Ergodic Convergence to a Zero of the Extended Sum

Abdellatif Moudafi, Michel Théra

Laboratoire d'Arithmétique, de Calcul formel et d'Optimisation
ESA - CNRS 6090

Ergodic Convergence to a Zero of the Extended Sum

Abdellatif Moudafi and Michel Théra

Rapport de recherche n° 2000-15

Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex
Tél. 05 55 45 73 23 - Fax. 05 55 45 73 22 - laco@unilim.fr

http://www.unilim.fr/laco/

ERGODIC CONVERGENCE TO A ZERO OF THE EXTENDED SUM

ABDELLATIF MOUDAFI AND MICHEL THÉRA

Abstract. In this note we show that the splitting scheme of Passty [7] as well as the barycentric-proximal method of Lehdili & Lemaire [4] can be used to approximate a zero of the extended sum of maximal monotone operators. When the extended sum is maximal monotone, we extend the convergence result obtained by Lehdili & Lemaire for convex functions to the case of maximal monotone operators. Moreover, we recover the main convergence results by Passty and Lehdili & Lemaire when the pointwise sum of the involved operators is maximal monotone.

1. Introduction and preliminaries

A wide range of problems in physics, economics and operation research can be formulated as a generalized equation $0 \in T(x)$ for a given set-valued mapping T on a Hilbert space X. Therefore, the problem of finding a zero of T, i.e., a point $\bar{x} \in X$ such that $0 \in T(x)$ is a fundamental problem in many areas of applied mathematics.

When T is a maximal monotone operator, a classical method for solving the problem $0 \in T(x)$ is the *Proximal Point Algorithm*, proposed by Rockafellar [12] which extends an earlier algorithm established by Martinet [5] for $T = \partial f$, i.e., when T is the subdifferential of a convex lower semicontinuous proper function. In this case, finding a zero of T is equivalent to the problem of finding a minimizer of f.

The case where T is the pointwise sum of two operators A and B is called a *splitting* of T. It is of fundamental interest in large-scale optimization since the objective function splits into the sum of two simpler functions and we can take advantage of this separable structure. For an overview of splitting methods of all kinds we refer to Eckstein [3]. Using the conjugate duality, splitting methods may apply in certain circumstances to the dual objective function.

Recall that the general framework of the conjugate duality is the following [11]: Consider f a convex lower semicontinuous function on the product $H \times U$ of let us say two Hilbert spaces H and U. Define

$$L(x, v) := \inf_u \{f(x, u) - \langle u, v\rangle\}$$

and

$$g(p, v) := \inf_x \{L(x, v) - \langle x, p\rangle\}.$$

1991 *Mathematics Subject Classification.* 49J52, 90C30.

Key words and phrases. maximal monotone operators, Yosida regularization, enlargements, variational sum, extended sum, splitting methods.

Setting $f_0(x) := f(x,0)$ and $g_0(v) := g(0,v)$, a well-known method to solve $\inf f_0$ is the *method of multipliers* which consists in solving the dual problem $\max g_0$ using the *Proximal Point Algorithm* [13].

It has been observed that in certain situations the study of a problem, with monotone operators involved, leads to an operator that turns out to be larger than the pointwise sum. Consequently, there have been several attempts for generalizing the usual pointwise sum of two monotone operators such as for instance the well-known extension was based on the Trotter-Lie formula. More recently, in 1994, the notion of *variational sum* of two maximal monotone operators was introduced in [1] by Attouch, Baillon and Théra using the Yosida regularization of operators. Recently, another notion of extended sum was proposed by Revalski and Théra [8] relying on the so-called enlargements of operators.

Our focus in this paper is on finding a zero of the extended sum of two monotone operators when this extended sum is a maximal monotone operator. Since, when the pointwise sum is maximal monotone, the extended sum and the pointwise sum coincides, the proposed algorithm will subsume the classical *Passty scheme* and the *barycentric-proximal method* of Lehdili and Lemaire which are related with weak ergodic type convergence.

