\quad + \|\bar{x} - \bar{x}_k\|. \end{aligned} \quad (2.5)$$

The later implies, by Lemma 1.2 and the fact that (2.2) insures $\lim_{k \rightarrow +\infty} \|\bar{x} - \bar{x}_k\| = 0$, that the positive real sequence $(\|x_k - \bar{x}\|^2)_{k \in \mathbb{N}}$ converges to some limit $l(\bar{x})$, that is,

$$l(\bar{x}) = \lim_{k \rightarrow +\infty} \|x_k - \bar{x}\|^2 < +\infty, \quad (2.6)$$

and also assures that

$$\begin{aligned} \lim_{k \rightarrow +\infty} \|Ax_{k-1} - A\bar{x}\|^2 &= 0, \\ \lim_{k \rightarrow +\infty} \|x_{k-1} - x_k - \lambda_k (Ax_{k-1} - A\bar{x}) + \bar{x}_k - \bar{x}\|^2 &= 0. \end{aligned} \quad (2.7)$$

Combining the two latter equalities, we infer that

$$\lim_{k \rightarrow +\infty} \|x_{k-1} - x_k\|^2 = 0. \quad (2.8)$$

Now, (1.9) can be written equivalently as

$$\frac{x_{k-1} - x_k}{\lambda_k} + Ax_k - Ax_{k-1} \in (A + \beta_k \partial \Psi)(x_k). \quad (2.9)$$

By virtue of Lemma 1.1, we have $(\beta_k \partial \Psi)$ graph converges to $N_{\text{argmin} \Psi}$ because

$$(\partial \Psi)^{-1}(0) = \partial \Psi^*(0) = \text{argmin} \Psi. \quad (2.10)$$

Furthermore, the Lipschitz continuity of A (see, e.g., [8]) clearly ensures that the sequence $(A + \beta_k \partial \Psi)$ graph converges in turn to $A + N_{\text{argmin} \Psi}$.

Now, let x^* be a cluster point of $\{x_k\}$. Passing to the limit in (2.9), on a subsequence still denoted by $\{x_k\}$, and taking into account the fact that the graph of a maximal monotone operator is weakly strongly closed in $H \times H$, we then conclude that

$$0 \in (A + N_C)x^*, \quad (2.11)$$

because A is Lipschitz continuous, (x_k) is asymptotically regular thanks to (2.8), and (λ_k) is bounded away from zero.

It remains to prove that there is no more than one cluster point, our argument is classical and is presented here for completeness.

Let \tilde{x} be another cluster of $\{x_k\}$, we will show that $\tilde{x} = x^*$. This is a consequence of (2.6). Indeed,

$$l(x^*) = \lim_{k \rightarrow +\infty} \|x_k - x^*\|^2, \quad l(\tilde{x}) = \lim_{k \rightarrow +\infty} \|x_k - \tilde{x}\|^2, \quad (2.12)$$

from

$$\|x_k - \tilde{x}\|^2 = \|x_k - x^*\|^2 + \|x^* - \tilde{x}\|^2 + 2\langle x_k - x^*, x^* - \tilde{x} \rangle, \quad (2.13)$$

we see that the limit of $\langle x_k - x^*, x^* - \tilde{x} \rangle$ as $k \rightarrow +\infty$ must exist. This limit has to be zero because x^* is a cluster point of $\{x_k\}$. Hence at the limit, we obtain

$$l(\tilde{x}) = l(x^*) + \|x^* - \tilde{x}\|^2. \quad (2.14)$$

Reversing the role of \tilde{x} and x^* , we also have

$$l(x^*) = l(\tilde{x}) + \|x^* - \tilde{x}\|^2. \quad (2.15)$$

That is $\tilde{x} = x^*$, which completes the proof. \square

Remark 2.2. (i) Note that, we can remove condition (2.2), but in this case we obtain that there exists a subsequence of (x_k) such that every weak cluster point is a solution of problem (1.2). This follows by Lemma 1.2 combined with the fact that $\bar{x} = J_{\lambda^*}^{\partial \delta_C}(\bar{x} - \lambda^* A \bar{x})$ and that

$(\beta_k \partial \Psi)$ graph converges to $\partial \delta_C$. The later is equivalent, see for example [6], to the pointwise convergence of $J_{\lambda_k}^{\beta_k \partial \Psi}$ to $J_{\lambda^*}^{\partial \delta_C}$ and therefore ensures that

$$\lim_{k \rightarrow +\infty} \left\| \bar{x} - J_{\lambda_k}^{\beta_k \partial \Psi}(\bar{x} - \lambda_k A \bar{x}) \right\| = 0. \quad (2.16)$$

(ii) In the special case $\Psi(x) = (1/2) \text{dist}(x, C)^2$, (2.2) reduces to $\sum_{k=0}^{\infty} 1/\beta_k < +\infty$, see Application (2) of Section 3.

