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Jean-André Marti

Centre d'Etude et de Recherche en Economie, Gestion, Modélisation et Informatique Appliquée

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Jean-André Marti

Université des Antilles et de la Guyane. Faculté de Droit et d'Economie de la Martinique. Campus de Schoelcher - Martinique FWI
B.P. 7209 - 97275 Schoelcher Cedex - Tél. : 0596. 72.74.00 - Fax. : 0596. 72.74.03

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J.-A. Marti
Laboratoire CEREGMIA, Université des Antilles

Abstract

I propose a self contained research paper. I hope it adds some news ideas and results to the fixed point theory in the framework of generalized functions algebras, with application to the Cauchy-Lipschitz problem in a generalized formulation including strongly irregular cases. This leads to the transport equation with distributions as coefficients we wish to treat later.

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1 Introduction

The paper is structured in three following sections (2, 3, 4).

Section 2 is devoted to the meaning of contraction in locally convex spaces or algebras. Fixed points of some operators F with a contraction property in some spaces (or algebras) E are involved to solve many problems in functional analysis. There are at least four journals on Fixed Point Theory, many publications on the subject (between [12] to [2]) and many books as the monograph [7]. However we are interested in the classical application to the Cauchy-Lipschitz theorem locally or globally formulated. Then the definition of a contraction we need is only a slight generalization (we suppose that E is locally convex) of the Frigon-Granas one ([6]) given when E is a Fréchet space. It leads to the expected result given in Theorem 1: *Any contraction $F : E \to E$ has a fixed point. If E is Hausdorff, this fixed point is unique.*

But the irregular cases of the Cauchy-Lipschitz theorem suggests a generalized formulation which is the subject of **Section 3** and invites to define some operator Φ in a factor algebra $\mathcal{A}$ of generalized functions. $\mathcal{A}$ is constructed ([10]) from a basic locally convex algebra $(\mathcal{E}, \tau)$. The elements $x \in \mathcal{A}$ are classes $[x_\lambda]$ of some families $(x_\lambda)_{\lambda\in\Lambda}$ with "moderateness" linked to a factor ring $\mathcal{C}$ of so-called generalized numbers. Under some hypotheses, Φ is well defined by

$$\mathcal{A} \ni [x_\lambda] = x \to \Phi(x) = [\Phi_\lambda(x_\lambda)] \in \mathcal{A}.$$

for some operator Φ_λ in $\mathcal{E}$. We suppose in addition that each Φ_λ is a contraction in $(\mathcal{E}, \tau_\lambda)$ endowed with a locally convex toplology τ_λ depending on λ. Then Φ_λ has a fixed point z_λ in $\mathcal{E}$ and unique if $(\mathcal{E}, \tau_\lambda)$ is Haudorff. This leads to define Φ as a contraction in $\mathcal{A}$ (Definition 4). I don't see any similar idea in the framework of generalized functions. I hope that it a good enough (or not too bad) one! Moreover with some additional hypotheses, we can prove the moderateness of $(z_\lambda)_\lambda$ and find (Theorem 6) a fixed point z of Φ through

$$\mathcal{A} \ni [z_\lambda] = z = \Phi(z) = [\Phi_\lambda(z_\lambda)] \in \mathcal{A} \tag{1}$$

But the uniqueness of z_λ is not sufficient to prove that $[z_\lambda] = z$ is the unique fixed point of Φ. Nevertheless we can obtain this uniqueness when taking, as in Theorem 10

$$\Phi_\lambda(x)(t) = x_0 + \int_0^t f_\lambda(s, x(s))ds$$

for x_0 given in $\mathbb{R}$, $x \in \mathrm{C}^0(\mathbb{R}_+, \mathbb{R})$, $t \in \mathbb{R}_+$ and $f_\lambda \in \mathrm{C}^0(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ from what it is clear that Φ_λ is a map $\mathcal{E} \to \mathcal{E}$ with $\mathcal{E} = \mathrm{C}^0(\mathbb{R}_+, \mathbb{R})$.

In a Subsection (of Section 3) we extend the results to the case where Φ is an operator in the product $\mathcal{A}^m$ of algebras constructed on $\mathcal{E}^m$. The natural topology (denoted τ^m) on the product $\mathcal{E}^m$ is defined by the family $\left(p_i^{(m)}\right)_{i\in I}$ of seminorms such that $p_i^{(m)}(x) = p_i^{(m)}(x_1 ... x_m) = \sum_{k=1}^{k=m} p_i(x_k)$. We denote by $(\mathcal{E}^m, \tau_\lambda^m)$ the topological space $\mathcal{E}^m$ endowed by the family $\left(q_{\lambda,i}^{(m)}\right)_{i\in I}$ with $q_{\lambda,i}^{(m)}(x) = \sum_{k=1}^{k=m} q_{\lambda,i}(x_k)$ for a given family $(q_{\lambda,i})_{i\in I}$ of seminorm on $\mathcal{E}$. The main result of that section is Theorem 8: *Any contraction $\Phi : \mathcal{A}^m \to \mathcal{A}^m$ has a fixed point in $\mathcal{A}^m$*

The expected application is the Cauchy-Lipschitz generalized problem studied in **Section 4**. Thanks to an embedding $\left(\mathfrak{C}_{\mathcal{C}}^1(J,\mathbb{R})\right)^m \to \left(\mathfrak{C}_{\mathcal{C}}^0(J,\mathbb{R})\right)^m$ with $\mathrm{Im}\left(\mathfrak{C}_{\mathcal{C}}^1(J,\mathbb{R})\right)^m$ as image, it is to solve

$$(2) \qquad \left\{ \begin{array}{c} \partial x = f(.,x) \\ x(t_0) = \xi \end{array} \right.$$

with $x \in \mathrm{Im}\left(\mathfrak{C}_{\mathcal{C}}^1(J,\mathbb{R})\right)^m \subset \left(\mathfrak{C}_{\mathcal{C}}^0(J,\mathbb{R})\right)^m$ and $f \in \left(\mathfrak{C}_{\tau,\mathcal{C}}^0(J \times \mathbb{R}^m, \mathbb{R})\right)^m$ globally Lipschitz, for some ring of generalized numbers $\mathcal{C} = A/I_A$, with $t_0 \in J$ and ξ is a given element $\in \widetilde{\mathbb{R}^m}$. The "derivation" ∂ is a map from $\mathrm{Im}\left(\mathfrak{C}_{\mathcal{C}}^1(J,\mathbb{R})\right)^m$ to $\left(\mathfrak{C}_{\mathcal{C}}^0(J,\mathbb{R})\right)^m$. The algebra $\left(\mathfrak{C}_{\mathcal{C}}^0(J,\mathbb{R})\right)^m$ (resp. $\left(\mathfrak{C}_{\mathcal{C}}^1(J,\mathbb{R})\right)^m$) generalize $\left(\mathrm{C}^0(J,\mathbb{R})\right)^m$ (resp. $\left(\mathrm{C}^1(J,\mathbb{R})\right)^m$and $\left(\mathfrak{C}_{\tau,\mathcal{C}}^0(J \times \mathbb{R}^m, \mathbb{R})\right)^m$is a generalization of $\left(\mathrm{C}^0(J \times \mathbb{R}^m, \mathbb{R})\right)^m$without use of derivatives.