Throughout we will assume that X is a real Hilbert space. The inner product and the associated norm will be designated respectively by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Given a (multivalued) operator $A : X \rightrightarrows X$, as usual the *graph* of A is denoted by $\mathrm{Gr}(A) := \{(x, x^*) \in X \times X : x^* \in Ax\}$, its *domain* by $\mathrm{Dom}\,(A) := \{x \in X : \ Ax \neq \emptyset\}$ and its inverse operator is $A^{-1} : X \rightrightarrows X$, $A^{-1}x^* := \{x \in X : x^* \in Ax\}$, $x^* \in X$. The operator A is called *monotone* if $\langle y - x, y^* - x^* \rangle \geq 0$, whenever $(x, x^*) \in \mathrm{Gr}\, A$ and $(y, y^*) \in \mathrm{Gr}\, A$. We denote by $\bar{A}$ the operator $\bar{A}x := \overline{Ax}$, $x \in X$, where the overbar means the norm-closure of a given set. The monotone operator A is said to be *maximal* if its graph is not contained properly in the graph of any other monotone operator from X to X. The graph $\mathrm{Gr}\,(A)$ is a closed subset with respect to the product of the norm topologies in $X \times X$.

Finally, given a maximal monotone operator $A : X \rightrightarrows X$ and a positive λ, recall that the *Yosida regularization* of A of order λ is the operator $A_\lambda := (A^{-1} + \lambda I)^{-1}$, and that the *resolvent* of A of order λ is the operator $J_\lambda^A := (I + \lambda A)^{-1}$, where I is the identity mapping. For any $\lambda > 0$, the Yosida regularization A_λ and the resolvent J_λ^A are everywhere defined single-valued maximal monotone operators.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued lower semicontinuous convex function in X which is *proper* (i.e. the *domain* $\mathrm{dom}\, f := \{x \in X : f(x) < +\infty\}$ of f is non-empty). Given $\varepsilon \geq 0$, the well-known ε*-subdifferential* of f is defined at $x \in \mathrm{dom}\, f$ by:

$$\partial_\varepsilon f(x) := \Big\{ x^* \in X^* : f(y) - f(x) \geq \langle y - x, x^* \rangle - \varepsilon \quad \text{for every } y \in X \Big\},$$

and $\partial_\varepsilon f(x) := \emptyset$, if $x \notin \mathrm{dom}\, f$. When $\varepsilon = 0$, $\partial_0 f$ is the *subdifferential* ∂f of f, which, as it is well-known, is a maximal monotone operator.

The concept of approximate subdifferential leads to similar enlargements for monotone operators. The next one has been investigated intensively in the last years: for the monotone operator $A : X \rightrightarrows X$ and $\varepsilon \geq 0$, the ε*-enlargement* of A is $A^\varepsilon : X \rightrightarrows X$, defined by $A^\varepsilon x := \{x^* \in X : \langle y - x, y^* - x^* \rangle \geq -\varepsilon$ for any $(y, y^*) \in \mathrm{Gr}\,(A)\}$. A^ε has closed convex images for any $\varepsilon \geq 0$ and due to the monotonicity of A, one has $Ax \subset A^\varepsilon x$ for every $x \in X$ and every $\varepsilon \geq 0$. In the case $A = \partial f$ one has $\partial_\varepsilon f \subset (\partial f)^\varepsilon$ and the inclusion can be strict.

2. Generalized Sums and Splitting Methods

We start by recalling different types of sums of monotone operators and we present two splitting methods for finding a zero of the extended sum.

Let $A, B : X \rightrightarrows X$ be two monotone operators. As usual $A + B : X \rightrightarrows X$ denotes the pointwise sum of A and B: $(A + B)x = Ax + Bx$, $x \in X$. $A + B$ is a monotone operator with $\mathrm{Dom}\,(A + B) = \mathrm{Dom}\,A \cap \mathrm{Dom}\,B$. Even if A and B are maximal monotone operators, their sum $A + B$ may fail to be maximal monotone.

