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Jean-André Marti
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Abstract

I propose a self contained research paper. I hope it adds some news ideas and results to the fixed point theory in the framework of generalized functions algebras, with application to the Cauchy-Lipschitz problem in a generalized formulation including strongly irregular cases. This leads to the transport equation with distributions as coefficients we wish to treat later.

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1 Introduction

The paper is structured in three following sections (2, 3, 4).

Section 2 is devoted to the meaning of contraction in locally convex spaces or algebras. Fixed points of some operators $F$ with a contraction property in some spaces (or algebras) $E$ are involved to solve many problems in functional analysis. There are at least four journals on Fixed Point Theory, many publications on the subject (between [12] to [2]) and many books as the monograph [7]. However we are interested in the classical application to the Cauchy-Lipschitz theorem locally or globally formulated. Then the definition of a contraction we need is only a slight generalization (we suppose that $E$ is locally convex) of the Frigon-Granas one ([6]) given when $E$ is a Fréchet space. It leads to the expected result given in Theorem 1: Any contraction $F : E \rightarrow E$ has a fixed point. If $E$ is Hausdorff, this fixed point is unique.

But the irregular cases of the Cauchy-Lipschitz theorem suggests a generalized formulation which is the subject of Section 3 and invites to define some operator $\Phi$ in a factor algebra $\mathcal{A}$ of generalized functions. $\mathcal{A}$ is constructed ([10]) from a basic locally convex algebra $(\mathcal{E}, \tau)$. The elements $x \in \mathcal{A}$ are classes $[x_\lambda]$ of some families $(x_\lambda)_{\lambda \in \Lambda}$ with "moderateness" linked to a factor ring $\mathcal{C}$ of so-called generalized numbers. Under some hypotheses, $\Phi$ is well defined by

$$\mathcal{A} \ni [x_\lambda] = x \rightarrow \Phi (x) = [\Phi_\lambda (x_\lambda)] \in \mathcal{A}.$$  

for some operator $\Phi_\lambda$ in $\mathcal{E}$. We suppose in addition that each $\Phi_\lambda$ is a contraction in $(\mathcal{E}, \tau_\lambda)$ endowed with a locally convex topology $\tau_\lambda$ depending on $\lambda$. Then $\Phi_\lambda$ has a fixed point $z_\lambda$ in $\mathcal{E}$ and unique if $(\mathcal{E}, \tau_\lambda)$ is Hausdorff. This leads to define $\Phi$ as a contraction in $\mathcal{A}$ (Definition 4). I don’t see any similar idea in the framework of generalized functions. I hope that it a good enough (or not too bad) one! Moreover with some additional hypotheses, we can prove the moderateness of $(z_\lambda)_{\lambda}$ and find (Theorem 6) a fixed point $z$ of $\Phi$ through

$$\mathcal{A} \ni [z_\lambda] = z \rightarrow \Phi (z) = [\Phi_\lambda (z_\lambda)] \in \mathcal{A}.$$  

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But the uniqueness of \( z_\lambda \) is not sufficient to prove that \([z_\lambda] = z\) is the unique fixed point of \( \Phi \). Nevertheless we can obtain this uniqueness when taking, as in Theorem 10
\[
\Phi_\lambda (x) (t) = x_0 + \int_0^t f_\lambda (s, x(s)) ds
\]
for \( x_0 \) given in \( \mathbb{R} \), \( x \in C^0 (\mathbb{R}_+, \mathbb{R}) \), \( t \in \mathbb{R}_+ \) and \( f_\lambda \in C^0 (\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}) \) from what it is clear that \( \Phi_\lambda \) is a map \( \mathcal{E} \to \mathcal{E} \) with \( \mathcal{E} = C^0 (\mathbb{R}_+, \mathbb{R}) \).

In a Subsection (of Section 3) we extend the results to the case where \( \Phi \) is an operator in the product \( \mathcal{A}^m \) of algebras constructed on \( \mathcal{E}^m \). The natural topology (denoted \( \tau^m \)) on the product \( \mathcal{E}^m \) is defined by the family \( \left( p_i ^{(m)} \right)_{i \in I} \) of seminorms such that
\[
p_i ^{(m)} (x) = p_i ^{(m)} (x_1 \ldots x_m) = \sum_{k=1}^{k=m} p_i (x_k).
\]
We denote by \( (\mathcal{E}^m, \tau^m) \) the topological space \( \mathcal{E}^m \) endowed by the family \( \left( q_{\lambda, i} ^{(m)} \right)_{i \in I} \) with \( q_{\lambda, i} ^{(m)} (x) = \sum_{k=1}^{k=m} q_{\lambda, i} (x_k) \) for a given family \( (q_{\lambda, i})_{i \in I} \) of seminorm on \( \mathcal{E} \). The main result of that section is Theorem 8: Any contraction \( \Phi : \mathcal{A}^m \to \mathcal{A}^m \) has a fixed point in \( \mathcal{A}^m \).

The expected application is the Cauchy-Lipschitz generalized problem studied in Section 4. Thanks to an embedding \( (\mathcal{C} _1^1 (J, \mathbb{R}))^m \to (\mathcal{C} _1^1 (J, \mathbb{R}))^m \) with \( \text{Im} (\mathcal{C} _1^1 (J, \mathbb{R}))^m \) as image, it is to solve
\[
(2) \begin{cases}
\partial x = f(\cdot, x) \\
x(t_0) = \xi
\end{cases}
\]
with \( x \in \text{Im} (\mathcal{C} _1^1 (J, \mathbb{R}))^m \subset (\mathcal{C} _1^1 (J, \mathbb{R}))^m \) and \( f \in \left( \mathcal{C} ^0_\tau (J \times \mathbb{R} ^m, \mathbb{R}) \right)^m \) globally Lipschitz, for some ring of generalized numbers \( C = A / I_A \), with \( t_0 \in J \) and \( \xi \) is a given element in \( \mathcal{E}^m \).

The "derivation" \( \partial \) is a map from \( \text{Im} (\mathcal{C} _1^1 (J, \mathbb{R}))^m \) to \( (\mathcal{C} _1^1 (J, \mathbb{R}))^m \). The algebra \( (\mathcal{C} _1^1 (J, \mathbb{R}))^m \) (resp. \( (\mathcal{C} _1^1 (J, \mathbb{R}))^m \)) generalize \( (C^0 (J, \mathbb{R}))^m \) (resp. \( (C^1 (J, \mathbb{R}))^m \) and \( (\mathcal{C} ^0_\tau (J \times \mathbb{R} ^m, \mathbb{R}))^m \) is a generalization of \( (C^0 (J \times \mathbb{R} ^m, \mathbb{R}))^m \) without use of derivatives.

The main result of that section (Theorem 10) is that it exists a ring of generalized numbers \( C = A / I_A \), such that \( f \in \left( \mathcal{C} ^0_\tau (J \times \mathbb{R} ^m, \mathbb{R}) \right)^m \) and a map \( \Phi : (\mathcal{C} _1^1 (J, \mathbb{R}))^m \to (\mathcal{C} _1^1 (J, \mathbb{R}))^m \) with an unique fixed point solving the Cauchy-Lipschitz problem \( (2) \) with \( t_0 \in \mathbb{R}_+ \) and \( \xi \in \mathbb{R}^m \).

All our ideas, technics and results are explicitely detailed and summarized in the final example (Example 2).

The last subsection shows a link between the Cauchy-Lipschitz theorem and the transport equation. We cite some results when the coefficients have a weak regularity of Sobolev type ([5]) or with controlled irregularities [3]. But it is not the case of distributions we wish to treat later with our generalized methods.

2 Contractions in locally convex and complete spaces

We suppose here tat the (seminormed with \( \mathcal{P} = (p_i)_{i \in I} \) space \( E \) is sequentially complete. A basis of \( 0 \)-neighbourhood is the set of all "balls" of the seminorms \( (p_i)_{i \in I} \)
\[
\beta (i, r) = \{ x \in E / p_i (x) < r \}
\]
for all \( i \in I \) and \( r > 0 \). Then, \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence iff
\[
(\forall \varepsilon > 0) (\forall i \in I) (\exists n_0) (\forall n, p) (n > n_0, p > 0 \Rightarrow p_i (x_{n+p} - x_n) < \varepsilon)
\]
and \( E \) is sequentially complete if any Cauchy sequence converges to an element \( e \in E \).
Definition 1 A map \( F : E \to E \) is called a contraction if for all \( i \in I \) it exists \( k_i < 1 \) such that
\[
\forall (x, y) \in E \times E, p_i(F(x) - F(y)) \leq k_i p_i(x - y).
\]

This definition is an obvious generalization of the Frigon-Granas one [6] given when \( \mathcal{P} \) is a countable family of semi norms \( \{p_i\}_{i \in \mathbb{N}} \) rendering \( E \) a Fréchet space.

In this case, \( F \) is not necessary a contraction in the usual sense when \( E \) is endowed with the metric \( d(x, y) = \sum_{i \in \mathbb{N}} p_i(x - y)/(1 + p_i(x - y)) \).

Theorem 1 Any contraction \( F : E \to E \) has a fixed point. If \( E \) is Hausdorff, this fixed point is unique.

Proof. Starting from \( x_0 \in E \), define \( x_{n+1} = F(x_n) \) by induction. It is easy to verify that \( x_n \) is a Cauchy sequence in the complete space \( E \) and converges to some \( x \in E \). The contraction property of the map \( F \) implies obviously its continuity. Then, passing to the limit in \( x_{n+1} = F(x_n) \), we obtain that \( x \) is a fixed point of \( E \). If \( E \) is Hausdorff, for all \( z \neq 0 \) it exists \( V \in \mathcal{V}(0) \) such that \( z \notin V \). Then it exists \( i \) (depending on \( z \)) such that \( p_i(z) > 0 \). If \( x \) and \( y \) are two different fixed points of \( F \), it exists \( j \) (depending on \( x - y \)) such that
\[
0 < p_j(x - y) = p_j(F(x) - F(y)) \leq k_j p_j(x - y) < p_i(x - y)
\]
which gives a contradiction. ■

Notation 1 We denote by
- \( \Lambda \) a set of indices
- \( (E, \tau) \) the space \( E \) endowed with the topology \( \tau \) of the previous family \( \mathcal{P} = \{p_i\}_{i \in I} \)
- \( (E, \tau_\lambda)_{\lambda \in \Lambda} \) the family of spaces \( (E, \tau_\lambda) \) seminormed by the family \( \mathcal{Q}_\lambda = \{q_{\lambda,i}\}_{i \in I} \)
- \( (F_\lambda)_{\lambda \in \Lambda} \) a family of contractions \( F_\lambda : (E, \tau_\lambda) \to (E, \tau_\lambda) \).