The main result of that section (Theorem 10) is *that it exists a ring of generalized numbers $C = A/I_A$, such that $f \in \left(\mathfrak{C}_{\tau,\mathcal{C}}^0(J \times \mathbb{R}^m, \mathbb{R})\right)^m$and a map $\Phi : \left(\mathfrak{C}_{\mathcal{C}}^0(J,\mathbb{R})\right)^m \to \left(\mathfrak{C}_{\mathcal{C}}^0(J,\mathbb{R})\right)^m$ with an unique fixed point solving the Cauchy-Lipschitz problem* (2) *with $t_0 \in R_+$ and $\xi \in \widetilde{\mathbb{R}^m}$.*

All our ideas, technics and results are explicitely detailed and summarized in the final example (Example 2).

The last subsection shows a link between the Cauchy-Lipschitz theorem and the transport equation. We cite some results when the coefficients have a weak regularity of Sobolev type ([5]) or with controlled irregularities [3]. But it is not the case of distributions we wish to treat later with our generalized methods.

2 Contractions in locally convex and complete spaces

We suppose here tat the (seminormed with $\mathcal{P} = (p_i)_{i\in I}$) space E is sequentially complete. A basis of 0-neighbourhood is the set of all "balls" of the seminorms $(p_i)_{i\in I}$

$$\beta(i,r) = \{x \in E / p_i(x) < r\}$$

for all $i \in I$ and $r > 0$. Then, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence iff

$$(\forall \varepsilon > 0)\,(\forall i \in I)\,(\exists n_0)\,(\forall n, p)\,(n > n_0, p > 0 \Longrightarrow p_i(x_{n+p} - x_n) < \varepsilon)$$

and E is sequentially complete if any Cauchy sequence converges to an element e in E.

Definition 1 *A map $F : E \to E$ is called a* ***contraction*** *if for all $i \in I$ it exists $k_i < 1$ such that*

$$\forall (x,y) \in E \times E, p_i(F(x) - F(y)) \leq k_i p_i(x-y).$$

This definition is an obvious generalization of the Frigon-Granas one [6] given when $\mathcal{P}$ is a countable family of semi norms $(p_i)_{i\in\mathbb{N}}$ rending E a Fréchet space.

In this case, F is not necessary a contraction in the usual sense when E is endowed with the metric $d(x,y) = \sum_{i\in N} p_i(x-y)/\left(1 + p_i(x-y)\right)$.

Theorem 1 *Any contraction $F : E \to E$ hax a fixed point. If E is Hausdorff, this fixed point is unique.*

Proof. Starting from $x_0 \in E$, define $x_{n+1} = F(x_n)$ by induction. It is easy to verify that x_n is a Cauchy sequence in the complete space E and converges to some $x \in E$. The contraction property of the map F implies obviously its continuity. Then, passing to the limit in $x_{n+1} = F(x_n)$, we obtain that x is a fixed point of E. If E is Haudorff, for all $z \neq 0$ it exists $V \in \mathcal{V}(0)$ such that $z \notin V$. Then it exists i (depending on z) such that $p_i(z) > 0$. If x and y are two different fixed points of F, it exists j (depending on $x-y$) such that

$$0 < p_j(x-y) = p_j(F(x) - F(y)) \leq k_j p_j(x-y) < p_i(x-y)$$

which gives a contradiction ■

Notation 1 *We denote by*
- Λ *a set of indices*
- (E,τ) *the space E endowed with the toplogy τ of the previous family $\mathcal{P} = (p_i)_{i\in I}$*
- $(E,\tau_\lambda)_{\lambda\in\Lambda}$ *the family of spaces (E,τ_λ) seminormed by the family $\mathcal{Q}_\lambda = (q_{\lambda,i})_{i\in I}$*
- $(F_\lambda)_{\lambda\in\Lambda}$ *a family of contractions $F_\lambda : (E,\tau_\lambda) \to (E,\tau_\lambda)$.*

Theorem 2 *Each F_λ has a fixed point $z_\lambda \in E$. If in addition we suppose that (E,τ) is Hausdorff and for each $i \in I$ and $\lambda \in \Lambda$ it exists a strictly positive constant $a_{\lambda,i}$ such that*

$$a_{\lambda,i} p_i \leq q_{\lambda,i}$$

then z_λ is unique

Proof. From Theorem 1 we know that each F_λ has a fixed point $z_\lambda \in E$. If (E,τ) is Hausdorff, for each $x \neq 0$ in E, it exists $i(x) \in I$ such that $p_i(x) > 0$. Then we have $q_{\lambda,i}(x) > 0$ which implies that (E,τ_λ) is Hausdorff. As $F_\lambda : (E,\tau_\lambda) \to (E,\tau_\lambda)$ is a contraction, z_λ is unique. ■

3 Contractions in generalized spaces or algebras

3.1 The $(\mathcal{C},\mathcal{E},\mathcal{P})$ setting

We consider the setting of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras which is an extension of Colombeau's one. It allows to construct multiparametrized generalized spaces or algebras where the "asymptotic $\mathcal{C}$" is given independantly from the basis topological space or algebra $\mathcal{C}$. To summarize the definitions and results given in [10, 11] the asymptotics is given by
(1) Λ: a set of indices;
(2) A: a solid subring of the ring $\mathbb{K}^\Lambda$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$); this means that whenever $(|s_\lambda|)_\lambda \leq (|r_\lambda|)_\lambda$ for some $((s_\lambda)_\lambda, (r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times A$, that is, $|s_\lambda| \leq |r_\lambda|$ for all λ, it follows that $(s_\lambda)_\lambda \in A$;

(3) I_A: a solid ideal of A.

Then $\mathcal{C}$ is defined as the factor ring A/I_A. On the other hand we give

(4) $\mathcal{E}$: a $\mathbb{K}$-topological space endowed with a family $\mathcal{P} = (p_i)_{i\in I}$ of semi-norms.