The above lack of maximality of the pointwise sum inspired the study of possible generalized sums of monotone operators. Recently, the *variational sum* was proposed in [1] using the Yosida approximation. More precisely, let $A, B : X \rightrightarrows X$ be maximal monotone operators and $\mathcal{I} := \{(\lambda, \mu) \in \mathrm{I\!R}^2 : \lambda, \mu \geq 0, \quad \lambda + \mu \neq 0\}$. The idea of the variational sum, $A \underset{v}{+} B$, is to take as a sum of A and B the graph-convergence limit (i.e. the Painlevé-Kuratowski limit of the graphs) of $A_\lambda + B_\mu$, $(\lambda, \mu) \in \mathcal{I}$, when $(\lambda, \mu) \to 0$. Namely, $A \underset{v}{+} B$ is equal to

$$\liminf_{\mathcal{F}}(A_\lambda + B_\mu) =$$
$$\Big\{(x, x^*) : \forall \{(\lambda_n, \mu_n)\} \subset \mathcal{I}, \lambda_n, \mu_n \to 0, \exists (x_n, x_n^*) \in A_\lambda + B_\mu ; (x_n, x_n^*) \to (x, x^*)\Big\}.$$

By contrast to the pointwise sum, this definition takes into account the behaviour of the operators also at nearby points of the initial one. We have that $\mathrm{Dom}\,(A) \cap \mathrm{Dom}\,(B) \subset \mathrm{Dom}\,(A \underset{v}{+} B)$ and $A \underset{v}{+} B$ is a monotone operator. It was shown in [1] that if $A + B$ is a maximal monotone operator then, $A + B = A \underset{v}{+} B$. Moreover, the subdifferential of the sum of two proper convex lower semicontinuous functions is equal to the variational sum of their subdifferentials.

Another notion of generalized sum was proposed by Revalski and Théra [8] relying on the enlargements: the *extended sum* of two monotone operators $A, B : X \rightrightarrows X$ is defined in [8] for each $x \in X$ by

$$A \underset{e}{+} B(x) = \bigcap_{\varepsilon > 0} \overline{A^\varepsilon x + B^\varepsilon x},$$

where the closure on the right hand side is taken with respect to the weak topology. Evidently, $A + B \subset A \underset{e}{+} B$ and hence, $\mathrm{Dom}\,(A) \cap \mathrm{Dom}\,(B) \subset \mathrm{Dom}\,(A \underset{e}{+} B)$. As it was shown in [8] (Corollary 3.2), if $A + B$ is a maximal monotone operator then, $A + B = A \underset{e}{+} B$. Furthermore, the subdifferential of the sum of two convex proper lower semicontinuous functions is equal to the extended sum of their subdifferentials ([8], Theorem 3.3.)

Let us now recall two splitting methods for the problem of finding a zero of the sum of two maximal monotone operators with maximal monotone sum. The first one have been proposed by Passty relying on a regularization of one of the operator. Actually, replacing the problem

$$(\mathcal{P}) \quad \text{find} \quad x \in X \quad \text{such that} \quad 0 \in (A + B)x$$

by

$$(\mathcal{P}_\lambda) \quad \text{find} \quad x \in X \quad \text{such that} \quad 0 \in (A + B_\lambda)x$$

leads to the following equivalent fixed-point formulation

$$\text{find} \quad x \in X \quad \text{such that} \quad x = J_\lambda^A \circ J_\lambda^B x. \tag{1}$$

Indeed,

$$\begin{aligned} & 0 \in (A + B_\lambda)x \\ & \qquad \iff 0 \in Ax + \frac{x - J_\lambda^B x}{\lambda} \\ & \qquad \iff J_\lambda^B x \in (I + \lambda A)x \\ & \qquad \iff x = J_\lambda^A \circ J_\lambda^B x. \end{aligned}$$