Theorem 2 Each \( F_\lambda \) has a fixed point \( z_\lambda \in E \). If in addition we suppose that \( (E, \tau) \) is Hausdorff and for each \( i \in I \) and \( \lambda \in \Lambda \) it exists a strictly positive constant \( a_{\lambda,i} \) such that
\[
a_{\lambda,i} p_i \leq q_{\lambda,i}
\]
then \( z_\lambda \) is unique.

Proof. From Theorem 1 we know that each \( F_\lambda \) has a fixed point \( z_\lambda \in E \). If \( (E, \tau) \) is Hausdorff, for each \( x \neq 0 \) in \( E \), it exists \( i(x) \in I \) such that \( p_i(x) > 0 \). Then we have \( q_{\lambda,i}(x) > 0 \) which implies that \( (E, \tau_\lambda) \) is Hausdorff. As \( F_\lambda : (E, \tau_\lambda) \to (E, \tau_\lambda) \) is a contraction, \( z_\lambda \) is unique. ■

3 Constructions in generalized spaces or algebras

3.1 The \((\mathcal{C}, \mathcal{E}, \mathcal{P})\) setting

We consider the setting of \((\mathcal{C}, \mathcal{E}, \mathcal{P})\)-algebras which is an extension of Colombeau’s one. It allows to construct multiparametrized generalized spaces or algebras where the "asymptotic \( \mathcal{C} \)" is given independantly from the basis topological space or algebra \( \mathcal{C} \). To summarize the definitions and results given in [10, 11] the asymptotics is given by

(1) \( \Lambda \): a set of indices;
(2) \( A \): a solid subring of the ring \( \mathbb{K}^\Lambda \) (\( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \)); this means that whenever \( \{(s_\lambda)_{\lambda}\} \leq \{(r_\lambda)_{\lambda}\} \) for some \( ((s_\lambda)_{\lambda}, (r_\lambda)_{\lambda}) \in \mathbb{K}^\Lambda \times A \), that is, \( |s_\lambda| \leq |r_\lambda| \) for all \( \lambda \), it follows that \( (s_\lambda)_{\lambda} \in A \);
(3) $I_A$: a solid ideal of $A$.

Then $C$ is defined as the factor ring $A/I_A$. On the other hand we give

(4) $E$: a $K$-topological space endowed with a family $P = \{p_t\}_{t \in I}$ of semi-norms.

Define $|B| = \{(|r\lambda|)_\lambda, (r\lambda)_\lambda \in B\}$, $B = A$ or $I_A$, and set

$$H_{(A, E, P)} = \{(u\lambda)_\lambda \in [E]^A \mid \forall i \in I, (p_i(u\lambda))_\lambda \in |A|\}$$

$$H_{(I_A, E, P)} = \{(u\lambda)_\lambda \in [E]^A \mid \forall i \in I, (p_i(u\lambda))_\lambda \in |I_A|\}$$

The following result summarize some results recalled in [9].

**Theorem 3** If $A$ is a solid subring of $K^A$, then $H_{(A, E, P)}$ is an $A$-module, and an $A$-algebra if $E$ is a topological algebra.

Moreover, if $I_A$ is an ideal of $A$ sharing the solidness property, then $H_{(I_A, E, P)}$ is an $A$-linear subspace of $H_{(A, E, P)}$, and an ideal of $H_{(A, E, P)}$ if $E$ is a topological algebra.

As a consequence, the factor space $H_{(A, E, P)}/H_{(I_A, E, P)}$ is again an $A$-module, but also an $A/I_A$-module (and of course an algebra, if $E$ is a topological algebra). For $(E, P) = (K, \{|\cdot|\})$, we get $H_{(A, K, |\cdot|)}/H_{(I_A, K, |\cdot|)} = A/I_A$.

**Remark 1** If we require $E$ to be a topological algebra, this means that multiplication in $E$ is continuous for the topology defined by the family of seminorms.

**Definition 2** The factor ring $C = A/I_A$ is called the ring of generalized numbers (associated to $A$ and $I_A$), and the $C$-algebra

$$A_C(E, P) := H_{(A, E, P)}/H_{(I_A, E, P)}$$

is called the $(C, E, P)$-algebra of $C$-generalized functions. We denote by $[x\lambda]_\lambda \in H_{(A, E, P)}$ the class in $A_C(E, P)$ of the family $(x\lambda)_\lambda \in H_{(A, E, P)}$.

**Example 1** Let us define

$$A = \{(r\varepsilon)_e \in R^{0,1} \mid \exists m \in N : |u\varepsilon| = o(\varepsilon^{-m}), \text{ as } \varepsilon \to 0\}$$

$$I_A = \{(r\varepsilon)_e \in R^{0,1} \mid \forall q \in N : |u\varepsilon| = o(\varepsilon^q), \text{ as } \varepsilon \to 0\}.$$  

In this case (with $E = C^\infty(R^n)$ and $P = \{p_{K,\alpha} : f \to \|K\| \partial^n f \|_K \in C^n\}$), the algebra $A = H_{(A, E, P)}/J_{(I_A, E, P)}$ is exactly the so-called special Colombeau algebra $G(R^n)$.

**Remark 2** If $E$ is a sheaf of $K$-topological algebras over a topological space $X$, we can prove that the factor $H_{(A, E, P)}/H_{(I_A, E, P)}$ is at least a presheaf satisfying the localization principle. But we don’t need the sheaf structure in the sequel.

We are going now to the the concept of "Overgenerated algebras". It is useful when constructing a $(C, E, P)$-structure to solve some problems with irregular data or coefficients.

Choose $B_p$ a finite family of $p$ nets in $(R^*_+)^A$ (usually given by the asymptotic structure of the problem.) Consider $B$ the subset of elements in $(R^*_+)^A$ obtained as rational fractions with coefficients in $R^*_+$, of elements in $B_p$ as variables.

**Definition 3** Define

$$A = \{(a\lambda)_\lambda \in K^A \mid \exists (b\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda < \lambda_0 : |a\lambda| \leq b\lambda\}.$$
We say that $A$ is overgenerated by $B_p$ (and it is easy to see that $A$ is a solid subring of $\mathbb{K}^\Lambda$). If $I_A$ is some solid ideal of $A$, we also say that $C = A/I_A$ is overgenerated by $B_p$. For example, as a “canonical” ideal of $A$, we can take

$$I_A = \{(a\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \forall (b\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda < \lambda_0 : |a\lambda| \leq b\lambda \}. $$

This definition implies that $B$ is stable by inverse.

### 3.2 Contraction operator in $\mathcal{A}_C(\mathcal{E}, \mathcal{P})$

First, we are looking if it is possible to define a map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ by means of a given family $(\Phi_\lambda)_{\lambda \in \Lambda}$ of maps $\Phi_\lambda : \mathcal{E} \rightarrow \mathcal{E}$. The general requirement is given in the following

**Lemma 4** Let $(\Phi_\lambda)_{\lambda \in \Lambda}$ be a given family of maps $\mathcal{E} \rightarrow \mathcal{E}$.

Suppose that for each $(x\lambda)_\lambda \in \mathcal{H}(A,\mathcal{E},\mathcal{P})$ and $(i\lambda)_\lambda \in \mathcal{H}(I_A,\mathcal{E},\mathcal{P})$ we have

1. $(\Phi_\lambda (x\lambda))_\lambda \in \mathcal{H}(A,\mathcal{E},\mathcal{P})$.
2. $(\Phi_\lambda (x\lambda + i\lambda))_\lambda - (\Phi_\lambda (x\lambda))_\lambda \in \mathcal{H}(I_A,\mathcal{E},\mathcal{P})$.

Then, $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is well defined by

$$\mathcal{A} \ni [x\lambda] = x \rightarrow \Phi(x) = [\Phi_\lambda (x\lambda)] \in \mathcal{A}.$$ 

**Proof.** From (i) we see that the class $[\Phi_\lambda (x\lambda)]$ lies in $\mathcal{A}$. Let $x\lambda + i\lambda$ be another representative of $x = [x\lambda]$. From (ii) we have $[\Phi_\lambda (x\lambda + i\lambda)] = [\Phi_\lambda (x\lambda)]$. Then, $\Phi$ is well defined. $\blacksquare$

**Theorem 5** Let be fulfilled the following hypotheses: It exists a family $(a_{n,\lambda})_{n,\lambda \in \mathbb{N} \times \Lambda}$ of positive numbers with $(a_{n,\lambda})_\lambda \in A$, verifying: for each $i \in I$ there exists $N(i)$ and $j(i) \in I$ such that, for each $\lambda \in \Lambda$ and $e \in \mathcal{E}$

$$p_i(\Phi_\lambda (e)) \leq \sum_{n=0}^{N(i)} a_{n,\lambda} p_{j(i)}^n(e).$$

Then for each $(x\lambda)_\lambda \in \mathcal{H}(A,\mathcal{E},\mathcal{P})$ we have $(\Phi_\lambda (x\lambda))_\lambda \in \mathcal{H}(A,\mathcal{E},\mathcal{P})$.

**Proof.** $(x\lambda)_\lambda \in \mathcal{H}(A,\mathcal{E},\mathcal{P})$ implies $\forall i \in I, (p_i (x\lambda))_\lambda \in |A|$. Then, $\sum_{n=0}^{N(i)} a_{n,\lambda} p_{j(i)}^n (x\lambda) \in |A|$. As $A$ is a solid subring of $\mathbb{K}^\Lambda$, it follows that $(p_i (\Phi_\lambda (x\lambda))_\lambda \in |A|$. $\blacksquare$

**Definition 4** The following hypotheses permit to well define a map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ from the family $(\Phi_\lambda)_{\lambda \in \Lambda}$ and to call it a contraction.

(a) for each $(x\lambda)_\lambda \in \mathcal{H}(A,\mathcal{E},\mathcal{P})$, $(\Phi_\lambda (x\lambda))_\lambda \in \mathcal{H}(A,\mathcal{E},\mathcal{P})$.

(b) Each $\Phi_\lambda$ is a contraction in $(\mathcal{E}, \tau_\Lambda)$ endowed with the family $Q_\lambda = (q_{i\lambda})_{i\in I}$, and the corresponding contraction constants are denoted by $k_{\lambda,i} < 1$.