Define $|B| = \{(|r_\lambda|)_\lambda\,,\ (r_\lambda)_\lambda \in B\}$, $B = A$ or I_A, and set

$$\mathcal{H}_{(A,\mathcal{E},\mathcal{P})} = \left\{(u_\lambda)_\lambda \in [\mathcal{E}]^\Lambda \mid \forall i \in I, ((p_i(u_\lambda))_\lambda \in |A|\right\}$$
$$\mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})} = \left\{(u_\lambda)_\lambda \in [\mathcal{E}]^\Lambda \mid \forall i \in I, (p_i(u_\lambda))_\lambda \in |I_A|\right\}$$

The following result summarize some results recalled in [9].

Theorem 3 *If A is a solid subring of $\mathbb{K}^\Lambda$, then $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ is an A–module, and an A–algebra if $\mathcal{E}$ is a topological algebra.*

Moreover, if I_A is an ideal of A sharing the solidness property, then $\mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}$ is an A–linear subspace of $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$, and an ideal of $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ if $\mathcal{E}$ is a topological algebra.

As a consequence, the factor space $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}$ is again an A–module, but also an A/I_A–module (and of course an algebra, if $\mathcal{E}$ is a topological algebra).

For $(\mathcal{E},\mathcal{P}) = (\mathbb{K},\{|\cdot|\})$, we get $\mathcal{H}_{(A,\mathbb{K},|\cdot|)}/\mathcal{H}_{(I_A,\mathbb{K},|\cdot|)} = A/I_A$.

Remark 1 *If we require $\mathcal{E}$ to be a topological algebra, this means that multiplication in $\mathcal{E}$ is continuous for the topology defined by the family of seminorms.*

Definition 2 *The factor ring $\mathcal{C} = A/I_A$ is called the ring of generalized numbers (associated to A and I_A), and the $\mathcal{C}$–algebra*

$$\mathcal{A}_\mathcal{C}(\mathcal{E},\mathcal{P}) := \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}$$

is called the $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebra of $\mathcal{C}$–generalized functions. We denote by $[x_\lambda]$ the class in $\mathcal{A}_\mathcal{C}(\mathcal{E},\mathcal{P})$ of the family $(x_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$.

Example 1 *Let us define*

$$A = \left\{(r_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]} \mid \exists m \in \mathbb{N}\ :\ |u_\varepsilon| = \mathrm{o}(\varepsilon^{-m}),\ \textit{as}\ \varepsilon \to 0\right\}$$
$$I_A = \left\{(r_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]} \mid \forall q \in \mathbb{N}\ :\ |u_\varepsilon| = \mathrm{o}(\varepsilon^{q}),\ \textit{as}\ \varepsilon \to 0\right\}.$$

In this case (with $\mathcal{E} = \mathrm{C}^\infty(\mathbb{R}^n)$ and $\mathcal{P} = \{p_{K,\alpha} : f \to \| \partial^\alpha f \|_{K \Subset \mathbb{R}^n, \alpha\in\mathbb{N}^n}\}$), the algebra $\mathcal{A} = \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$ is exactly the so-called special Colombeau algebra $\mathcal{G}(\mathbb{R}^n)$.

Remark 2 *If $\mathcal{E}$ is a sheaf of $\mathbb{K}$-topological algebras over a topological space X, we can prove that the factor $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}$ is at least a presheaf satisfying the localization principle. But we don't need the sheaf structure in the sequel.*

We are going now to the the concept of "Overgenerated algebras". It is useful when constructing a $(\mathcal{C},\mathcal{E},\mathcal{P})$-structure to solve some problems with irregular data or coefficients.

Choose B_p a finite family of p nets in $(\mathbb{R}_+^*)^\Lambda$ (usually given by the asymptotic structure of the problem.) Consider B the subset of elements in $(\mathbb{R}_+^*)^\Lambda$ obtained as **rational fractions** with coefficients in $\mathbb{R}_+^*$, of elements in B_p as variables.

Definition 3 *Define*

$$A = \left\{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \exists\,(b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda\right\}.$$

*We say that A **is overgenerated** by B_p (and it is easy to see that A is a solid subring of $\mathbb{K}^{\Lambda}$).If I_A is some solid ideal of A, we also say that $\mathcal{C} = A/I_A$ **is overgenerated** by B_p. For example, as a "canonical" ideal of A, we can take*

$$I_A = \left\{ (a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \forall (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \le b_\lambda \right\}.$$

This definition implies that B is stable by inverse.

3.2 Contraction operator in $\mathcal{A}_\mathcal{C}(\mathcal{E}, \mathcal{P})$

First, we are looking if it is possible to define a map $\Phi : \mathcal{A} \to \mathcal{A}$ by means of a given family $(\Phi_\lambda)_{\lambda\in\Lambda}$of maps $\Phi_\lambda : \mathcal{E} \to \mathcal{E}$. The general requirement is given in the following

Lemma 4 *Let $(\Phi_\lambda)_{\lambda\in\Lambda}$ be a given family of maps $: \mathcal{E} \to \mathcal{E}$.*
Suppose that for each $(x_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ and $(i_\lambda)_\lambda \in \mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}$ we have
(i) $(\Phi_\lambda (x_\lambda))_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$,
(ii) $(\Phi_\lambda (x_\lambda + i_\lambda))_\lambda - (\Phi_\lambda (x_\lambda))_\lambda \in \mathcal{I}_{(A,\mathcal{E},\mathcal{P})}$.
Then, $\Phi : \mathcal{A} \to \mathcal{A}$ is well defined by

$$\mathcal{A} \ni [x_\lambda] = x \to \Phi(x) = [\Phi_\lambda (x_\lambda)] \in \mathcal{A}.$$

Proof. From (i) we see that the class $[\Phi_\lambda (x_\lambda)]$ lies in $\mathcal{A}$. Let $x_\lambda + i_\lambda$ be another representative of $x = [x_\lambda]$. From (ii) we have $[\Phi_\lambda (x_\lambda + i_\lambda)] = [\Phi_\lambda (x_\lambda)]$. Then, Φ is well defined. ∎

Theorem 5 *Let be fullfilled the following hypotheses: It exists a family $(a_{n,\lambda})_{n,\lambda\in\mathbb{N}\times\Lambda}$ of positive numbers with $(a_{n,\lambda})_\lambda \in A$, verifying: for each $i \in I$ there exists $N(i)$ and $j(i) \in I$ such that, for each $\lambda \in \Lambda$ and $e \in \mathcal{E}$*

$$p_i(\Phi_\lambda(e)) \le \sum_{n=0}^{N(i)} a_{n,\lambda} p_{j(i)}^n (e) .$$

Then for each $(x_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ we have $(\Phi_\lambda (x_\lambda))_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$.