Iterating the above relation with variable λ_n tending to zero gives the scheme of Passty:

$$x_n = J_{\lambda_n}^A \circ J_{\lambda_n}^B x_{n-1} \quad \forall n \in \mathbb{N}^*. \tag{2}$$

Another approach called the *barycentric-proximal* method, based on a complete regularization of the two operators under consideration was proposed by Lehdili and Lemaire [4]. It consists in replacing problem $(\mathcal{P})$ by problem $(\mathcal{P}_{\lambda,\mu})$:

$$(\mathcal{P}_{\lambda,\mu}) \quad \text{find} \quad x \in X \quad \textit{such that} \quad 0 \in (A_\lambda + B_\mu)x.$$

Problem $(\mathcal{P}_{\lambda,\mu})$ is equivalent to the fixed-point problem

$$\text{find} \quad x \in X \quad \text{such that} \quad x = \frac{\mu}{\lambda + \mu} J_\lambda^A x + \frac{\lambda}{\lambda + \mu} J_\mu^B x. \tag{3}$$

Indeed,

$$\begin{aligned} & 0 = (A_\lambda + B_\mu)x \\ & \qquad \iff 0 = \frac{x - J_\lambda^A x}{\lambda} + \frac{x - J_\mu^B x}{\mu} \\ & \qquad \iff (\frac{1}{\lambda} + \frac{1}{\mu})x = \frac{1}{\lambda} J_\lambda^A x + \frac{1}{\mu} J_\mu^B x \\ & \qquad \iff x = \frac{\mu}{\lambda + \mu} J_\lambda^A x + \frac{\mu}{\lambda + \mu} J_\mu^B x. \end{aligned}$$

The barycentric-proximal method is nothing but the iteration method for (3) with variable parameters. More precisely, the iteration is given by:

$$\tilde{x}_n = \frac{\mu_n}{\lambda_n + \mu_n} J_{\lambda_n}^A \tilde{x}_{n-1} + \frac{\lambda_n}{\lambda_n + \mu_n} J_{\mu_n}^B \tilde{x}_{n-1} \quad \forall n \in \mathbb{N}^*. \tag{4}$$

For sake of simplicity we suppose that $\lambda_n = \mu_n$ for all $n \in \mathbb{N}^*$.

3. The Main Result

In what follows, we show that these methods allow to approximate a solution of the following problem

$$(\mathcal{Q}) \quad \text{find} \quad x \in X \quad \textit{such that} \quad 0 \in (A \underset{e}{+} B)x.$$

In the case where $A+B$ is maximal monotone we recover the results by Lehdili and Lemaire for (4) and Passty for (2). Moreover, in the case of convex minimization our result generalizes a theorem of Lehdili and Lemaire. Indeed in this setting, the extended and the variational sums coincide.
To prove our main result we need the following variant of Opial's lemma [6]:

Lemma 1. *Let $\{\lambda_n\}$ be a sequence of positive reals such that $\sum_{n=0}^{+\infty} \lambda_n = +\infty$ and $\{x_n\}$ be a sequence with weighted average $\{z_n\}$ given by $z_n := \frac{\sum_{k=1}^{n} \lambda_k x_k}{\sum_{k=1}^{n} \lambda_k}$. Let us assume that there exists a nonempty closed convex subset S of X such that*

- *any weak limit of a subsequence of $\{z_n\}$ is in S;*
- *$\lim_{n\to+\infty} \|x_n - u\|$ exists for all $u \in S$.*

Then $\{z_n\}$ weakly converges to an element of S.

Theorem 2. *Let us assume that $\operatorname{Dom} A \cap \operatorname{Dom} B \neq \emptyset$ and that $A \underset{e}{+} B$ is a maximal monotone operator. Further we suppose that problem $(\mathcal{Q})$ has a solution. Let $\{x_n\}$ (resp. $\{\tilde{x}_n\}$) be a sequence generated by (2) (resp. by (4)) and $\{z_n\}$ (resp. $\{\tilde{z}_n\}$) be the corresponding weighted average. Let us assume that*

$$\sum_{n=1}^{+\infty} \lambda_n^2 < +\infty \text{ and } \sum_{n=1}^{+\infty} \lambda_n = +\infty.$$

Then any weak limit point of a subsequence of $\{z_n\}$ (resp. $\{\tilde{z}_n\}$) is a zero of the extended sum. Moreover, if A^ϵ is locally bounded [1] on S, then the whole sequence weakly converges to some zero of the extended sum.