(c) For each $i \in I$ and $\lambda \in \Lambda$ it exist some strictly positive constants $\alpha_{\lambda,i}$ and $\beta_{\lambda,i}$ such that

$$\alpha_{\lambda,i} p_i \leq q_{\lambda,i} \leq \beta_{\lambda,i} p_i.$$ 

(d) For each $i \in I$, the families $\left(\frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} \right)_\lambda$ and $\left(\frac{1}{1 - k_{\lambda,i}} \right)_\lambda$ lies in $|A|$.

**Theorem 6** Any contraction $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ has a fixed point in $\mathcal{A}$. 

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Proof. Remark that condition (a) which is (i) in Lemma 4 can be fulfilled when giving the hypotheses in Theorem 5, that is to say $(a_{n,\lambda})_\lambda \in A$. Now we have from (c)

$$p_i(\Phi_\lambda (x_\lambda + i_\lambda) - (\Phi_\lambda (x_\lambda)) \leq \frac{1}{\alpha_{\lambda,i}} k_{i,\lambda} q_{\lambda,i} (i_\lambda) \leq \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} p_i(i_\lambda)$$

from what we deduce that $(p_i(\Phi_\lambda (x_\lambda + i_\lambda) - (\Phi_\lambda (x_\lambda))_\lambda \in A$ and the condition (ii) in Lemma 4 is verified. Then $\Phi$ is well defined. From Theorem 2 we know that each $\Phi_\lambda$ has a fixed point $z_\lambda$ obtained as limit of the Cauchy sequence $z_{n,\lambda}$ defined by induction by $z_{n+1,\lambda} = \Phi_\lambda(z_{n,\lambda})$. Starting from $z_0 = [z_0]_\lambda \in A$, we deduce that $z_1 = [\Phi_\lambda(z_0)]_\lambda \in A$, and $z_1 - z_0 \in A$. That is to say $p_i(z_1,\lambda - z_0,\lambda) \in |A|$. 

By induction we can compute for all $n, p \in \mathbb{N}$

$$q_{\lambda,i}(z_{n+p,\lambda} - z_{n,\lambda}) \leq \frac{k_{\lambda,i}^n}{1 - k_{\lambda,i}} q_{\lambda,i}(z_{1,\lambda} - z_{0,\lambda})$$

giving

$$q_{\lambda,i}(z_{p,\lambda} - z_{0,\lambda}) \leq \frac{1}{1 - k_{\lambda,i}} q_{\lambda,i}(z_{1,\lambda} - z_{0,\lambda}) .$$

When taking the limit $z_\lambda$ of $z_{p,\lambda}$ in $(\mathcal{E}, \tau_\lambda)$ when $p \to \infty$, we get

$$q_{\lambda,i}(z_\lambda - z_{0,\lambda}) \leq \frac{1}{1 - k_{\lambda,i}} q_{\lambda,i}(z_{1,\lambda} - z_{0,\lambda}) .$$

Writing now $q_{\lambda,i}(z_\lambda) \leq q_{\lambda,i}(z_\lambda - z_{0,\lambda}) + q_{\lambda,i}(z_{0,\lambda})$ we have

$$p_i(z_\lambda) \leq \frac{1}{\alpha_{\lambda,i}} q_{\lambda,i}(z_\lambda) \leq \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} \left[ \frac{1}{1 - k_{\lambda,i}} (p_i(z_{1,\lambda} - z_{0,\lambda}) + p_i(z_{0,\lambda})) \right] .$$

Then, from the hypotheses $(p_i(z_\lambda))_\lambda \in |A|$, that is to say $((p_i(z_\lambda)))_\lambda \in \mathcal{H}_{(\mathcal{A}, \mathcal{E}, \mathcal{P})}$. If $z = [z_\lambda]$, then we have $\Phi(z) = [\Phi_\lambda(z_\lambda)] = [\lambda] = z$. Then $z$ is a fixed point of $\Phi$.

However following the under Remark 3 we cannot prove the uniqueness of $z$ without other hypotheses than uniqueness of fixed points of $\Phi_\lambda$.

3.3 Constructions in product of algebras

Definition 5 For $m \in \mathbb{N}$, we define

$$\mathcal{H}_{(\mathcal{A}, \mathcal{E}, \mathcal{P})} = \left\{ (u_\lambda)_\lambda = (u_{1,\lambda}, ... u_{m,\lambda}) \in [\mathcal{E}^m]^\mathcal{A} \mid \forall i \in I, \forall k = 1, ... m, ((p_i(u_{k,\lambda}))_\lambda \in |A|) \right\} ,$$

$$\mathcal{H}_{(I_\mathcal{A}, \mathcal{E}, \mathcal{P})} = \left\{ (u_\lambda)_\lambda = (u_{1,\lambda}, ... u_{m,\lambda}) \in [\mathcal{E}^m]^\mathcal{A} \mid \forall i \in I, \forall k = 1, ... m, ((p_i(u_{k,\lambda}))_\lambda \in |I_A|) \right\} .$$

According to the results and definitions in Subsection 3.1, if $\mathcal{A}$ is a solid subring of $\mathbb{K}^\mathcal{A}$, then $\mathcal{H}_{(\mathcal{A}, \mathcal{E}, \mathcal{P})}$ is an $\mathcal{A}$-module, and an $\mathcal{A}$-algebra if $\mathcal{E}$ is a topological algebra. Moreover, if $I_\mathcal{A}$ is an ideal of $\mathcal{A}$ sharing the solidness property, then $\mathcal{H}_{(I_\mathcal{A}, \mathcal{E}, \mathcal{P})}$ is an $\mathcal{A}$-linear subspace of $\mathcal{H}_{(\mathcal{A}, \mathcal{E}, \mathcal{P})}$, and an ideal of $\mathcal{H}_{(\mathcal{A}, \mathcal{E}, \mathcal{P})}$ if $\mathcal{E}$ is a topological algebra. As a consequence, the factor space $\mathcal{H}_{(\mathcal{A}, \mathcal{E}, \mathcal{P})}/\mathcal{H}_{(I_\mathcal{A}, \mathcal{E}, \mathcal{P})}$ is again an $\mathcal{A}$-module, but also an $\mathcal{A}/I_\mathcal{A}$-module (and of course an algebra, if $\mathcal{E}$ is a topological algebra).

Definition 6 We pose $\mathcal{A}_{\mathcal{E}, \mathcal{P}} = \mathcal{H}_{(\mathcal{A}, \mathcal{E}, \mathcal{P})}/\mathcal{H}_{(I_\mathcal{A}, \mathcal{E}, \mathcal{P})}$ and denote by $[x_\lambda]$ the class in $\mathcal{A}_{\mathcal{E}, \mathcal{P}}$ of the family $(x_\lambda)_\lambda = (x_{1,\lambda}, ... x_{m,\lambda})_\lambda \in \mathcal{H}_{(\mathcal{A}, \mathcal{E}, \mathcal{P})}$.
First, it is possible as previously, to define a map $\Phi : \mathcal{A}^m \to \mathcal{A}^m$ by means of a given family $(\Phi_\lambda)_{\lambda \in \Lambda}$ of maps $\Phi_\lambda : \mathcal{E}^m \to \mathcal{E}^m$. The general requirement is similar to the previous one given in Lemma 4.

Suppose that for each $(x_\lambda)_\lambda \in \mathcal{H}^m_{(\lambda,E,P)}$ and $(i_\lambda)_\lambda \in \mathcal{H}^m_{(\lambda,E,P)}$ we have

(i) $(\Phi_\lambda (x_\lambda))_\lambda \in \mathcal{H}^m_{(\lambda,E,P)}$,

(ii) $(\Phi_{\lambda + i_\lambda})_\lambda - (\Phi_\lambda (x_\lambda))_\lambda \in \mathcal{H}^m_{(\lambda,E,P)}$.

Then, $\Phi : \mathcal{A}^m \to \mathcal{A}^m$ is well defined by

$$\mathcal{A}^m \ni [x_\lambda] = x \to \Phi (x) = [\Phi_\lambda (x_\lambda)] \in \mathcal{A}^m.$$ 

Now, we are defining a contraction property for the given family $(\Phi_\lambda)_{\lambda \in \Lambda}$ of maps $\Phi_\lambda : \mathcal{E}^m \to \mathcal{E}^m$.

**Notation 2** The natural topology (denoted $\tau^m$) on the product $\mathcal{E}^m$ is defined by the family $(p_i^{(m)})_{i \in I}$ of seminorms such that $p_i^{(m)}(x) = p_i(x_1, \ldots, x_m) = N(p_i(x_1), \ldots, p_i(x_n))$ where $N$ is any norm on $\mathbb{R}^n$. For example, we can choose $p_i^{(m)}(x) = \sum_{k=1}^{k=m} p_i(x_k)$. Denote by $(\mathcal{E}^m, \tau^m)$ the topological space $\mathcal{E}^m$ endowed by the family $(q_{\lambda,i})_{i \in I}$ with $q_{\lambda,i}^{(m)}(x) = \sum_{k=1}^{k=m} q_{\lambda,i}(x_k)$ for a given family $(q_{\lambda,i})_{i \in I}$ of seminorm on $\mathcal{E}$.

**Definition 7** $(\Phi_\lambda)_{\lambda \in \Lambda}$ is called a family of contractions in $\mathcal{E}^m$ if for each $(i, \lambda) \in I \times \Lambda$ it exists a semi norm $q_{i,\lambda}$ and a constant $k_{i,\lambda} < 1$ such that for all $(x, y) \in \mathcal{E}^m \times \mathcal{E}^m$

$$(3) \quad q_{i,\lambda}^{(m)} (\Phi_\lambda (x) - \Phi_\lambda (y)) \leq k_{i,\lambda} \sum_{k=1}^{k=m} q_{i,\lambda}(x_k - y_k).$$

**Proposition 7** Each contraction $\Phi_\lambda$ in $\mathcal{E}^m$ has a fixed point $z_\lambda$. Moreover if $(\mathcal{E}, \tau)$ is Hausdorff and for each $(i, \lambda) \in I \times \Lambda$ it exists a strictly positive constant $a_{i,\lambda}$ such that $a_{i,\lambda}q_i \leq q_{i,\lambda}$, then $z_\lambda$ is unique.