Proof. $(x_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ implies $\forall i \in I, (p_i (x_\lambda))_\lambda \in |A|$. Then, $\sum_{n=0}^{N(i)} a_{n,\lambda} p_{j(i)}^n (x_\lambda) \in |A|$. As A is a solid subring of $\mathbb{K}^\Lambda$, it follows that $(p_i (\Phi_\lambda (x_\lambda))_\lambda \in |A|$ ∎

Definition 4 *The following hypotheses permit to well define a map $\Phi : \mathcal{A} \to \mathcal{A}$ from the family $(\Phi_\lambda)_{\lambda\in\Lambda}$ and to call it a **contraction.***
(a) for each $(x_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$, $(\Phi_\lambda (x_\lambda))_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$,
(b) Each Φ_λ is a contraction in $(\mathcal{E}, \tau_\lambda)$ endowed with the family $\mathcal{Q}_\lambda = (q_{\lambda,i})_{i\in I}$, and the corresponding contraction constants are denoted by $k_{\lambda,i} < 1$,
(c) For each $i \in I$ and $\lambda \in \Lambda$ it exist some strictly positive constants $\alpha_{\lambda,i}$ and $\beta_{\lambda,i}$ such that

$$\alpha_{\lambda,i} p_i \le q_{\lambda,i} \le \beta_{\lambda,i} p_i,$$

(d) For each $i \in I$, the families $\left(\dfrac{\beta_{\lambda,i}}{\alpha_{\lambda,i}}\right)_\lambda$ and $\left(\dfrac{1}{1 - k_{\lambda,i}}\right)_\lambda$ lies in $|A|$.

Theorem 6 *Any contraction Φ $\mathcal{A} \to \mathcal{A}$ has a fixed point in $\mathcal{A}$.*

Proof. Remark that condition (a) which is (i) in Lemma 4 can be fulfilled when giving the hypotheses in Theorem 5, that is to say $(a_{n,\lambda})_\lambda \in A$. Now we have from (c)

$$p_i(\Phi_\lambda(x_\lambda + i_\lambda) - (\Phi_\lambda(x_\lambda)) \leq \frac{1}{\alpha_{\lambda,i}} k_{i,\lambda} q_{\lambda,i}(i_\lambda) \leq \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} p_i(i_\lambda)$$

from what we deduce that $(p_i(\Phi_\lambda(x_\lambda + i_\lambda) - (\Phi_\lambda(x_\lambda))_\lambda \in A$ and the condition (ii) in Lemma 4 is verified. Then Φ is well defined. From Theorem 2 we know that each Φ_λ has a fixed point z_λ obtained as limit of the Cauchy sequence $z_{n,\lambda}$ defined by induction by $z_{n+1,\lambda} = \Phi_\lambda(z_{n,\lambda})$. Starting from $z_0 = [z_{0,\lambda}] \in \mathcal{A}$, we deduce that $z_1 = [\Phi_\lambda(z_{0,\lambda})] \in \mathcal{A}$, and $z_1 - z_0 \in \mathcal{A}$. That is to say $p_i(z_{1,\lambda} - z_{0,\lambda})_\lambda \in |A|$.

By induction we can compute for all $n, p \in \mathbb{N}$

$$q_{\lambda,i}(z_{n+p,\lambda} - z_{n,\lambda}) \leq \frac{k_{\lambda,i}^n}{1 - k_{\lambda,i}} q_{\lambda,i}(z_{1,\lambda} - z_{0,\lambda}) \text{ giving}$$

$$q_{\lambda,i}(z_{p,\lambda} - z_{0,\lambda}) \leq \frac{1}{1 - k_{\lambda,i}} q_{i,\lambda}(z_{1,\lambda} - z_{0,\lambda}).$$

When taking the limit z_λ of $z_{p,\lambda}$ in $(\mathcal{E}, \tau_\lambda)$ when $p \to \infty$, we get

$$q_{\lambda,i}(z_\lambda - z_{0,\lambda}) \leq \frac{1}{1 - k_{\lambda,i}} q_{\lambda,i}(z_{1,\lambda} - z_{0,\lambda}).$$

Writing now $q_{\lambda,i}(z_\lambda) \leq q_{\lambda,i}(z_\lambda - z_{0,\lambda}) + q_{\lambda,i}(z_{0,\lambda})$ we have

$$p_i(z_\lambda) \leq \frac{1}{\alpha_{\lambda,i}} q_{i,\lambda}(z_\lambda) \leq \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} \left[\frac{1}{1 - k_{\lambda,i}} (p_i(z_{1,\lambda} - z_{0,\lambda}) + p_i(z_{0,\lambda})) \right].$$

Then, from the hypotheses $(p_i(z_\lambda))_\lambda \in |A|$, that is to say $((p_i(z_\lambda)))_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$. If $z = [z_\lambda]$, then we have $\Phi(z) = [\Phi_\lambda(z_\lambda)] = [z_\lambda] = z$. Then z is a fixed point of Φ.

However following the under Remark 3 we cannot prove the uniqueness of z without other hypotheses than uniqueness of fixed points of Φ_λ. ■

3.3 Contractions in product of algebras

Definition 5 *For $m \in \mathbb{N}$, we define*

$$\mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})} = \left\{ (u_\lambda)_\lambda = (u_{1,\lambda}, ... u_{m,\lambda},) \in [\mathcal{E}^m]^\Lambda \mid \forall i \in I, \forall k = 1, ...m, ((p_i(u_{k,\lambda}))_\lambda \in |A| \right\},$$
$$\mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})} = \left\{ (u_\lambda)_\lambda = (u_{1,\lambda}, ... u_{m,\lambda},) \in [\mathcal{E}^m]^\Lambda \mid \forall i \in I, \forall k = 1, ...m, ((p_i(u_{k,\lambda}))_\lambda \in |I_A| \right\}.$$

According to the results and definitions in Subsection 3.1, if A is a solid subring of $\mathbb{K}^\Lambda$, then $\mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})}$ is an A–module, and an A–algebra if $\mathcal{E}$ is a topological algebra. Moreover, if I_A is an ideal of A sharing the solidness property, then $\mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})}$ is an A–linear subspace of $\mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})}$, and an ideal of $\mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})}$ if $\mathcal{E}$ is a topological algebra. As a consequence, the factor space $\mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})} / \mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})}$ is again an A–module, but also an A/I_A–module (and of course an algebra, if $\mathcal{E}$ is a topological algebra).