Proof: Let us show that all the assumptions of Lemma 1 are satisfied for $S = (A \underset{e}{+} B)^{-1}(0)$. First let us remark that thanks to the maximal monotonicity of $A \underset{e}{+} B$, the set S is closed and convex and nonempty (by assumption).
Take $(x,y) \in A \underset{e}{+} B$. By definition of the extended sum, it amounts to saying that for each $\epsilon > 0$, $y \in \overline{A^\epsilon(x) + B^\epsilon(x)}$. Equivalently, for all $\epsilon > 0$, there exists a sequence $\{y_{p,\epsilon}\}$ weakly converging to y, with

$$y_{p,\epsilon} = y_{1,p,\epsilon} + y_{2,p,\epsilon}, \; y_{1,p,\epsilon} \in A^\epsilon(x) \text{ and } y_{2,p,\epsilon} \in B^\epsilon(x).$$

Define inductively x_n and v_n by $x_n = J_{\lambda_n}^A v_n$ and $v_n = J_{\lambda_n}^B x_{n-1}$. Equivalently we have,

$$\frac{v_n - x_n}{\lambda_n} \in A(x_n) \quad \text{and} \quad \frac{x_{n-1} - v_n}{\lambda_n} \in B(v_n). \tag{5}$$

Definition of ϵ-enlargments combined to relations (5) yields

$$\left\langle \frac{v_n - x_n}{\lambda_n} - y_{1,p,\epsilon}, x_n - x \right\rangle \geq -\epsilon \quad \text{and} \quad \left\langle \frac{x_{n-1} - v_n}{\lambda_n} - y_{2,p,\epsilon}, v_n - x \right\rangle \geq -\epsilon.$$

[1] Recall that $A : X \rightrightarrows X$ is *locally bounded* if for each point x from the norm-closure of $\operatorname{Dom}(A)$ there is a neighborhood U of x such that $A(U)$ is a norm-bounded subset in X

In other words

$$\langle v_n - x_n, x_n - x\rangle \geq \lambda_n \langle y_{1,p,\epsilon}, x_n - x\rangle - \lambda_n \epsilon \tag{6}$$

and

$$\langle x_{n-1} - v_n, v_n - x\rangle \geq \lambda_n \langle y_{2,p,\epsilon}, v_n - x\rangle - \lambda_n \epsilon. \tag{7}$$

From (6) and (7), using the general equality

$$2 < a - b, b - c >= \|a - c\|^2 - \|a - b\|^2 - \|b - c\|^2$$

we obtain

$$\|x_{n-1} - x\|^2 - \|v_n - x\|^2 \geq \|x_{n-1} - v_n\|^2 + 2\lambda_n \langle y_{2,p,\epsilon}, v_n - x\rangle - 2\lambda_n \epsilon$$

and

$$\|v_n - x\|^2 - \|x_n - x\|^2 \geq \|v_n - x_n\|^2 + 2\lambda_n \langle y_{1,p,\epsilon}, x_n - x\rangle - 2\lambda_n \epsilon.$$