**Proof.** We deduce from (3) that $q_{i,\lambda}^{(m)} (\Phi_\lambda (x) - \Phi_\lambda (y)) \leq k_{i,\lambda}q_{i,\lambda}^{(m)}(x - y)$. From Theorem 2 we know that each $\Phi_\lambda$ has a fixed point $z_\lambda \in \mathcal{E}^m$. If $(\mathcal{E}, \tau)$ is Hausdorff, for each nonnull $r \in \mathcal{E}$, it exists $i \in I$ such that $p_i(r) > 0$. Then $q_{i,\lambda}(r) > 0$ which implies that $(\mathcal{E}, \tau_\lambda)$ is Hausdorff. If $y_\lambda \neq z_\lambda$ is another fixed point of $\Phi_\lambda$, it exists at least $h \in \mathbb{N}$ with $1 \leq h \leq m$ such that $y_{\lambda,h} - z_{\lambda,h} \neq 0$. Therefore, exists $j \in I$ such that

$$0 < q_{j,\lambda}(y_{\lambda,h} - z_{\lambda,h}) \leq q_{j,\lambda}^{(m)}(y_{\lambda} - z_\lambda) = q_{j,\lambda}^{(m)} (\Phi_\lambda (y_\lambda) - \Phi_\lambda (z_\lambda)) \leq k_{\lambda,j}q_{\lambda,j}^{(m)}(y_{\lambda} - z_\lambda) < q_{\lambda,j}^{(m)} (y_{\lambda} - z_\lambda)$$

which leads to a contradiction.

And now, to achieve the construction of $\Phi : \mathcal{A}^m \to \mathcal{A}^m$ and prove the existence of a fixed point in the same way as in Theorem 6, we propose the following

**Definition 8** The following hypotheses permit to well define a map $\Phi : \mathcal{A}^m \to \mathcal{A}^m$ from the family $(\Phi_\lambda)_{\lambda \in \Lambda}$ and to call it a contraction.

(a) for each $(x_\lambda)_\lambda \in \mathcal{H}^m_{(\lambda,E,P)}$, $(\Phi_\lambda (x_\lambda))_\lambda \in \mathcal{H}^m_{(\lambda,E,P)}$,

(b) The map $\Phi_\lambda : \mathcal{E}^m \to \mathcal{E}^m$ is a contraction of $\mathcal{E}^m$ following Definition 7,

(c) For each $i \in I$ and $\lambda \in \Lambda$ it exist some strictly positive constants $\alpha_{i,\lambda}$ and $\beta_{i,\lambda}$ such that

$$\alpha_{i,\lambda} p_i \leq q_{i,\lambda} \leq \beta_{i,\lambda} p_i,$$

(d) For each $i \in I$, the families $(\frac{\beta_{i,\lambda}}{\alpha_{i,\lambda}})^{\lambda} \lambda$ and $(\frac{1}{1 - k_{i,\lambda}})^{\lambda} \lambda$ lies in $|\Lambda|$. 7
Theorem 8 Any contraction $\Phi \mathcal{A}^m \to \mathcal{A}^m$ has a fixed point in $\mathcal{A}^m$.

Proof. First we can re-write $\mathcal{H}^m_{(A, \varepsilon, p)}$ and $\mathcal{H}^m_{(I, A, \varepsilon, p)}$ as

$$\mathcal{H}^m_{(A, \varepsilon, p)} = \left\{ (u_\lambda)_{\lambda} \in [\mathcal{E}^m]^A \mid \forall i \in I, \left( p_i^{(m)}(u_\lambda) \right)_\lambda \in [A] \right\}.$$

$$\mathcal{H}^m_{(I, A, \varepsilon, p)} = \left\{ (u_\lambda)_{\lambda} \in [\mathcal{E}^m]^A \mid \forall i \in I, \left( p_i^{(m)}(u_\lambda) \right)_\lambda \in [I_A] \right\}.$$

We deduce from (c) that

$$\alpha_{\lambda, i}p_i^{(m)} \leq q_{\lambda, i}^{(m)} \leq \beta_{\lambda, i}p_i^{(m)}.$$

For $(x_\lambda)_\lambda \in \mathcal{H}^m_{(A, \varepsilon, p)}$ and $(i_\lambda)_\lambda \in \mathcal{H}^m_{(I, A, \varepsilon, p)}$ we have

$$p_i^{(m)}(\Phi_\lambda(x_\lambda + i_\lambda) - \Phi_\lambda(x_\lambda)) \leq \frac{1}{\alpha_{\lambda, i}}k_{\lambda, i}q_{\lambda, i}^{(m)}(i_\lambda) \leq \frac{\beta_{\lambda, i}}{\alpha_{\lambda, i}}p_i^{(m)}(i_\lambda) = \frac{\beta_{\lambda, i}}{\alpha_{\lambda, i}} \sum_{k=1}^{k=m} p_i(i_k).$$

Then, $\left( p_i^{(m)}(\Phi_\lambda(x_\lambda + i_\lambda) - \Phi_\lambda(x_\lambda)) \right)_\lambda \in [I_A]$ and $(\Phi_\lambda(x_\lambda + i_\lambda))_\lambda - (\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{H}^m_{(I, A, \varepsilon, p)}$. It follows that $\Phi : \mathcal{A}^m \to \mathcal{A}^m$ is well defined by

$$\mathcal{A}^m \ni [x_\lambda] = x \to \Phi(x) = [\Phi_\lambda(x_\lambda)] \in \mathcal{A}^m.$$

From (b) and Theorem 2 we know that each $\Phi_\lambda$ has a fixed point $z_\lambda = (z_{1,1}, ..., z_{m,1}) \in \mathcal{E}^m$ and from Theorem 1 we know that $z_\lambda$ is obtained as limit of the Cauchy sequence $z_{n, \lambda}$ defined by induction by $z_{n+1, \lambda} = \Phi_\lambda(z_{n, \lambda})$. Starting from $z_0 = [z_0, \lambda] \in \mathcal{A}^m$, we deduce that $z_1 = [\Phi_\lambda(z_0, \lambda)] \in \mathcal{A}^m$, and $z_1 - z_0 \in \mathcal{A}^m$. That is to say $p_i^{(m)}(z_{1, \lambda} - z_{0, \lambda}) \in [A]$.

By induction we can compute for all $n, p \in \mathbb{N}$

$$q_{\lambda, i}^{(m)}(z_{n+p, \lambda} - z_{n, \lambda}) \leq \frac{k_{\lambda, i}^n}{1 - k_{\lambda, i}} q_{\lambda, i}^{(m)}(z_{1, \lambda} - z_{0, \lambda})$$

giving

$$q_{\lambda, i}^{(m)}(z_{p, \lambda} - z_{0, \lambda}) \leq \frac{1}{1 - k_{\lambda, i}} q_{\lambda, i}^{(m)}(z_{1, \lambda} - z_{0, \lambda}).$$

When taking the limit $z_\lambda$ of $z_{p, \lambda}$ in $(\mathcal{E}^m, \tau^m_\lambda)$ when $p \to \infty$, we get

$$q_{\lambda, i}^{(m)}(z_\lambda - z_{0, \lambda}) \leq \frac{1}{1 - k_{\lambda, i}} q_{\lambda, i}^{(m)}(z_{1, \lambda} - z_{0, \lambda}).$$

Writing now $q_{\lambda, i}^{(m)}(z_\lambda) \leq q_{\lambda, i}^{(m)}(z_\lambda - z_{0, \lambda}) + q_{\lambda, i}^{(m)}(z_{0, \lambda})$ we have

$$p_i^{(m)}(z_\lambda) \leq \frac{1}{\alpha_{\lambda, i}} q_{\lambda, i}^{(m)}(z_\lambda) \leq \frac{\beta_{\lambda, i}}{\alpha_{\lambda, i}} \left[ \frac{1}{1 - k_{\lambda, i}} (p_i^{(m)}(z_{1, \lambda} - z_{0, \lambda}) + p_i^{(m)}(z_{0, \lambda})) \right].$$

From the hypotheses of Definition 8, we have $\left( p_i^{(m)}(z_\lambda) \right)_\lambda \in [A]$, then $((z_\lambda))_\lambda \in \mathcal{H}^m_{(A, \varepsilon, p)}$. Therefore, if $z = [z_\lambda]$, we have $\Phi(z) = [\Phi_\lambda(z_\lambda)] = [z_\lambda] = z$. Then $z$ is a fixed point of $\Phi$. ■

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Remark 3 If \( y = [y_\lambda] \) is any fixed point of \( \Phi \), that point verifies \( y_\lambda = \Phi_\lambda (y_\lambda) + i_\lambda \) for some \((i_\lambda)_\lambda \in \mathcal{H}^{m}_{(I_\lambda, \mathcal{E}, \mathcal{P})}\). We try to prove that \([y_\lambda] = [z_\lambda]\) that is to say \((y_\lambda - z_\lambda)_\lambda \in \mathcal{H}^{m}_{(I_\lambda, \mathcal{E}, \mathcal{P})}\). Then, it exists at least \( \mu \in \Lambda \) such that \( y_\mu - z_\mu \neq 0 \). As \((\mathcal{E}^m, r^m_\mu)\) is Hausdorff it exists \( j \in I \) (depending on \( y_\mu - z_\mu \)) such that \( q^{(m)}_{j,\mu}(y_\mu - z_\mu) > 0 \). Writing

\[
0 < q^{(m)}_{j,\mu}(y_\mu - z_\mu) \leq q^{(m)}_{j,\mu}(\Phi_\mu(y_\mu) - \Phi_\mu(z_\mu) + i_\mu) \leq k_{j,\mu}q^{(m)}_{j,\mu}(y_\mu - z_\mu) + q^{(m)}_{j,\mu}(i_\mu)
\]

We don’t see a contradiction and uniqueness cannot be proved without other hypotheses.

4 The Cauchy-Lipschitz theorem

We try to give a generalized formulation of the Cauchy-Lipschitz theorem close to the classical one. We can limit the order of derivatives to one or even zero, as in the globally Lipschitz problem.

Let \( J \) be an interval of \( \mathbb{R} \) and \( f \) a continuous function \( \in C^0(J \times \mathbb{R}^m, \mathbb{R}^m) = (C^0(J \times \mathbb{R}^m, \mathbb{R}))^m \) satisfying a global Lipschitz condition as

\[
\forall K \in J, \exists k > 0, \forall t \in J, \forall y, z \in \mathbb{R}^m, \| f(t, y) - f(t, z) \| \leq k \| y - z \| .
\]

Then, the Cauchy problem

\[
\begin{cases}
x'(t) = f(t, x(t)) \\
x(t_0) = x_0
\end{cases}
\]

with \( x_0 \in \mathbb{R}^m \) and \( t_0 \in J \) admits one unique global solution \( x \in C^1(J, \mathbb{R}^m) = (C^1(J, \mathbb{R}))^m \).