Definition 6 *We pose $\mathcal{A}^m_\mathcal{C}(\mathcal{E}, \mathcal{P}) := \mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})} / \mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})}$ and denote by $[x_\lambda]$ the class in $\mathcal{A}^m_\mathcal{C}(\mathcal{E}, \mathcal{P})$ of the family $(x_\lambda)_\lambda = (x_{1,\lambda}, ... x_{m,\lambda},)_\lambda \in \mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})}$.*

First, it is possible as previously, to define a map $\Phi : \mathcal{A}^m \to \mathcal{A}^m$ by means of a given family $(\Phi_\lambda)_{\lambda\in\Lambda}$ of maps $\Phi_\lambda : \mathcal{E}^m \to \mathcal{E}^m$. The general requirement is similar to the previous one given in Lemma 4.

Suppose that for each $(x_\lambda)_\lambda \in \mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})}$ and $(i_\lambda)_\lambda \in \mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})}$ we have

(i) $(\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})}$,

(ii) $(\Phi_\lambda(x_\lambda + i_\lambda))_\lambda - (\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})}$.

Then, $\Phi : \mathcal{A}^m \to \mathcal{A}^m$ is well defined by

$$\mathcal{A}^m \ni [x_\lambda] = x \to \Phi(x) = [\Phi_\lambda(x_\lambda)] \in \mathcal{A}^m.$$

Now, we are defining a contraction property for the given family $(\Phi_\lambda)_{\lambda\in\Lambda}$ of maps $\Phi_\lambda : \mathcal{E}^m \to \mathcal{E}^m$.

Notation 2 *The natural topology (denoted τ^m) on the product $\mathcal{E}^m$ is defined by the family $\left(p_i^{(m)}\right)_{i\in I}$ of seminorms such that $p_i^{(m)}(x) = p_i^{(m)}(x_1...x_m) = N(p_i(x_1),..p_i(x_n))$ where N is any norm on $\mathbb{R}^n$. For example, we can choose $p_i^{(m)}(x) = \sum\limits_{k=1}^{k=m} p_i(x_k)$. Denote by $(\mathcal{E}^m, \tau_\lambda^m)$ the topological space $\mathcal{E}^m$ endowed by the family $\left(q_{\lambda,i}^{(m)}\right)_{i\in I}$ with $q_{\lambda,i}^{(m)}(x) = \sum\limits_{k=1}^{k=m} q_{\lambda,i}(x_k)$ for a given family $(q_{\lambda,i})_{i\in I}$ of seminorm on $\mathcal{E}$*

Definition 7 $(\Phi_\lambda)_{\lambda\in\Lambda}$ *is called a family of* ***contractions in*** $\mathcal{E}^m$ *if for each $(i,\lambda) \in I \times \Lambda$ it exists a semi norm $q_{i,\lambda}$ and a constant $k_{\lambda,i} < 1$ such that for all $(x,y) \in \mathcal{E}^m \times \mathcal{E}^m$*

$$q_{\lambda,i}^{(m)}(\Phi_\lambda(x) - \Phi_\lambda(y)) \leq k_{\lambda,i} \sum_{k=1}^{k=m} q_{\lambda,i}(x_k - y_k). \tag{3}$$

Proposition 7 *Each contraction Φ_λ in $\mathcal{E}^m$ has a fixed point z_λ Moreover if $(\mathcal{E},\tau)$ is Hausdorff and for each $(i,\lambda) \in I \times \Lambda$ it exists a strictly positive constant $a_{i,\lambda}$ such that $a_{i,\lambda} p_i \leq q_{i,\lambda}$, then z_λ is unique.*

Proof. We deduce from (3) that $q_{\lambda,i}^{(m)}(\Phi_\lambda(x) - \Phi_\lambda(y)) \leq k_{\lambda,i} q_{\lambda,i}^{(m)}(x - y)$. From Theorem 2 we know that each Φ_λ has a fixed point $z_\lambda \in \mathcal{E}^m$. If $(\mathcal{E},\tau)$ is Hausdorff , for each nonnull $r \in \mathcal{E}$, it exists $i \in I$ such that $p_i(r) > 0$. Then $q_{\lambda,i}(r) > 0$ which implies that $(\mathcal{E},\tau_\lambda)$ is Hausdorff. If $y_\lambda \neq z_\lambda$ is another fixed point of Φ_λ, it exists at least $h \in \mathbb{N}$ with $1 \leq h \leq m$ such that $y_{\lambda,h} - z_{\lambda,h} \neq 0$. Therefore, exists $j \in I$ such that

$$0 < q_{\lambda,j}(y_{\lambda,h} - z_{\lambda,h}) \leq q_{\lambda,j}^{(m)}(y_\lambda - z_\lambda) = q_{\lambda,j}^{(m)}(\Phi_\lambda(y_\lambda) - \Phi_\lambda(z_\lambda)) \leq k\lambda_{\lambda,j} q_{i,\lambda}^{(m)}(y_\lambda - z_\lambda) < q_{\lambda,j}^{(m)}(y_\lambda - z_\lambda)$$

which leads to a contradiction ■

And now, to achieve the construction of $\Phi : \mathcal{A}^m \to \mathcal{A}^m$ and prove the existence of a fixed point in the same way as in Theorem 6, we propose the following

Definition 8 *The following hypotheses permit to well define a map $\Phi : \mathcal{A}^m \to \mathcal{A}^m$ from the family $(\Phi_\lambda)_{\lambda\in\Lambda}$ and to call it a* ***contraction.***

(a) *for each $(x_\lambda)_\lambda \in \mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})}$, $(\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})}$,*

(b) *The map $\Phi_\lambda : \mathcal{E}^m \to \mathcal{E}^m$ is a contraction of $\mathcal{E}^m$ following Definition 7,*

(c) *For each $i \in I$ and $\lambda \in \Lambda$ it exist some strictly positive constants $\alpha_{\lambda,i}$ and $\beta_{\lambda,i}$ such that*

$$\alpha_{\lambda,i} p_i \leq q_{\lambda,i} \leq \beta_{\lambda,i} p_i,$$

(d) *For each $i \in I$, the families $\left(\dfrac{\beta_{\lambda,i}}{\alpha_{\lambda,i}}\right)_\lambda$ and $\left(\dfrac{1}{1-k_{\lambda,i}}\right)_\lambda$ lies in $|A|$.*

Theorem 8 *Any contraction Φ $\mathcal{A}^m \rightarrow \mathcal{A}^m$ has a fixed point in $\mathcal{A}^m$.*