By adding the two last inequalities, we infer

$$\begin{aligned}
\|x_{n-1} - x\|^2 - \|x_n - x\|^2 \quad & \geq \|x_{n-1} - v_n\|^2 + \|v_n - x_n\|^2 + 2\lambda_n \langle y_{2,p,\epsilon}, v_n - x\rangle \\
& +2\lambda_n \langle y_{1,p,\epsilon}, x_n - x\rangle - 4\lambda_n \epsilon \\
& \geq \|v_n - x_n\|^2 + 2\lambda_n \langle y_{2,p,\epsilon}, v_n - x\rangle \\
& +2\lambda_n \langle y_{1,p,\epsilon}, x_n - x\rangle - 4\lambda_n \epsilon \\
& = \|v_n - x_n\|^2 + 2\lambda_n \langle y_{2,p,\epsilon}, v_n - x\rangle - 2\lambda_n \langle y_{2,p,\epsilon}, x_n - x\rangle \\
& +2\lambda_n \langle y_{p,\epsilon}, x_n - x\rangle - 4\lambda_n \epsilon \\
& = \|v_n - x_n\|^2 + 2\lambda_n \langle y_{2,p,\epsilon}, v_n - x_n\rangle \\
& +2\lambda_n \langle y_{p,\epsilon}, x_n - x\rangle - 4\lambda_n \epsilon.
\end{aligned}$$

Therefore,

$$\|x_{n-1} - x\|^2 - \|x_n - x\|^2 \ \geq -\lambda_n^2 \|y_{2,p,\epsilon}\|^2 + 2\lambda_n \langle y_{p,\epsilon}, x_n - x\rangle - 4\lambda_n \epsilon. \tag{8}$$

After summation and division by $\sum_{k=1}^n \lambda_k$, we get

$$2\langle y_{p,\epsilon}, z_n - x\rangle \leq \frac{\|x_0 - x\|^2}{\sum_{k=1}^n \lambda_k} + \|y_{2,p,\epsilon}\|^2 \frac{\sum_{k=1}^n \lambda_k^2}{\sum_{k=1}^n \lambda_k} + 4\epsilon. \tag{9}$$

Passing to the limit as $n \to +\infty$ for a subsequence, gives

$$\langle y_{p,\epsilon}, \bar{z} - x\rangle \leq 2\epsilon.$$

Then passing to the limit as $p \to +\infty$ and as ϵ goes to zero yields

$$\langle y, x - \bar{z}\rangle \geq 0.$$

Thus, thanks to the maximality of $A \underset{e}{+} B$, we derive that $0 \in (A \underset{e}{+} B)\bar{z}$, that is every weak limit point of a subsequence of $\{z_n\}$ is a solution of (Q). Therefore the first assumption of Lemma 1 is satisfied.
Now taking $x \in (A \underset{e}{+} B)^{-1}(0)$ in (8), we obtain

$$\|x_n - x\|^2 \leq \|x_{n-1} - x\|^2 + \lambda_n^2 \|y_{2,p,\epsilon}\|^2 - 2\lambda_n \langle y_{p,\epsilon}, x_n - x\rangle + 4\lambda_n \epsilon \quad \forall \epsilon > 0. \tag{10}$$

Local boundedness of A^ϵ and relation $y_{2,p,\epsilon} \in A^\epsilon(x)$ show that $\{y_{2,p,\epsilon}\}$ is bounded when $p \to +\infty$ and when $\epsilon \to 0$. This combined to the fact that $\{y_{p,\epsilon}\} \to 0$ as $p \to +\infty$ gives

$$\|x_n - x\|^2 \leq \|x_{n-1} - x\|^2 + C\lambda_n^2. \tag{11}$$

So, as $\sum_{n=1}^{+\infty} \lambda_n^2 < +\infty$, it is well-known that $\lim_{n\to+\infty} \|x_n - x\|^2$ exists, that is, the second assumption of Lemma 1 is also verified.
Let us now establish that $\{\tilde{z}_n\}$ weakly converges to some $\tilde{z} \in (A \underset{e}{+} B)^{-1}(0)$.