Suppose now that $\Psi(x) = \text{dist}(x, C)$, it well-known that $\text{prox}_{\gamma \Psi}(x) = P_C(x)$ if $\text{dist}(x, C) \leq \gamma$. Consequently,

$$J_{\lambda_k}^{\beta_k \partial \Psi}(x) = P_C(x) \quad \text{if } \text{dist}(x, C) \leq \lambda_k \beta_k, \quad (2.17)$$

which is the case for all $k \geq \kappa$ for some $\kappa \in \mathbb{N}$ because (λ_k) is bounded and $\lim_{k \rightarrow +\infty} \beta_k = +\infty$. Hence $\lim_{k \rightarrow +\infty} \left\| \bar{x} - J_{\lambda_k}^{\beta_k \partial \Psi}(\bar{x} - \lambda_k A \bar{x}) \right\| = 0$, for all $k \geq \kappa$, and thus (2.2) is clearly satisfied.

The particular case $\Psi = 0$ corresponds to the unconstrained case, namely, $C = H$. In this context the resolvent associated to $\beta_k \partial \Psi$ is the identity, and condition (2.2) is trivially satisfied.

2.2. Strong Convergence

Now, we would like to stress that we can guarantee strong convergence by reinforcing assumptions on A .

Proposition 2.3. *Assume that A is strong monotone with constant $\alpha > 0$, that is,*

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2 \quad \forall x, y \in H, \text{ for some } \alpha > 0, \quad (2.18)$$

and Lipschitz continuous with constant $L > 0$, that is,

$$\|Ax - Ay\| \leq L \|x - y\| \quad \forall x, y \in H, \text{ for some } L > 0. \quad (2.19)$$

If $\lambda_k \in]\varepsilon, 2\alpha/L^2 - \varepsilon[$ (where $\varepsilon > 0$ is a small enough constant) and $\lim_{k \rightarrow +\infty} \lambda_k = \lambda^* > 0$, then the sequence generated by (1.9) strongly converges to the unique solution of (1.2).

Proof. Indeed, by replacing inverse strong monotonicity of A by strong monotonicity and Lipschitz continuity, it is easy to see from the first part of the proof of Theorem 2.1 that the operator of $I - \lambda_k A$ satisfies

$$\|(I - \lambda_k A)(x) - (I - \lambda_k A)(y)\|^2 \leq \left(1 - 2\lambda_k \alpha + \lambda_k^2 L^2\right) \|x - y\|^2. \quad (2.20)$$

Following the arguments in the proof of Theorem 2.1 to obtain

$$\|x_k - \bar{x}\| \leq \sqrt{1 - 2\lambda_k\alpha + \lambda_k^2L^2}\|x_{k-1} - \bar{x}\| + \delta_k(\bar{x}) \quad \text{with } \delta_k(\bar{x}) := \left\| \bar{x} - J_{\lambda_k}^{\beta_k\partial\Psi}(\bar{x} - \lambda_kA\bar{x}) \right\|. \tag{2.21}$$

Now, by setting $\Theta(\lambda) = \sqrt{1 - 2\lambda\alpha + \lambda^2L^2}$, we can check that $0 < \Theta(\lambda) < 1$ if and only if $\lambda_k \in]0, 2\alpha/L^2[$, and a simple computation shows that $0 < \Theta(\lambda_k) \leq \Theta^* < 1$ with $\Theta^* = \max\{\Theta(\varepsilon), \Theta(2\alpha/L^2 - \varepsilon)\}$. Hence,

$$\|x_k - \bar{x}\| \leq (\Theta^*)^k \|x_0 - \bar{x}\| + \sum_{j=0}^{k-1} (\Theta^*)^j \delta_{k-j}(\bar{x}). \tag{2.22}$$