The problem reduces to finding a fixed point of the map \( F \in C^0(J, \mathbb{R}^m) \rightarrow C^0(J, \mathbb{R}^m) \) such that

\[
\forall t \in J, F(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds
\]

that is to say, for each \( k = 1, 2...m \) and \( x = (x_1, x_m) \), \( F = (F_1...F_m) \), \( f = (f_1...f_m) \in C^0(J \times \mathbb{R}^m, \mathbb{R})^m \)

\[
\forall t \in J, k = 1, 2...m : F_k(x)(t) = x_{0,k} + \int_{t_0}^{t} f_k(s, x(s))ds.
\]

The standard proof gives a convenient norm on \( C^0(J, \mathbb{R}^m) \) for which \( F \) is a contraction, and the Picard procedure gives its fixed point.

To formulate (4) in a generalized setting we have to introduce some convenient algebras.

4.1 The generalized framework

Definition 9 When \( l = 0 \) or 1, with \( \mathcal{E} = C^l(J, \mathbb{R}) \) and \( \mathcal{P}^l = (p_{K,l})_K \) such that

\[
p_{K,l}(u) = \sup_{t \in [K, 0] \cup [l]} |u^{(j)}(t)|,
\]

we can write \( \mathcal{H}(\mathcal{A}, \mathcal{E}, \mathcal{P}) \) and \( \mathcal{H}(I_\lambda, \mathcal{E}, \mathcal{P}) \) as

\[
\mathcal{H}(\mathcal{A}, \mathcal{E}, \mathcal{P}) = \left\{ (u_\lambda)_\lambda \in [\mathcal{E}]^\Lambda \mid \forall K \in J, (p_{K,l}(u_\lambda))_\lambda \in |A| \right\}
\]

\[
\mathcal{H}(I_\lambda, \mathcal{E}, \mathcal{P}) = \left\{ (u_\lambda)_\lambda \in [\mathcal{E}]^\Lambda \mid \forall K \in J, (p_{K,l}(u_\lambda))_\lambda \in |I_\lambda| \right\}.
\]

and define \( \mathcal{H}^\infty_J \) or \( \mathcal{H}^\infty_{I_\lambda} \) by replacing \( l \) by \( \infty \) in the previous definitions, with
\[ \mathcal{P}^\infty = (p_{KL})_{K \in J, L \in N^*}. \]
We begin to pose:

\[ \mathfrak{C}^0_1 (J, \mathbb{R}) = \mathcal{H}^A_1(J, \mathbb{R}) / \mathcal{H}^0_1(J, \mathbb{R}), \]
\[ \mathfrak{C}^0_0 (J, \mathbb{R}) = \mathcal{H}^A_1(J, \mathbb{R}) / \mathcal{H}^0_1(J, \mathbb{R}) \cap \mathcal{H}^1_1(J, \mathbb{R}). \]

**Remark 4** We have the classical embedding: \( C^1(J, \mathbb{R}) \to C^0(J, \mathbb{R}) \) which inspires our generalized requirements. But the only way to embed the factor algebra \( \mathfrak{C}^0_1 (J, \mathbb{R}) \) defined from \( \mathcal{H}^A_1(J, \mathbb{R}) \) into \( \mathfrak{C}^0_0 (J, \mathbb{R}) \) is to define \( \mathfrak{C}^0_1 (J, \mathbb{R}) = \mathcal{H}^A_1(J, \mathbb{R}) \cap \mathcal{H}^1_1(J, \mathbb{R}) \), following a well-known result on the embedding of factor algebras. We would like to define \( \mathfrak{C}^0_1 (J, \mathbb{R}) = \mathcal{H}^A_1(J, \mathbb{R}) / \mathcal{H}^1_1(J, \mathbb{R}). \)

In contrast, in this case the natural mapping \( \mathfrak{C}^0_1 (J, \mathbb{R}) \to \mathfrak{C}^0_0 (J, \mathbb{R}) \) is neither injective nor surjective. Indeed we cannot prove here that \( \mathcal{H}^A_1(J, \mathbb{R}) \cap \mathcal{H}^1_1(J, \mathbb{R}) \not= \mathcal{H}^1_1(J, \mathbb{R}) \), in contrary to the "well known"equality \( \mathcal{H}^0_1(J, \mathbb{R}) \cap \mathcal{H}^1_1(J, \mathbb{R}) = \mathcal{H}^1_1(J, \mathbb{R}) \), proved in Lemma 4.4 in [9] which generalize 1.23 Theorem in [8]. However, we can define a map \( \mathfrak{p} \) from \( \text{Im} \mathfrak{C}^0_1(J, \mathbb{R}) \) to \( \mathfrak{C}^0_0(J, \mathbb{R}) \) which looks like a derivation.

**Definition 10** If \( i(x) \in \text{Im} \mathfrak{C}^0_1(J, \mathbb{R}) \) the embedding \( i \) verifies

\[ i : (x_e)_{e} + \mathcal{H}^A_1(J) \cap \mathcal{H}^0_1(J) \to (x_e)_{e} + \mathcal{H}^0_1(J) \]
which implies that \( (x_e')_{e} \in \mathcal{H}^0_1(J) \) and allows to define

\[ \mathfrak{p} : \mathfrak{C}^0_1(J, \mathbb{R}) \ni i^{-1}(x) = (x_e)_{e} + \mathcal{H}^A_1(J) \cap \mathcal{H}^0_1(J) \to (x_e')_{e} + \mathcal{H}^0_1(J) \in \mathfrak{C}^0_0(J, \mathbb{R}) \]
which leads to define the map \( \mathfrak{p} = \mathfrak{p} \circ i^{-1} \) from \( \text{Im} \mathfrak{C}^0_1(J, \mathbb{R}) \) to \( \mathfrak{C}^0_0(J, \mathbb{R}) \).

For \( f \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{K}) \) with \( K \subseteq J, p > 0 \), we pose

\[ q_{K, -p}(f) = \sup_{t \in K, y \in \mathbb{R}} (1 + |y|)^{-p} |f(t, y)|, \]
then for any solid unitary subring \( A \) with ideal \( I_A \) of \( \mathbb{K}^A \), we define

\[ \mathcal{H}^0_1(A \times \mathbb{K}^m, \mathbb{R}) = \left\{ (f_x)^{\lambda} \in \mathcal{H}^0_1(A \times \mathbb{K}^m, \mathbb{R}) \mid \forall K \subseteq J, \forall p > 0, (q^{(m)}_{K, -p}(f_x)^{\lambda})_{\lambda} \in |A| \right\}, \]
\[ \mathcal{H}^1_1(A \times \mathbb{K}^m, \mathbb{R}) = \left\{ (f_x)^{\lambda} \in \mathcal{H}^0_1(A \times \mathbb{K}^m, \mathbb{R}) \mid \forall K \subseteq J, \forall p > 0, (q^{(m)}_{K, -p}(f_x)^{\lambda})_{\lambda} \in |I_A| \right\}. \]

For \( f = (f_1, \ldots, f_m) \in (\mathcal{C}^0(\mathbb{R} \times \mathbb{R}^m, \mathbb{K}))^m \), \( K \subseteq J, p > 0 \), we pose

\[ q^{(m)}_{K, -p}(f) = \sup_{t \in K, y \in \mathbb{R}^m} (1 + |y|)^{-p} |f(t, y)|^{(m)} \text{ with } |f(t, y)|^{(m)} = \sum_{k=1}^{m} |f_k(t, y)|, \]
and define, for \( f_{\lambda} = (f_{1, \lambda}, \ldots, f_{m, \lambda}) \)

\[ (\mathcal{H}^0_1(A \times \mathbb{K}^m, \mathbb{R}))^m = \left\{ (f_{x})^{\lambda} \in \left[ \mathcal{H}^0_1(A \times \mathbb{K}^m, \mathbb{R}) \right]^m \mid \forall K \subseteq J, \forall p > 0, (q^{(m)}_{K, -p}(f_{x})^{\lambda})_{\lambda} \in |I_A| \right\}, \]
\[ (\mathcal{H}^1_1(A \times \mathbb{K}^m, \mathbb{R}))^m = \left\{ (f_{x})^{\lambda} \in \left[ \mathcal{H}^0_1(A \times \mathbb{K}^m, \mathbb{R}) \right]^m \mid \forall K \subseteq J, \forall p > 0, (q^{(m)}_{K, -p}(f_{x})^{\lambda})_{\lambda} \in |I_A| \right\}. \]

In the same way when \( l = 0 \) or 1, with \( \mathfrak{E} = C^0(J, \mathbb{R}) \) and \( \mathfrak{P}^l = (p_{KL}^{(m)})_{K \subseteq J} \) such that

\[ p_{KL}^{(m)}(u) = \sup_{t \in K, 0 \leq j \leq l} |u_j(t)|, \]
we can re-write \( \mathcal{H}^{m^l}_{(A, \mathfrak{E}, \mathfrak{P}^l)} \) and \( \mathcal{H}^m_{(I_A, \mathfrak{E}, \mathfrak{P}^l)} \) as

\[ \left( \mathcal{H}_1^A(J, \mathbb{R}) \right)^m = \mathcal{H}^{m^l}_{(A, \mathfrak{E}, \mathfrak{P}^l)} = \left\{ (u_x)^{\lambda} \in \left[ \mathcal{E}^m \right]^A \mid \forall K \subseteq J, \left( p_{KL}^{(m)}(u_x)^{\lambda} \right)_{\lambda} \in |I_A| \right\}, \]
\[ \left( \mathcal{H}_{I_A}^A(J, \mathbb{R}) \right)^m = \mathcal{H}^{m^l}_{(I_A, \mathfrak{E}, \mathfrak{P}^l)} = \left\{ (u_x)^{\lambda} \in \left[ \mathcal{E}^m \right]^A \mid \forall K \subseteq J, \left( p_{KL}^{(m)}(u_x)^{\lambda} \right)_{\lambda} \in |I_A| \right\}. \]
Definition 11  Summarizing, we finally define for any \( m \in \mathbb{N} \)

\[
\begin{align*}
&\left( \mathcal{C}_{t,C}^0(J \times \mathbb{R}^m, \mathbb{R}) \right)^m = \left( \mathcal{H}_{t,A}^0(J \times \mathbb{R}^m, \mathbb{R}) \right)^m / \left( \mathcal{H}_{t,I}^0(J \times \mathbb{R}^m, \mathbb{R}) \right)^m, \\
&\left( \mathcal{C}_A^0(J, \mathbb{R}) \right)^m = \left( \mathcal{H}_A^0(J, \mathbb{R}) \right)^m / \left( \mathcal{H}_{I}^0(J, \mathbb{R}) \right)^m, \\
&\left( \mathcal{C}_J^0(J, \mathbb{R}) \right)^m = \left( \mathcal{H}_J^0(J, \mathbb{R}) \right)^m / \left( \mathcal{H}_{I}^0(J, \mathbb{R}) \right)^m \cap \left( \mathcal{H}_I^0(J, \mathbb{R}) \right)^m.
\end{align*}
\]

which leads to define as previously the map \( \partial = \partial \circ \iota^{-1} \) from \( \text{Im} \left( \mathcal{C}_A^0(J, \mathbb{R}) \right)^m \) to \( \left( \mathcal{C}_J^0(J, \mathbb{R}) \right)^m \).