Set $u_n = J^A_{\lambda_n}\tilde{x}_{n-1}, v_n = J^B_{\lambda_n}\tilde{x}_{n-1}$ and take again $(x, y) \in A \underset{e}{+} B$. By definition of the extended sum, for all $\epsilon > 0$, there exists a sequence $\{y_{p,\epsilon}\}$ weakly converging to y, with $y_{p,\epsilon} = y_{1,p,\epsilon} + y_{2,p,\epsilon},\ y_{1,p,\epsilon} \in A^\epsilon(x)$ and $y_{2,p,\epsilon} \in B^\epsilon(x)$. Definition of ϵ-enlargements combined to relation (12)

$$\frac{\tilde{x}_{n-1} - u_n}{\lambda_n} \in A(u_n) \quad \text{and} \quad \frac{\tilde{x}_{n-1} - v_n}{\lambda_n} \in B(v_n) \tag{12}$$

yields

$$\left\langle \frac{\tilde{x}_{n-1} - u_n}{\lambda_n} - y_{1,p,\epsilon}, u_n - x \right\rangle \geq -\epsilon \quad \text{and} \quad \left\langle \frac{\tilde{x}_{n-1} - v_n}{\lambda_n} - y_{2,p,\epsilon}, v_n - x \right\rangle \geq -\epsilon.$$

Equivalently,

$$\langle \tilde{x}_{n-1} - u_n, u_n - x\rangle \geq \lambda_n\langle y_{1,p,\epsilon}, u_n - x\rangle - \lambda_n\epsilon \tag{13}$$

and

$$\langle \tilde{x}_{n-1} - v_n, v_n - x\rangle \geq \lambda_n\langle y_{2,p,\epsilon}, v_n - x\rangle - \lambda_n\epsilon. \tag{14}$$

From (13) and (14), we obtain:

$$\|\tilde{x}_{n-1} - x\|^2 - \|u_n - x\|^2 \geq \|\tilde{x}_{n-1} - u_n\|^2 + 2\lambda_n\langle y_{1,p,\epsilon}, u_n - x\rangle - 2\lambda_n\epsilon$$

and

$$\|\tilde{x}_{n-1} - x\|^2 - \|v_n - x\|^2 \geq \|\tilde{x}_{n-1} - v_n\|^2 + 2\lambda_n\langle y_{2,p,\epsilon}, v_n - x\rangle - 2\lambda_n\epsilon$$

Therefore,

$$\|\tilde{x}_{n-1} - x\|^2 - \|u_n - x\|^2 \geq -\lambda_n^2\|y_{1,p,\epsilon}\|^2 + 2\lambda_n\langle y_{1,p,\epsilon}, \tilde{x}_{n-1} - x\rangle - 2\lambda_n\epsilon \tag{15}$$

and

$$\|\tilde{x}_{n-1} - x\|^2 - \|v_n - x\|^2 \geq -\lambda_n^2\|y_{2,p,\epsilon}\|^2 + 2\lambda_n\langle y_{2,p,\epsilon}, \tilde{x}_{n-1} - x\rangle - 2\lambda_n\epsilon \tag{16}$$

As $\tilde{x}_n = \frac{1}{2}(u_n + v_n)$, using convexity of the norm, we obtain

$$-\|\tilde{x}_n - x\|^2 \geq -\frac{1}{2}\|u_n - x\|^2 - \frac{1}{2}\|v_n - x\|^2. \tag{17}$$

Mutiplying (15) and (16) by $\frac{1}{2}$, summing up and adding (17) gives:

$$\|\tilde{x}_{n-1} - x\|^2 - \|\tilde{x}_n - x\|^2 \geq -\frac{1}{2}\lambda_n^2(\|y_{1,p,\epsilon}\|^2 + \|y_{2,p,\epsilon}\|^2) + 2\lambda_n\langle y_p, \tilde{x}_{n-1} - x\rangle - 2\lambda_n\epsilon.$$

Summing the last inequality from $k = 1$ to n and dividing by $\sum_{k=1}^n \lambda_k$, gives

$$2\langle y_{p,\epsilon}, \tilde{z}_n - x\rangle \leq \frac{\|x_0 - x\|^2}{\sum_{k=1}^n \lambda_k} + \frac{1}{2}(\|y_{1,p,\epsilon}\|^2 + \|y_{2,p,\epsilon}\|^2)\frac{\sum_{k=1}^n \lambda_k^2}{\sum_{k=1}^n \lambda_k} + 2\epsilon. \tag{18}$$

We finish the proof by proceeding similarly to the first part.