The result follows from Ortega and Rheinboldt [9, page 338] and the fact that $\lim_{k \rightarrow +\infty} \delta_k(\bar{x}) = 0$. The later follows thanks to the equivalence between graph convergence of the sequence of operators $(\beta_k\partial\Psi)$ to $\partial\delta_C$ and the pointwise convergence of their resolvent operators combined with the fact that $\lim_{k \rightarrow +\infty} \lambda_k = \lambda^*$. □

3. Applications

(1) Hierarchical Convex Minimization Problems

Having in mind the connection between monotone operators and convex functions, we may consider the special case $A = \nabla\Phi$, Φ being a proper lower semicontinuous differentiable convex function. Differentiability of Φ ensures that $\nabla\Phi + N_{\text{argmin}\Psi} = \partial(\Phi + \delta_{\text{argmin}\Psi})$ and (1.2) reads as

$$\min_{x \in \text{argmin}\Psi} \Phi(x). \tag{3.1}$$

Using definition of the Moreau-Yosida approximate, algorithm (1.9) reads as

$$x_k = \underset{y \in H}{\text{argmin}} \left\{ f(y) + \frac{1}{2\lambda_k} \|y - (I - \lambda_kA)x_{k-1}\|^2 \right\}. \tag{3.2}$$

In this case, it is well-known that the assumption (2.1) of inverse strong monotonicity of $\nabla\Phi$ is equivalent to its L -Lipschitz continuity. If further we assume $\sum_{k=1}^{\infty} \delta_k(\bar{x}) < +\infty$ for all $\bar{x} \in \text{VI}(\nabla\Phi, C)$ and $\lambda_k \in]\varepsilon, 2/L - \varepsilon[$, then by Theorem 2.1 we obtain weak convergence

of algorithm (3.2) to a solution of (3.1). The strong convergence is obtained, thanks to Proposition 2.3, if in addition Ψ is strongly convex (i.e., there is $\alpha > 0$;

$$(1 - \mu)\Psi(x_1) + \mu\Psi(x_2) \geq \Psi((1 - \mu)x_1 + \mu x_2) + \frac{\alpha}{2}\mu(1 - \mu)\|x_1 - x_2\|^2 \quad (3.3)$$

for all $\mu \in [0, 1]$, all $x_1, x_2 \in H$) and (λ_k) a convergent sequence with $\lambda_k \in]\varepsilon, 2\alpha/L^2 - \varepsilon[$. Note that strong convexity of Ψ is equivalent to α -strong monotonicity of its gradient. A concrete example in signal recovery is the Projected Land weber problem, namely,

$$\min_{x \in C} \Phi(x) := \frac{1}{2}\|Lx - z\|^2, \quad (3.4)$$

L being a linear-bounded operator. Set $A(x) := \nabla\Phi(x) = L^*(Lx - z)$. Consequently,

$$\forall x, y \in H \quad \|A(x) - A(y)\| = \|L^*L(x - y)\| \leq \|L\|^2\|x - y\|, \quad (3.5)$$

and A is therefore Lipschitz continuous with constant $\|L\|^2$. Now, it is well-known that the problem possesses exactly one solution if L is bounded below, that is,

$$\exists \kappa > 0 \quad \forall x \in H \quad \|L(x)\| \geq \kappa\|x\|. \quad (3.6)$$

In this case, A is strongly monotone. Indeed, it is easily seen that f is strongly convex: consider $x, y \in H$ and $\mu \in]0, 1[$, one has

$$\frac{\|\mu(Lx - z) + (1 - \mu)(Ly - z)\|^2}{2} \leq \frac{\mu\|Lx - z\|^2}{2} + \frac{(1 - \mu)\|Ly - z\|^2}{2} - \frac{\kappa^2\mu(1 - \mu)\|x - y\|^2}{2}. \quad (3.7)$$