Proof. First we can re-write $\mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})}$ and $\mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})}$ as

$$\mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})} = \left\{(u_\lambda)_\lambda \in [\mathcal{E}^m]^\Lambda \mid \forall i \in I, \left((p_i^{(m)}(u_\lambda)\right)_\lambda \in |A|\right\},$$
$$\mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})} = \left\{(u_\lambda)_\lambda \in [\mathcal{E}^m]^\Lambda \mid \forall i \in I, \left((p_i^{(m)}(u_\lambda)\right)_\lambda \in |I_A|\right\}.$$

We deduce from (c) that

$$\alpha_{\lambda,i} p_i^{(m)} \leq q_{\lambda,i}^{(m)} \leq \beta_{\lambda,i} p_i^{(m)}.$$

For $(x_\lambda)_\lambda \in \mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})}$ and $(i_\lambda)_\lambda \in \mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})}$ we have

$$p_i^{(m)}\left(\Phi_\lambda\left(x_\lambda + i_\lambda\right) - \Phi_\lambda\left(x_\lambda\right)\right) \leq \frac{1}{\alpha_{\lambda,i}} k_{\lambda,i} q_{\lambda,i}^{(m)}(i_\lambda) \leq \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} p_i^{(m)}(i_\lambda) = \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} \sum_{k=1}^{k=m} p_i(i_k).$$

Then, $\left(p_i^{(m)}\left(\Phi_\lambda\left(x_\lambda + i_\lambda\right) - \Phi_\lambda\left(x_\lambda\right)\right)\right)_\lambda \in |I_A|$ and $\left(\Phi_\lambda\left(x_\lambda + i_\lambda\right)\right)_\lambda - \left(\Phi_\lambda\left(x_\lambda\right)\right)_\lambda \in \mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})}$. It follows that $\Phi : \mathcal{A}^m \rightarrow \mathcal{A}^m$ is well defined by

$$\mathcal{A}^m \ni [x_\lambda] = x \rightarrow \Phi\left(x\right) = \left[\Phi_\lambda\left(x_\lambda\right)\right] \in \mathcal{A}^m.$$

From (b) and Theorem 2 we know that each Φ_λ has a fixed point $z_\lambda = (z_{1,\lambda}, ...z_{m,\lambda}) \in \mathcal{E}^m$ and from Theorem 1 we know that z_λ is obtained as limit of the Cauchy sequence $z_{n,\lambda}$ defined by induction by $z_{n+1,\lambda} = \Phi_\lambda(z_{n,\lambda})$. Starting from $z_0 = [z_{0,\lambda}] \in \mathcal{A}^m$, we deduce that $z_1 = [\Phi_\lambda(z_{0,\lambda})] \in \mathcal{A}^m$, and $z_1 - z_0 \in \mathcal{A}^m$. That is to say $p_i^{(m)}\left(z_{1,\lambda} - z_{0,\lambda}\right)_\lambda \in |A|$.

By induction we can compute for all $n, p \in \mathbb{N}$

$$q_{\lambda,i}^{(m)}\left(z_{n+p,\lambda} - z_{n,\lambda}\right) \leq \frac{k_{\lambda,i}^n}{1 - k_{\lambda,i}} q_{\lambda,i}^{(m)}\left(z_{1,\lambda} - z_{0,\lambda}\right) \text{ giving}$$

$$q_{\lambda,i}^{(m)}\left(z_{p,\lambda} - z_{0,\lambda}\right) \leq \frac{1}{1 - k_{\lambda,i}}\, q_{\lambda,i}^{m}\left(z_{1,\lambda} - z_{0,\lambda}\right).$$

When taking the limit z_λ of $z_{p,\lambda}$ in $(\mathcal{E}^m, \tau_\lambda^m)$ when $p \rightarrow \infty$, we get

$$q_{\lambda,i}^{(m)}\left(z_\lambda - z_{0,\lambda}\right) \leq \frac{1}{1 - k_{\lambda,i}} q_{\lambda,i}^{m}\left(z_{1,\lambda} - z_{0,\lambda}\right).$$

Writing now $q_{\lambda,i}^{(m)}\left(z_\lambda\right) \leq q_{\lambda,i}^{(m)}\left(z_\lambda - z_{0,\lambda}\right) + q_{\lambda,i}^{(m)}\left(z_{0,\lambda}\right)$ we have

$$p_i^{(m)}\left(z_\lambda\right) \leq \frac{1}{\alpha_{\lambda,i}} q_{\lambda,i}^{(m)}\left(z_\lambda\right) \leq \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}}\left[\frac{1}{1 - k_{\lambda,i}}(p_i^{(m)}\left(z_{1,\lambda} - z_{0,\lambda}\right) + p_i^{(m)}\left(z_{0,\lambda}\right))\right].$$

From the hypotheses of Definition 8, we have $\left(p_i^{(m)}\left(z_\lambda\right)\right)_\lambda \in |A|$, then $((z_\lambda))_\lambda \in \mathcal{H}^m_{(A,\mathcal{E},\mathcal{P})}$. Therefore, if $z = [z_\lambda]$, we have $\Phi\left(z\right) = \left[\Phi_\lambda\left(z_\lambda\right)\right] = [z_\lambda] = z$. Then z is a fixed point of Φ. ■

Remark 3 *If $y = [y_\lambda]$ is any fixed point of Φ, that point verifies $y_\lambda = \Phi_\lambda(y_\lambda) + i_\lambda$ for some $(i_\lambda)_\lambda \in \mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})}$. We try prove that $[y_\lambda] = [z_\lambda]$ that is to say . $(y_\lambda - z_\lambda)_\lambda \in \mathcal{H}^m_{(I_A,\mathcal{E},\mathcal{P})}$. Then, it exists at least $\mu \in \Lambda$ such that $y_\mu - z_\mu \neq 0$. As $\left(\mathcal{E}^m, \tau^m_\mu\right)$ is Hausdorff it exists $j \in I$ (depending on $y_\mu - z_\mu$) such that $q^{(m)}_{\mu,j}(y_\mu - z_\mu) > 0$. Writing*

$$\begin{aligned}0 &< q^{(m)}_{j,\mu}(y_\mu - z_\mu) \leq q^{(m)}_{j,\mu}(\Phi_\mu(y_\mu) - \Phi_\mu(z_\mu) + i_\mu) \leq k_{j,\mu} q^{(m)}_{j,\mu}(y_\mu - z_\mu) + q^{(m)}_{j,\mu}(i_\mu) \\ &< q^{(m)}_{j,\mu}(y_\mu - z_\mu) + q^{(m)}_{j,\mu}(i_\mu).\end{aligned}$$

We don't see a contadiction and uniqueness cannot be proved without other hypotheses.