Theorem 9  Let \( F = (F_1, \ldots, F_m) \in \left( \mathcal{C}_{t,C}^0(J \times \mathbb{R}^m, \mathbb{R}) \right)^m \) and \( u = (u_1, u_2, \ldots, u_m) \in \left( \mathcal{C}_J^0(J, \mathbb{R}) \right)^m \).

Then, \( F(., u) \) is a well defined element of \( \left( \mathcal{C}_J^0(J, \mathbb{R}) \right)^m \).

Proof.  \( F \) has for representatives \( (F_\lambda)_{\lambda} \) with \( F_\lambda : t \rightarrow F_\lambda(t, y) \). If \( u_\lambda \) is a representative of \( u \in \left( \mathcal{A}^0(J) \right)^m \), we have to prove that the family \( (t \rightarrow F_\lambda(t, u_\lambda(t)))_{\lambda} \) lies in \( \left( \mathcal{H}_{t,A}^0(J \times \mathbb{R}^m) \right)^m = \mathcal{H}^m_{(A, \mathcal{C}_0(J, \mathbb{R}), \mathcal{P}^0)} \). To simplify the proof, suppose that \( m = 1 \) for basic estimates.

With one hand, if \( u = [u_\lambda] \in \mathcal{A}^0(J) \), we have: \( \forall K \subseteq J, \left( \sup_{t \in K} |u_\lambda(t)| \right)_\lambda \in |A|, \) then

\[
\forall t \in K, |u_\lambda(t)| \leq \sup_{t \in K} |u_\lambda(t)| = |a_\lambda| \text{ with } a_\lambda \in A.
\]

As \( A \) is solid, \( (u_\lambda(t))_\lambda \) is in \( |A| \) (\( |u_\lambda(t)|_\lambda \) in \( |A| \)). Then, for any \( p > 0, ((1 + |u_\lambda(t)|)^p)_\lambda \in |A| \).

On the other hand, \( F \in \mathcal{A}_{t,C}^0(J \times \mathbb{R}) \) has for representatives \( (F_\lambda)_{\lambda} \) such that

\[
\exists p > 0, \sup_{t \in K, y \in \mathbb{R}} (1 + |y|)^{-p} |F_\lambda(t, y)| = |b_\lambda| \text{ with } (b_\lambda)_\lambda \in |A|.
\]

Then, \( \exists p > 0, \forall t \in K, |F_\lambda(t, u_\lambda(t))| \leq (1 + |u_\lambda(t)|)^p |b_\lambda| = |c_{p,\lambda}| \) with \( (c_{p,\lambda})_\lambda \in A \)

and from solidness of \( A, ((F_\lambda(t, u_\lambda(t)))_\lambda \in |A|. \) Then \( (t \mapsto (F_\lambda(t, u_\lambda(t)))_\lambda \in \mathcal{H}(A, \mathcal{C}^0(J, \mathbb{R}), \mathcal{P}^0). \)

If \( (v_\lambda)_{\lambda} \) is another representative of \( u \) and \( (G_\lambda) \) another representative of \( F \), it is easy to show that \( (t \mapsto (F_\lambda(t, u_\lambda(t)))_\lambda - (t \mapsto (G_\lambda(t, v_\lambda(t)))_\lambda \in \mathcal{H}(I_A, \mathcal{C}^0(J, \mathbb{R}), \mathcal{P}^0). \) Then \( F(., u) \) is a well defined element of \( \mathcal{C}_J^0(J) \). Similar estimates permits to replace \( \mathbb{R} \) by \( \mathbb{R}^m \), \( \mathcal{C}_0^0(J) \) by \( \left( \mathcal{C}^0_J(J, \mathbb{R}) \right)^m \) and \( \mathcal{C}_{t,C}^0(J \times \mathbb{R}) \) by \( \left( \mathcal{C}_{t,C}^0(J \times \mathbb{R}^m) \right)^m \). \( \blacksquare \)

Definition 12  Let \( f \in \left( \mathcal{C}_{t,C}^0(\mathbb{R}^+, \mathbb{R}^m) \right)^m \). We tell that \( f \) is globally Lipschitz if for any representative \( (f_\varepsilon)_\varepsilon \) of \( f \) we have (with \( ||x|| = \sum_{k=1}^{k=m} |x_k| \) when \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m \))

\[
\forall t \in \mathbb{R}^+, \forall \varepsilon \in [0, 1], \exists k_\varepsilon(t) > 0, \forall (v, w) \in \mathbb{R}^m \times \mathbb{R}^m, \left( ||f_\varepsilon((t, v) - f_\varepsilon((t, w))|| \leq k_\varepsilon(t)||v - w|| \right)
\]

with

\[
\forall T \in \mathbb{R}^+, \sup_{t \in [0, T]} k_\varepsilon(t) = M_{T,\varepsilon} < +\infty.
\]
4.2 The generalized Cauchy-Lipschitz problem

In terms of generalized functions the Cauchy-Lipschitz problem (4) can be defined as

Definition 13 The Cauchy-Lipschitz generalized problem is to solve (2) that is

\[
\begin{align*}
\dot{x} &= f(t, x) \\
x(t_0) &= \xi
\end{align*}
\]

with \( x \in \text{Im} \left( \mathcal{C}_0^1(J) \right)^m \subset \left( \mathcal{C}_0^0(J) \right)^m \) and \( f \in \left( \mathcal{C}_{t,C}^0(J \times \mathbb{R}^m) \right)^m \) globally Lipschitz, for some ring of generalized numbers \( \mathcal{C} = A/I_A \), with \( t_0 \in J \) and \( \xi \) is a given element in \( \mathbb{R}^m \).

It is classical to choose \( J = \mathbb{R}_+ \) when \( t \) is the time parameter, then we do that, without loss of generality. But we have exactly the same result when \( J \) is an open subset of \( \mathbb{R} \) and taking \( J = \mathbb{R} \) simplifies some estimates.

Theorem 10 It exists a ring of generalized numbers \( \mathcal{C} = A/I_A \), such that \( f \in \left( \mathcal{C}_{t,C}^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}) \right)^m \) and a map \( \Phi : \left( \mathcal{C}_0^0(\mathbb{R}_+, \mathbb{R}) \right)^m \to \left( \mathcal{C}_0^0(\mathbb{R}_+, \mathbb{R}) \right)^m \) with a fixed point solving the Cauchy-Lipschitz problem (2) with \( t_0 \in \mathbb{R}_+ \) and \( \xi \in \mathbb{R}^m \).

Proof. To construct the convenient ring of generalized numbers \( \mathcal{C} = A/I_A \) interacting with the construction of \( \Phi \) and its fixed point, we follow four steps to verify the hypotheses of Definition of a generalized contraction

• (a) For \( x_0 \in \mathbb{R}^m \), \( x \in \left( \mathcal{C}_0^0(\mathbb{R}_+, \mathbb{R}) \right)^m \), \( t \in \mathbb{R}_+ \) and \( f_\varepsilon \in \left( \mathcal{C}_0^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}) \right)^m \) we pose

\[
\Phi_\varepsilon (x)(t) = x_0 + \int_0^t f_\varepsilon(s, x(s))ds
\]

from what it is clear that \( \Phi_\varepsilon \) is a map \( \mathcal{E}^m \to \mathcal{E}^m \) with \( \mathcal{E} = \mathcal{C}_0^0(\mathbb{R}_+, \mathbb{R}) \).

\( (\mathcal{E}^m, \tau^m) \) is here a topological space where \( \tau^m \) is given by the family of norms \( \left( p_T^{(m)} \right)_{T \in \mathbb{R}_+} \) such that \( p_T^{(m)}(x) = \sup_{t \in [0, T]} ||x(t)|| \). We suppose that it exists a family \( (a_\varepsilon)_\varepsilon \in \mathcal{E}^{[0,1]} = \mathcal{C}_0^0(\mathbb{R}_+, \mathbb{R})^{[0,1]} \) such that

\[
\begin{align*}
\forall T &\in \mathbb{R}_+, (p_T (a_\varepsilon))_\varepsilon \in |A| \\
\forall t &\in \mathbb{R}_+, ||f_\varepsilon(t, x(t))|| \leq |a_\varepsilon(t)||x(t)||.
\end{align*}
\]

Then, we have

\[
||\Phi_\varepsilon (x)(t)|| \leq ||x_0|| + \int_0^t |a_\varepsilon(s)||x(s)||ds \leq ||x_0|| + |a_\varepsilon(r)||x_0|| + \int_0^t |x(s)||ds, r \in [0, t]
\]

which leads to \( p_T^{(m)}(\Phi_\varepsilon (x)) \leq ||x_0|| + T p_T (a_\varepsilon) p_T^{(m)}(x) \).

Then if \( (x_\varepsilon)_\varepsilon \in \mathcal{H}^m_{(A, \mathcal{C}_0^0(\mathbb{R}_+), \mathcal{P}^0)} \), that is to say \( \left( p_T^{(m)} (x_\varepsilon) \right)_\varepsilon \in |A| \), we have

\[
\left( p_T^{(m)} (\Phi_\varepsilon (x_\varepsilon)) \right)_\varepsilon \leq (||x_0||)_\varepsilon + T (p_T (a_\varepsilon))_\varepsilon \left( p_T^{(m)} (x_\varepsilon) \right)_\varepsilon
\]

which leads (from \( \exists n \in \mathbb{N} \) such that \( T(p_T (a_\varepsilon))_\varepsilon \leq n(p_T (a_\varepsilon))_\varepsilon \) to \( \left(p_T^{(m)}(\Phi_\varepsilon (x_\varepsilon))\right)_\varepsilon \in |A| \) that is to say \( (\Phi_\varepsilon (x_\varepsilon))_\varepsilon \in \mathcal{H}^m_{(A, \mathcal{C}_0^0(\mathbb{R}_+), \mathcal{P}^0)} \).