(2) Classical Penalization

In the special case where $\Psi(x) = (1/2) \text{dist}(x, C)^2$, we have

$$\partial\Psi(x) = x - \text{Proj}_C(x), \quad (3.8)$$

which is nothing but the classical penalization operator, see [10]. In this context, taking into account the fact that

$$((\partial f)_\lambda)_\mu = \nabla f_{\lambda+\mu}, \quad J_\lambda^{\partial f} = I - \lambda(\partial f)_\lambda = I - \lambda\nabla f_\lambda, \quad (\delta_C)_\lambda = \frac{1}{\lambda}\Psi, \quad (3.9)$$

and that \bar{x} solves (1.2), and thus $\bar{x} = P_C(\bar{x} - \lambda_k A\bar{x})$, we successively have

$$\begin{aligned}
\|\bar{x}_k - \bar{x}\| &= \left\| J_{\lambda_k}^{\beta_k \partial \Psi}(\bar{x} - \lambda_k A\bar{x}) - J_{\lambda_k}^{N_C}(\bar{x} - \lambda_k A\bar{x}) \right\| \\
&= \lambda_k \left\| (\beta_k \partial \Psi)_{\lambda_k}(\bar{x} - \lambda_k A\bar{x}) - (N_C)_{\lambda_k}(\bar{x} - \lambda_k A\bar{x}) \right\| \\
&= \lambda_k \left\| \beta_k (\partial \Psi)_{\lambda_k \beta_k}(\bar{x} - \lambda_k A\bar{x}) - \nabla(\delta_C)_{\lambda_k}(\bar{x} - \lambda_k A\bar{x}) \right\| \\
&= \lambda_k \left\| \beta_k (\partial(\delta_C)_1)_{\lambda_k \beta_k}(\bar{x} - \lambda_k A\bar{x}) - \nabla(\delta_C)_{\lambda_k}(\bar{x} - \lambda_k A\bar{x}) \right\| \\
&= \lambda_k \left\| \beta_k \nabla(\delta_C)_{1+\lambda_k \beta_k}(\bar{x} - \lambda_k A\bar{x}) - \nabla(\delta_C)_{\lambda_k}(\bar{x} - \lambda_k A\bar{x}) \right\| \\
&= \lambda_k \left(\frac{1}{\lambda_k} - \frac{\beta_k}{1 + \lambda_k \beta_k} \right) \|(\bar{x} - \lambda_k A\bar{x}) - P_C(\bar{x} - \lambda_k A\bar{x})\| \\
&= \frac{1}{1 + \lambda_k \beta_k} \|\lambda_k A\bar{x}\| \leq \frac{1}{\beta_k} \|A\bar{x}\|.
\end{aligned} \tag{3.10}$$

So condition on the parameters reduces to $\sum_{k=1}^{\infty} 1/\beta_k < +\infty$, and algorithm (1.9) is nothing but a relaxed projection-gradient method. Indeed, using (1.5) and the fact that $J_{\lambda}^{N_C} = P_C$, we obtain

$$x_k = \left(\frac{1}{1 + \lambda_k \beta_k} I + \frac{\lambda_k \beta_k}{1 + \lambda_k \beta_k} P_C \right) (I - \lambda_k A) x_{k-1}. \tag{3.11}$$

An inspection of the proof of Theorem 2.1 shows that the weak converges is assured with $\lambda_k \in]\varepsilon, 2/L - \varepsilon[$.

(3) A Hierarchical Fixed-Point Problem

Having in mind the connection between inverse strongly monotone operators and nonexpansive mappings, we may consider the following fixed-point problem:

$$(I - P)x + N_C(x) \ni 0, \tag{3.12}$$

with P a nonexpansive mapping, namely, $\|Px - Py\| \leq \|x - y\|$.

It is well-known that $A = I - P$ is inverse strongly monotone with $L = 2$. Indeed, by definition of P , we have

$$\|(I - A)x - (I - A)y\| \leq \|x - y\|. \tag{3.13}$$

On the other hand

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay \rangle. \tag{3.14}$$

Combining the two last inequalities, we obtain

$$\langle x - y, Ax - Ay \rangle \geq \frac{1}{2} \|Ax - Ay\|^2. \quad (3.15)$$

Therefore, by Theorem 2.1 we get the weak convergence of the sequence (x_k) generated by the following algorithm:

$$x_k = \text{prox}_{\beta_k \Psi}((I - \lambda_k)x_{k-1} + \lambda_k P x_{k-1}) \quad (3.16)$$