4 The Cauchy-Lipschitz theorem

We try to give a generalized formulation of the Cauchy-Lipschitz theorem close to the classical one. We can limit the order of derivatives to one or even zero, as in the globally Lipschitz problem.

Let J be an intervall of $\mathbb{R}$ and f a continuous function $\in C^0(J \times \mathbb{R}^m, \mathbb{R}^m) = \left(C^0(J \times \mathbb{R}^m, \mathbb{R})\right)^m$ satisfying a global Lipschitz condition as

$$\forall K \Subset J, \exists k > 0, \forall t \in J, \forall y, z \in \mathbb{R}^m, ||f(t,y) - f(t,z)|| \leq k\,||y - z||.$$

Then, the Cauchy problem

$$(4) \qquad \begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

with $x_0 \in \mathbb{R}^m$ and $t_0 \in J$ admits one unique global solution $x \in C^1(J, \mathbb{R}^m) = \left(C^1(J, \mathbb{R})\right)^m$.

The problem reduces to finding a fixed point of the map $F \in C^0(J, \mathbb{R}^m) \to C^0(J, \mathbb{R}^m)$ such that

$$\forall t \in J, F(x)(t) = x_0 + \textstyle\int_{t_0}^t f(s, x(s))ds$$

that is to say, for each $k = 1, 2...m$ and $x = (x_1, ...x_m)$, $F = (F_1, ...F_m)$, $f = (f_1, ...f_m) \in C^0(J \times \mathbb{R}^m, \mathbb{R})^m$

$$\forall t \in J, k = 1, 2...m : F_k(x)(t) = x_{0,k} + \textstyle\int_{t_0}^t f_k(s, x(s))ds.$$

The standard proof gives a convenient norm on $C^0(J, \mathbb{R}^m)$ for which F is a contraction, and the Picard procedure gives its fixed point.

To formulate (4) in a generalized setting we have to introduce some convenient algebras.

4.1 The generalized framework

Definition 9 *When $l = 0$ or 1, with $\mathcal{E} = \mathrm{C}^l(J, \mathbb{R})$ and $\mathcal{P}^l = (p_{K,l})_K$ such that $p_{K,l}(u) = \sup\limits_{t \in K, 0 \leq j \leq l} \left|u^{(j)}(t)\right|$, we can write $\mathcal{H}_{(A,\mathcal{E},\mathcal{P}^l)}$ and $\mathcal{H}_{(I_A,\mathcal{E},\mathcal{P}^l)}$ as*

$$\begin{aligned}\mathcal{H}^l_A(J, \mathbb{R}) &= \mathcal{H}_{(A,\mathrm{C}^l(J,\mathbb{R}),\mathcal{P}^l)} = \left\{(u_\lambda)_\lambda \in [\mathcal{E}]^\Lambda \mid \forall K \Subset J, ((p_{K,l}(u_\lambda))_\lambda \in |A|\right\} \\ \mathcal{H}^l_{I_A}(J, \mathbb{R}) &= \mathcal{H}_{(I_A,\mathrm{C}^l(J,\mathbb{R}),\mathcal{P}^l)} = \left\{(u_\lambda)_\lambda \in [\mathcal{E}]^\Lambda \mid \forall K \Subset J, ((p_{K,l}(u_\lambda))_\lambda \in |I_A|\right\}.\end{aligned}$$

and define $\mathcal{H}^\infty_A(J, \mathbb{R})$ or $\mathcal{H}^\infty_{I_A}(J, \mathbb{R})$ by replacing l by ∞ in the previous definitions, with

$\mathcal{P}^{\infty}=(p_{K,l})_{K\Subset J,l\in\mathbb{N}}$. *We begin to pose:*

$$\mathfrak{C}_{\mathcal{C}}^{0}(J,\mathbb{R})=\mathcal{H}_{A}^{0}(J,\mathbb{R})/\mathcal{H}_{I_A}^{0}(J,\mathbb{R}),$$
$$\mathfrak{C}_{\mathcal{C}}^{1}(J,\mathbb{R})=\mathcal{H}_{A}^{1}(J,\mathbb{R})/\mathcal{H}_{A}^{1}(J,\mathbb{R})\cap\mathcal{H}_{I_A}^{0}(J,\mathbb{R}).$$

Remark 4 *We have the classical embedding*: $\mathrm{C}^1(J,\mathbb{R})\to\mathrm{C}^0(J,\mathbb{R})$ *which inspires our generalized requirements. But the only way to embed the factor algebra* $\mathfrak{C}_{\mathcal{C}}^{1}(J,\mathbb{R})$ *(defined from* $\mathcal{H}_{A}^{1}(J,\mathbb{R})$*) into* $\mathfrak{C}_{\mathcal{C}}^{0}(J,\mathbb{R})$ *is to define* $\mathfrak{C}_{\mathcal{C}}^{1}(J,\mathbb{R})$ *as* $\mathcal{H}_{A}^{1}(J,\mathbb{R})/\mathcal{H}_{A}^{1}(J,\mathbb{R})\cap\mathcal{H}_{I_A}^{0}(J,\mathbb{R})$*, following a well-know result on the embedding of factor algebras. We would like to define* $\mathfrak{C}_{\mathcal{C}}^{1}(J,\mathbb{R})=\mathcal{H}_{A}^{1}(J,\mathbb{R})/\mathcal{H}_{I_A}^{1}(J,\mathbb{R})$*. Infortunately, in this case the natural mapping* $\mathfrak{C}_{\mathcal{C}}^{1}(J,\mathbb{R})\to\mathfrak{C}_{\mathcal{C}}^{0}(J,\mathbb{R})$ *is neither injective nor surjective. Indeed we cannot prove here that* $\mathcal{H}_{A}^{1}(J,\mathbb{R})\cap\mathcal{H}_{I_A}^{0}(J,\mathbb{R})=\mathcal{H}_{I_A}^{1}(J,\mathbb{R})$*, in contrary to the "well known"equality* $\mathcal{H}_{A}^{\infty}(J,\mathbb{R})\cap\mathcal{H}_{I_A}^{0}(J,\mathbb{R})=\mathcal{H}_{I_A}^{\infty}(J,\mathbb{R})$*, proved in Lemma* 4.4 *in [9] which generalize* 1.2.3 *Theorem in [8]. However, we can define a map* ∂ *from* $\operatorname{Im}\mathfrak{C}_{\mathcal{C}}^{1}(J,\mathbb{R})$ *to* $\mathfrak{C}_{\mathcal{C}}^{0}(J,\mathbb{R})$ *which looks like a derivation.*