• (b) Putting \( \Delta = [0, 1] \), \( t_0 = 0 \) and \( \lambda = \varepsilon \), we first have to write (2) in term of representatives \( k_\varepsilon (t) \)

\[
\begin{align*}
x_\varepsilon'(t) &= f_\varepsilon(t, x_\varepsilon(t)) \\
x_\varepsilon(0) &= \xi_\varepsilon
\end{align*}
\]

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where \(|\|\xi_\varepsilon\|\|_\varepsilon \in |A|\).

We recall that if \((f_\varepsilon)_\varepsilon\) is a representative of \(f \in \left(\mathcal{C}^0_{\mathcal{T},\mathcal{E}}(\mathbb{R}_+ \times \mathbb{R}^m)\right)^m\), we have
\[
f_\varepsilon \in \left(C^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R})\right)^m \text{ and } \forall K \in J, \exists p > 0, \left(q^{(m)}_{K, -p}(f_\varepsilon)_\varepsilon \right) \in |A| \text{ and indenpently of this estimate we have to add the following one}
\]
\[
\forall t \in \mathbb{R}_+, \forall \varepsilon \in [0, 1], \exists k_\varepsilon (t) > 0, \forall (v, w) \in \mathbb{R}^m \times \mathbb{R}^m, ||f_\varepsilon ((t, v) - f_\varepsilon ((t, w))|| \leq k_\varepsilon (t) ||v - w||
\]

with
\[
\forall T \in \mathbb{R}_+, \sup_{t \in [0, T]} k_\varepsilon (t) = M_{T, \varepsilon} < +\infty.
\]

For \(x_0\) given in \(\mathbb{R}^m, x \in \left(C^0(\mathbb{R}_+, \mathbb{R})\right)^m, t \in \mathbb{R}_+\) and \(f_\varepsilon \in \left(C^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R})\right)^m\) we pose
\[
\Phi_\varepsilon (x) (t) = x_0 + \int_0^t f_\varepsilon (s, x(s)) ds
\]
from what it is clear that \(\Phi_\varepsilon\) is a map \(\mathcal{E}^m \to \mathcal{E}^m\) with \(\mathcal{E} = C^0(\mathbb{R}_+, \mathbb{R})\).

The natural topology (denoted \(\tau^m\)) on the product \(\mathcal{E}^m\) is here defined by the family \(\left(p_T^{(m)}\right)_{T \in \mathbb{R}_+}\) of seminorms such that for \(x = (x_1, ..., x_m)\), \(p_T^{(m)} (x) = \sum_{k=1}^{k=m} p_T (x_k)\) with \(p_T (x_k) = \sup_{t \in [0, T]} |x_k (t)|\).

Denote by \((\mathcal{E}^m, \tau^m_\varepsilon)\) the topological space \(\mathcal{E}^m\) endowed by the family \(\left(q^{(m)}_{T, \varepsilon}\right)_{T \in \mathbb{R}_+}\) of seminorm on \(\mathcal{E} = C^0(\mathbb{R}_+, \mathbb{R})\) such that for each \(\varepsilon \in [0, 1]\) and \(y \in C^0(\mathbb{R}_+, \mathbb{R})\) we have \(q^{(m)}_{T, \varepsilon} (y) = \sup_{t \in [0, T]} (|y(t)| e^{-T M_{T, \varepsilon}})\). Then \(q^{(m)}_{T, \varepsilon} (x) = \sup_{t \in [0, T]} (||x(t)|| e^{-T M_{T, \varepsilon}})\)

We claim that the map \(\Phi_\varepsilon\) is a contraction in \((\mathcal{E}^m, \tau^m_\varepsilon)\). From Definition 12, we have to prove that for each \((T, \varepsilon) \in \mathbb{R}_+ \times [0, 1]\) it exists a constant \(k_{T, \varepsilon} \in [0, 1]\) such that for all \((x, y) \in \mathcal{E}^m \times \mathcal{E}^m\)
\[
q^{(m)}_{T, \varepsilon} \Phi_\varepsilon (x) - \Phi_\varepsilon (y) \leq k_{T, \varepsilon} q^{(m)}_{T, \varepsilon} (x - y).
\]
We have, for each \(t \in \mathbb{R}_+
\[
\Phi_\varepsilon (x) (t) - \Phi_\varepsilon (y) (t) = x_0 + \int_0^t (f_\varepsilon (s, x(s)) - f_\varepsilon (s, y(s))) ds
\]
from what we deduce
\[
e^{-T M_{T, \varepsilon}} ||\Phi_\varepsilon (x) (t) - \Phi_\varepsilon (y) (t)|| \leq e^{-T M_{T, \varepsilon}} \int_0^t ||f_\varepsilon (s, x(s)) - f_\varepsilon (s, y(s))|| ds
\]
and, for \(t \in [0, T]\)
\[
e^{-T M_{T, \varepsilon}} ||\Phi_\varepsilon (x) (t) - \Phi_\varepsilon (y) (t)|| \leq e^{-T M_{T, \varepsilon}} \int_0^t M_{T, \varepsilon} ||x(s) - y(s)|| ds.
\]

Writing now
\[
e^{-T M_{T, \varepsilon}} \int_0^t M_{T, \varepsilon} ||x(s) - y(s)|| ds = e^{-T M_{T, \varepsilon}} \int_0^t M_{T, \varepsilon} e^{s M_{T, \varepsilon}} (e^{-s M_{T, \varepsilon}} ||x(s) - y(s)||) ds
\]
we obtain
\[
e^{-T M_{T, \varepsilon}} \int_0^t M_{T, \varepsilon} e^{s M_{T, \varepsilon}} (e^{-s M_{T, \varepsilon}} ||x(s) - y(s)||) ds \leq e^{-T M_{T, \varepsilon}} \int_0^t M_{T, \varepsilon} e^{s M_{T, \varepsilon}} q^{(m)}_{T, \varepsilon} (x - y) ds
\]
which leads to
\[ e^{-i\lambda T, s} \| \Phi_\varepsilon (x) (t) - \Phi_\varepsilon (y) (t) \| \leq e^{-i\lambda T, s} q_{T, \varepsilon}^{(m)} (x - y) \left( e^{i\lambda T, s} - 1 \right) = (1 - e^{-i\lambda T, s}) q_{T, \varepsilon}^{(m)} (x - y). \]

When taking \( \sup_{t \in [0, T]} \) we finally obtain a constant \( k_{T, \varepsilon} = (1 - e^{-i\lambda T, s}) < 1 \) such that

\[ q_{T, \varepsilon}^{(m)} (\Phi_\varepsilon (x) - \Phi_\varepsilon (y)) \leq k_{T, \varepsilon} q_{T, \varepsilon}^{(m)} (x - y). \]

Then according to Definition 12, \( \Phi_\varepsilon \) is a contraction in \( E^m \) and has an unique fixed point \( z_\varepsilon \) from Theorem 6.

- (c) We can write for \( x \in E^m \)

\[ e^{-i\lambda T, s} \sup_{t \in [0, T]} \| x(t) \| \leq \sup_{t \in [0, T]} \left( \| x(t) \| e^{-i\lambda T, s} \right) \leq \sup_{t \in [0, T]} \| x(t) \|. \]

then

\[ e^{-i\lambda T, s} p_T^{(m)} \leq q_{T, \varepsilon}^{(m)} \leq p_T^{(m)}. \]

- (d). Assume now that for each \( T \in \mathbb{R}_+ \) the family \( (e^{i\lambda T, s})_\varepsilon \) lies in \( |A| \) and recall that we have asked in (a) that \( (p_T (a_\varepsilon))_\varepsilon \in |A| \). As \( k_{T, \varepsilon} = (1 - e^{-i\lambda T, s}) \) we have \( \left( \frac{1}{1 - k_{T, \varepsilon}} \right)_\varepsilon = (e^{i\lambda T, s})_\varepsilon. \)

Define now

\[ \Phi_\varepsilon (x_\varepsilon) (t) = \xi_\varepsilon + \int_0^t f_\varepsilon (s, x_\varepsilon (s)) ds \]

where \( (\xi_\varepsilon)_\varepsilon \) is a given representative of the given element \( \xi \in \mathbb{R}^m \)

Finally, from Definition 8, the map \( \Phi : (C_0^0 (\mathbb{R}_+, \mathbb{R}))^m \to (C_0^0 (\mathbb{R}_+, \mathbb{R}))^m \) such that

\[ [x_\varepsilon] \mapsto [\Phi_\varepsilon (x_\varepsilon)] \]

is a contraction, with \( z = [z_\varepsilon] \) as fixed point from Theorem 8, \( z_\varepsilon \) being the unique fixed point of \( \Phi_\varepsilon \) verifying

\[ z_\varepsilon (t) = \xi_\varepsilon + \int_0^t f_\varepsilon (s, z_\varepsilon (s)) ds. \]

Then, \( z_\varepsilon' (t) = f_\varepsilon (t, z_\varepsilon (t)) \). and \( z_\varepsilon (0) = \xi_\varepsilon \). Moreover \( z \) is solution to the Cauchy-Lipschitz problem (2) given in Definition 13.

We are going to prove that \( z \) is the unique fixed point of \( \Phi \), and therefore the unique solution of (1).

If \( y = [y_\varepsilon] \) is another fixed point of \( \Phi \), we have \( y_\varepsilon = \Phi_\varepsilon (y_\varepsilon) + i_\varepsilon \) with \( (i_\varepsilon)_\varepsilon \in \mathcal{H}^{(m)}_{(I_A, C^0 (\mathbb{R}_+, \mathbb{R}), \mathcal{P}^0)} \), that is to say \( (p_T^{(m)} (i_\varepsilon))_\varepsilon \in I_A \). Writing now

\[ z_\varepsilon (t) - y_\varepsilon (t) = \int_0^t (f_\varepsilon (s, z_\varepsilon (s)) - f_\varepsilon (s, y_\varepsilon (s))) ds + i_\varepsilon (t) \]

we have

\[ ||z_\varepsilon - y_\varepsilon (t)|| \leq \int_0^t ||f_\varepsilon (s, z_\varepsilon (s)) - f_\varepsilon (s, y_\varepsilon (s))|| ds + ||i_\varepsilon (t)|| \]

\[ ||z_\varepsilon - y_\varepsilon (t)|| \leq ||i_\varepsilon (t)|| + \int_0^t M_{T, \varepsilon} ||z_\varepsilon (s) - y_\varepsilon (s)|| ds \]

and, from Gronwall lemma, for \( t \in [0, T] \)

\[ ||z_\varepsilon (t) - y_\varepsilon (t)|| \leq ||i_\varepsilon (t)|| e^{T M_{T, s}} \]

\[ p_T^{(m)} (z_\varepsilon - y_\varepsilon) \leq p_T^{(m)} (i_\varepsilon) e^{T M_{T, s}}. \]
The Cauchy-Lipschitz condition.