to a solution of (3.12) provided that $\sum_{k=1}^{\infty} \delta_k(\bar{x}) < +\infty$ for all $\bar{x} \in \text{VI}(I - P, C)$ and $\lambda_k \in]\varepsilon, 1 - \varepsilon[$. The strong convergence of (1.9) is obtained, by applying Proposition 2.3, for P a contraction mapping, namely, $\|Px - Py\| \leq \gamma \|x - y\|$ for $0 < \gamma < 1$ which is equivalent to the $(1 - \gamma)$ -strong monotonicity of $(I - P)$, and (λ_k) is a convergent sequence with $\lambda_k \in]\varepsilon, 2(1 - \gamma)/(1 + \gamma)^2 - \varepsilon[$. It is easily seen that in this case $I - P$ is $(1 + \gamma)$ -Lipschitz continuous.

4. Towards the Multivalued Case

Now, we are interested in (1.2) when $A : H \rightarrow 2^H$ is a multi-valued maximal monotone operator. With the help of the Yosida approximate which is always inverse strongly monotone (and thus single-valued), we consider the following partial regularized version of (1.2):

$$A_\gamma x_\gamma^* + N_C(x_\gamma^*) \ni 0, \quad (4.1)$$

where A_γ stands for the Yosida approximate of A .

It is well-known that A_γ is inverse strongly monotone. More precisely, we have

$$\langle A_\gamma x - A_\gamma y, x - y \rangle \geq \gamma \|A_\gamma x - A_\gamma y\|^2. \quad (4.2)$$

Using definition of the Yosida approximate, algorithm (1.9) applied to (4.1) reads as

$$x_k^\gamma = (I + \lambda_k \beta_k \partial \Psi)^{-1} \left(\left(1 - \frac{\lambda_k}{\gamma} \right) x_{k-1}^\gamma + \frac{\lambda_k}{\gamma} J_\gamma^A(x_{k-1}^\gamma) \right). \quad (4.3)$$

From Theorem 2.1, we infer that x_k^γ converges weakly to a solution \bar{x}^γ provided that $\lambda_k \in]\varepsilon, 2\gamma - \varepsilon[$. Furthermore, it is worth mentioning that if A is strongly monotone, A_γ is also strongly monotone, and thus (4.1) has a unique solution \bar{x}^γ . By a result in [8, page 35], we have the following estimate:

$$\|\bar{x} - \bar{x}^\gamma\| \leq o(\sqrt{\gamma}). \quad (4.4)$$

Consequently, (4.3) provides approximate solutions to the variational inclusion (1.2) for small values of γ . Furthermore, when $A = \nabla \Phi$, we have

$$(\partial \Phi)_\gamma(\bar{x}) + N_C(\bar{x}) = \nabla \Phi_\gamma(\bar{x}) + N_C(\bar{x}) = \partial(\Phi_\gamma + \delta_C)(\bar{x}), \quad (4.5)$$

and thus (4.1) reduces to

$$\min_{x \in C} \Phi_\gamma(x). \quad (4.6)$$

If (3.1) and (4.1) are solvable, by ([11] Theorem 3.3), we have for all $\gamma > 0$

$$0 \leq \min_{x \in C} \Phi(x) - \min_{x \in C} \Phi_\gamma(x) \leq \gamma \|\bar{y}\|^2, \quad (4.7)$$

where $\bar{y} = \nabla \Phi(\bar{y}) (\in -N_C(\bar{x}))$ with \bar{x} a solution of (3.1). The value of (3.1) is thus close to those of (4.1) for small values of γ , and hence, this confirmed the pertinence of the proposed approximation idea to reach the multi-valued case. Observe that in this context, algorithm (4.3) reads as

$$x_k^\gamma = \text{prox}_{\beta_k \Psi} \left(\left(1 - \frac{\lambda_k}{\gamma} \right) x_{k-1}^\gamma + \frac{\lambda_k}{\gamma} \text{prox}_{\gamma \Phi} \left(x_{k-1}^\gamma \right) \right). \quad (4.8)$$

5. Conclusion

The authors have introduced a forward-backward penalization-gradient algorithm for solving variational inequalities and studied their asymptotic convergence properties. We have provided some applications to hierarchical fixed-point and optimization problems and also proposed an idea to reach monotone variational inclusions.

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