Corollary 3.
- *If $A + B$ is maximal monotone, we recover results by Passty (resp. Lehdili and Lemaire), that is $\{z_n\}$ (resp. $\{\tilde{z}_n\}$) converges weakly to some $\bar{z} \in (A + B)^{-1}(0)$.*
- *If $A \underset{v}{+} B$ and $A \underset{e}{+} B$ are both maximal monotone, we generalize the convergence result of Lehdili and Lemaire to maximal monotone operators, that is, $\{\tilde{z}_n\}$ converges weakly to some $\bar{z} \in (A \underset{v}{+} B)^{-1}(0)$. Indeed, in this case, our proof works without assuming that A^ϵ is locally bounded.*
- *In the case $A = \partial f$, $B = \partial g$, $\{z_n\}$ (resp. $\{\tilde{z}_n\}$) converges weakly to a solution of*

$$(OP) \quad \textit{find} \quad x \in X \quad \textit{such that} \quad 0 \in \partial(f + g)(x)$$

or equivalently

$$\textit{find} \quad x \in X \quad \textit{such that} \quad 0 \in argmin_{x \in X}(f + g)(x),$$

and we recover the convergence result of Lehdili and Lemaire for (4).

References

[1] H. Attouch, J.-B. Baillon and M. Théra, "Variational sum of monotone operators," J. Convex Anal., **1**, 1994, p.1–29.
[2] R.S. Burachik, A.N. Iusem and B. F. Svaiter, " Enlargements of maximal monotone operators with applications to variational inequalities", Set-Valued analysis, **5**, 1997, p. 159–180.
[3] J. Eckstein, " Splitting Methods for Monotone Operators with Applications to Parallel Optimization", Ph.D. Thesis, CICS-TH-140, MIT, 1989
[4] N. Lehdili, B. Lemaire, " The b arycentric proximal method ", Communications on Applied Nonlinear Analysis **6**, N0 2, 1999, p. 29–47.
[5] B. Martinet, *Algorithmes pour la résolution de problèmes d'optimisation et minimax*, Thèse d'Etat, Université de Grenoble, 1972.
[6] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mapping," Bull. Amer. Math. Soc., **7**, 1967, p.591–597.
[7] G. B. Passty, "Ergodic convergence to a zero of the sum of maximal monotone operators," J. Math. Anal. Appl.,**7**, 1978, p.591–597.
[8] J.-P. Revalski and M. Théra, "Enlargements and sums of monotone operators," Preprint 1999, to appear in Nonlinear Analysis, Theory, Methods and Applications.
[9] J.-P. Revalski and M. Théra, "Variational and extended sums of monotone operators," in : Ill-posed Variational Problems and Regularization Techniques, M. Théra and R. Tichatschke (eds.), *Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag, **477**, p. 229–246, 1999.
[10] J.-P. Revalski and M. Théra, "Generalized Sums of Monotone Operators," Comptes Rendus à l'Académie des Sciences, t. **329**, Série I, 4, 1999, p. 979–98.
[11] R. T. Rockafellar, *Conjugate duality and optimization*, SIAM, 1974.
[12] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," SIAM J. Control and Optimization, **14**, N5, 1975, p. 877–898.
[13] R. T. Rockafellar, "Augmented Lagrangians and applications of the proximal point algorithm in convex programming," Math. Oper. Res. **1**, 1976, p.97–116.

Univeristé des Antilles et de la Guyane, Département de Mathématiques EA 2431, Pointe-à-Pitre, Guadeloupe. Département Scientifique Interfacultaires, Martinique, France
E-mail address: abdellatif.moudafi@martinique.univ-ag.fr

LACO, UPRESSA 6090, Université de Limoges, 123, avenue Albert Thomas, 87060 Limoges Cedex France
E-mail address: michel.thera@unilim.fr