Existence of globally Lipschitz for with $M$

Construction of the factor ring $C = A/I_A$.

Example 2 The Cauchy-Lipschitz generalized problem to solve is given as

\[
\begin{aligned}
\partial x &= \Delta(\cdot, x) \\
x(0) &= \xi
\end{aligned}
\]  

with $x \in \mathcal{C}_0^0(\mathbb{R}, \mathbb{R})$, with $t_0 \in \mathbb{R}$ and $\xi$ is a given element $\in \mathbb{R}$. We define $\Delta$ from $\Delta_\varepsilon(t, x) = \varphi_\varepsilon(t) \varphi_\varepsilon(x)$ when $\varphi_\varepsilon$ is the "standard mollifier" $\varphi_\varepsilon(.) = \varphi(\frac{t}{\varepsilon})$, $\varphi \in C^1(\mathbb{R})$, $\text{supp}\varphi = [-1, 1]$, $\int \varphi(s) \, ds = 1$.

1. Existence of $\Delta$.

We begin to take $A$ as containing some elements such that $\Delta$ lies in $\mathcal{C}_0^0(\mathbb{R} \times \mathbb{R})$ and is globally Lipschitz for $C = A/I_A$. From $\Delta_\varepsilon(t, x) = \varphi_\varepsilon(t) \varphi_\varepsilon(x)$ we have, for any $p \geq 0$

\[
|\Delta_\varepsilon(t, x)| \leq \frac{1}{\varepsilon^2} \left| \varphi\left(\frac{t}{\varepsilon}\right) \varphi\left(\frac{x}{\varepsilon}\right) \right| \leq \frac{M^2}{\varepsilon^2} \leq \frac{M^2}{\varepsilon^2} (1 + |x|)^p
\]

with $M = \sup_{s \in \mathbb{R}} |\varphi(s)|$. Then

\[
\forall T \geq 0, \forall p \geq 0, q_{T-p}(\Delta_\varepsilon) \leq \frac{M^2}{\varepsilon^2}
\]

with $q_{T-p}(\Delta_\varepsilon) = \sup_{t \in [-T,T], x \in \mathbb{R}} (1 + |x|)^{-p} |\Delta_\varepsilon(t, x)|$.

We construct here $A$ as a set of families of real elements, being a solid unitary subring of $\mathbb{R}^{[0,1]}$, and such that the family $(\varepsilon)_\varepsilon \in A$ (or $|A|$). As exists $n \in \mathbb{N}$ such that $M^2 \leq n$ and $(\frac{n}{\varepsilon})_\varepsilon \in |A|$, then we have: $(q_{T-p}(\Delta_\varepsilon))_\varepsilon \leq |A|$ that is to say $\Delta$ lies in $\mathcal{C}_0^0(\mathbb{R} \times \mathbb{R})$.

2. The Cauchy-Lipschitz condition.

We have to prove that

\[
\forall \varepsilon > 0, \forall T \in \mathbb{R}, \exists M_{T,\varepsilon} > 0, \sup_{t \in [-T,T], (v, w) \in \mathbb{R}^2} |\Delta_\varepsilon(t, v) - \Delta_\varepsilon(t, w)| \leq M_{T,\varepsilon} |v - w|.
\]

Writing

\[
\Delta_\varepsilon(t, v) - \Delta_\varepsilon(t, w) = \frac{1}{\varepsilon^2} \varphi\left(\frac{t}{\varepsilon}\right) \left[ \varphi\left(\frac{v}{\varepsilon}\right) - \varphi\left(\frac{w}{\varepsilon}\right) \right] = \frac{1}{\varepsilon^2} \varphi\left(\frac{t}{\varepsilon}\right) \varphi'(\frac{v}{\varepsilon}) \frac{v - w}{\varepsilon}
\]

with $r \in \left(\frac{v}{\varepsilon}, \frac{w}{\varepsilon}\right)$ we obtain

\[
\sup_{t \in [-T,T], (v, w) \in \mathbb{R}^2} |\Delta_\varepsilon(t, v) - \Delta_\varepsilon(t, w)| \leq \frac{MM'}{\varepsilon^3} |v - w|
\]

with $M' = \sup_{s \in \mathbb{R}} |\varphi'(s)|$.

3. Construction of the factor ring $C = A/I_A$. 

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From the (d) point of Theorem 10, we have to choose $A$ containing $\left( e \frac{TMM'}{e_\varepsilon^2} \right)$. As it exists $n \in \mathbb{N}$ such that $TMM' \leq n$, we have $e \frac{TMM'}{e_\varepsilon^2} = \left( e \frac{1}{e_\varepsilon^2} \right)^n$. As $A$ is a solid subring, we only have to require that $\left( e \frac{1}{e_\varepsilon^2} \right) \in |A|$, that is to say $\left( e \frac{1}{e_\varepsilon^2} \right) \in |A|$, and as we have $\frac{1}{\varepsilon} \leq e \frac{1}{e_\varepsilon^2}$, then $\left( e \frac{1}{e_\varepsilon} \right)$ and $(\varepsilon)_\varepsilon$ also are in $|A|$. But $e \frac{1}{e_\varepsilon^2} \leq e \frac{1}{e_\varepsilon^2}$ and the previous statements are obviously verified if we require only that $\left( e \frac{1}{e_\varepsilon} \right) \in |A|$. Then it suffice to take $C$ "overgenerated " by the family $\left( e \frac{1}{e_\varepsilon} \right)$ in the meaning of Definition 3.

- Construction of the algebra $C^0(R, R) = H^0_A(R, R)/H^0_{IA}(R, R)$.

We recall that

$H^0_A(R, R) = \left\{ (u_\varepsilon)_\varepsilon \in \left( C^0(R, R) \right)^{0,1} \mid \forall K \in R, \left( (\rho K, 0)(u_\varepsilon)_\varepsilon \right) \in |A| \right\}$,

$H^0_{IA}(R, R) = \left\{ (u_\varepsilon)_\varepsilon \in \left( C^0(R, R) \right)^{0,1} \mid \forall K \in R, \left( (\rho K, 0)(u_\varepsilon)_\varepsilon \right) \in |IA| \right\}$.

We explicit the construction of $A$ and $I_A$ "overgenerated " by the family $\left( e \frac{1}{e_\varepsilon} \right)$. First, consider $B$ the subset of elements in $(R_+^*)^{0,1}$ obtained as rational fractions with coefficients in $R_+^*$, of $e_\varepsilon$ as variable. It follows that

$A = \left\{ (a_\varepsilon)_\varepsilon \in R^{0,1} \mid \exists (b_\varepsilon)_\varepsilon \in B, \exists \varepsilon_0 \in [0, 1], \forall \varepsilon \prec \varepsilon_0 : |a_\varepsilon| \leq b_\varepsilon \right\}$

$I_A = \left\{ (a_\varepsilon)_\varepsilon \in R^{0,1} \mid \forall (b_\varepsilon)_\varepsilon \in B, \exists \varepsilon_0 \in [0, 1], \forall \varepsilon \prec \varepsilon_0 : |a_\varepsilon| \leq b_\varepsilon \right\}$

- Fixed point of the map $\Phi$ and solution to (6).

We know that the map $\Phi : C^0(R_+, R) \to C^0(R_+, R)$ such that

$[x_\varepsilon] \mapsto [\Phi_\varepsilon(x_\varepsilon)]$

is a contraction, with $z = [z_\varepsilon]$ as fixed point, $z_\varepsilon$ being the unique fixed point of $\Phi_\varepsilon$ verifying

$z_\varepsilon(t) = \xi_\varepsilon + \int_0^t f_\varepsilon(s, z_\varepsilon(s))ds$

where $(\xi_\varepsilon)_\varepsilon$ is a given representative of the given element $\xi \in \widetilde{R}$. Moreover $z$ is the unique solution to the Cauchy-Lipschitz problem (6).

4.3 Towards the transport equation with irregular coefficients

We consider the following Cauchy problem for the transport equation in $(t, x)$-variables

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = f, \quad u(0, x) = u_0(x)$$

In a general case (typically the distribution one), the nonlinear term needs some regularity to be defined. And it is not the case if $\alpha$ is a distribution in $(t, x)$-variables. Under some hypotheses on $\alpha$: weak regularity (of Sobolev type), control of its uniform divergency in space and some
space increasing condition, Di Perna and Lions [5] have obtained some results with uniqueness and stability. More recently the paper of L. Ambrosio [1] studies the same subject.

However this context don’t permits to define and solve (7) when \( \alpha \) is a distribution in \((t, x)\)-variables.

In a simplified case where \( f = 0 \) and \( \alpha = a_t \otimes 1_x \) where \( a_t \in \mathcal{D}'(\mathbb{R}) \), the problem is posed and solved in [4]. When \( \alpha \in \mathcal{D}'(\mathbb{R}^2) \) we turn back to the regular case as the starting point of our generalized methods. It is well known that when \( \alpha, f \) and \( u_0 \) are of class \( C^1 \), the problem (7) admits a unique solution of class \( C^1 \) given by integrating along the characteristics

\[
(8) \quad u(t, x) = u_0(X(0, t, x) + \int_0^t f(s, X(s, t, x))ds.
\]

We can see that the regular solution of (7) is linked to the following Cauchy-Lipschitz problem of which \( X(s, t, x) \) is the unique solution

\[
\begin{align*}
\frac{dX}{ds}(s, t, x) &= \alpha(s, X(s, t, x)) \\
X(t, t, x) &= x.
\end{align*}
\]

When \( \alpha \) and \( f \) are not continuous, N. Caroff [3] propose an approach based on the approximation of discontinuous data by \( C^1 \) function \( \alpha_n \) and \( f_n \) and an Egorov theorem. She gives a result similar to (8) when the irregularities are controled by: \( u_0 \in \text{Lip}_0(\mathbb{R}, \mathbb{R}); \alpha \in L^\infty(\mathbb{R}^2, \mathbb{R}) \) and for some \( \delta > 0, \delta^{-1} \leq \alpha(t, x) \leq \delta; f \in L^\infty(\mathbb{R}^2, \mathbb{R}); \forall x \in \mathbb{R}, \alpha(., x) \) and \( f(., x) \) are locally Lipschitz uniformly in \( x \).

But as in the previous case, this context don’t permit to define and solve (7) when \( \alpha \) is a distribution in \((t, x)\)-variables.

With the generalized methods over exposed we are trying to generalize that result to the distributional case in consideration.

References