Definition 10 *If* $i(x)\in\operatorname{Im}\mathfrak{C}_{\mathcal{C}}^{1}(J,\mathbb{R})$ *the embedding* i *verifies*

$$i:(x_\varepsilon)_\varepsilon+\mathcal{H}_{A}^{1}(J)\cap\mathcal{H}_{I_A}^{0}(J)\to(x_\varepsilon)_\varepsilon+\mathcal{H}_{I_A}^{0}(J)$$

which implies that $(x'_\varepsilon)_\varepsilon\in\mathcal{H}_{A}^{0}(J)$ *and allows to define*

$$\widetilde{\partial}:\mathfrak{C}_{\mathcal{C}}^{1}(J,\mathbb{R})\ni i^{-1}(x)=(x_\varepsilon)_\varepsilon+\mathcal{H}_{A}^{1}(J)\cap\mathcal{H}_{I_A}^{0}(J)\to\left(x'_\varepsilon\right)_\varepsilon+\mathcal{H}_{I_A}^{0}(J)\in\mathfrak{C}_{\mathcal{C}}^{0}(J,\mathbb{R})$$

which leads to define the map $\partial=\widetilde{\partial}\circ i^{-1}$ *from* $\operatorname{Im}\mathfrak{C}_{\mathcal{C}}^{1}(J,\mathbb{R})$ *to* $\mathfrak{C}_{\mathcal{C}}^{0}(J,\mathbb{R})$.

For $f\in\mathrm{C}^0(J\times\mathbb{R},\mathbb{R}),K\Subset J,p>0$,we pose

$$q_{K,-p}(f)=\sup_{t\in K,y\in\mathbb{R}}(1+|y|)^{-p}\,|f(t,y)|\,,$$

then for any solid unitary subring A with ideal I_A of $\mathbb{K}^\Lambda$, we define

$$\mathcal{H}_{\tau,A}^{0}(J\times\mathbb{R},\mathbb{R})=\left\{(f_\lambda)_\lambda\in\left[\mathrm{C}^0(J\times\mathbb{R},\mathbb{R})\right]^\Lambda\mid\forall K\Subset J,\exists p>0,(q_{K,-p}(f_\lambda))_\lambda\in|A|\right\},$$
$$\mathcal{H}_{\tau,I_A}^{0}(J\times\mathbb{R},\mathbb{R})=\left\{(f_\lambda)_\lambda\in\left[\mathrm{C}^0(J\times\mathbb{R},\mathbb{R})\right]^\Lambda\mid\forall K\Subset J,\exists p>0,(q_{K,-p}(f_\lambda))_\lambda\in|I_A|\right\}.$$

For $f=(f_1,..f_m)\in\left(\mathrm{C}^0(J\times\mathbb{R}^m,\mathbb{R})\right)^m,K\Subset J,p>0$, we pose

$$q_{K,-p}^{(m)}(f)=\sup_{t\in K,y\in\mathbb{R}^m}(1+|y|)^{-p}\,|f(t,y)|^{(m)}\ \text{ with }\ |f(t,y)|^{(m)}=\sum_{k=1}^{k=m}|f_k(t,y)|\,,$$

and define, for $f_\lambda=(f_{1,\lambda},...f_{m,\lambda})$

$$\left(\mathcal{H}_{\tau,A}^{0}(J\times\mathbb{R}^m,\mathbb{R})\right)^m=\left\{(f_\lambda)_\lambda\in\left[\left(\mathrm{C}^0(J\times\mathbb{R}^m,\mathbb{R})\right)^m\right]^\Lambda\mid\forall K\Subset J,\exists p>0,\left(q_{K,-p}^{(m)}(f_\lambda)\right)_\lambda\in|A|\right\},$$
$$\left(\mathcal{H}_{\tau,I_A}^{0}(J\times\mathbb{R}^m,\mathbb{R})\right)^m=\left\{(f_\lambda)_\lambda\in\left[\left(\mathrm{C}^0(J\times\mathbb{R}^m,\mathbb{R})\right)^m\right]^\Lambda\mid\forall K\Subset J,\exists p>0,\left(q_{K,-p}^{(m)}(f_\lambda)\right)_\lambda\in|I_A|\right\}.$$

In the same way when $l=0$ or 1, with $\mathcal{E}=\mathrm{C}^l(J,\mathbb{R})$ and $\mathcal{P}^l=\left(p_{K,l}^{(m)}\right)_{K\Subset J}$ such that $p_{K,l}^{(m)}(u)=\sup_{t\in K,0\leq j\leq l}\left|\left|u^{(j)}(t)\right|\right|$, we can re-write $\mathcal{H}_{(A,\mathcal{E},\mathcal{P}^l)}^{m}$ and $\mathcal{H}_{(I_A,\mathcal{E},\mathcal{P}^l)}^{m}$ as

$$\left(\mathcal{H}_{A}^{l}(J,\mathbb{R})\right)^m=\mathcal{H}_{(A,\mathrm{C}^l(J,\mathbb{R}),\mathcal{P}^l)}^{m}=\left\{(u_\lambda)_\lambda\in\left[\mathcal{E}^m\right]^\Lambda\mid\forall K\Subset J,\left((p_{K,l}^{(m)}(u_\lambda)\right)_\lambda\in|A|\right\},$$
$$\left(\mathcal{H}_{I_A}^{l}(J,\mathbb{R})\right)^m=\mathcal{H}_{(I_A,\mathrm{C}^l(J,\mathbb{R}),\mathcal{P}^l)}^{m}=\left\{(u_\lambda)_\lambda\in\left[\mathcal{E}^m\right]^\Lambda\mid\forall K\Subset J,\left((p_{K,l}^{(m)}(u_\lambda)\right)_\lambda\in|I_A|\right\}.$$

Definition 11 *Summarizing, we finally define for any* $m \in \mathbb{N}$

- $\left(\mathfrak{C}^0_{\tau,\mathcal{C}}(J \times \mathbb{R}^m, \mathbb{R})\right)^m = \left(\mathcal{H}^0_{\tau,A}(J \times \mathbb{R}^m, \mathbb{R})\right)^m / \left(\mathcal{H}^0_{\tau,I_A}(J \times \mathbb{R}^m, \mathbb{R})\right)^m$,
- $\left(\mathfrak{C}^0_{\mathcal{C}}(J, \mathbb{R})\right)^m = \left(\mathcal{H}^0_A(J, \mathbb{R})\right)^m / \left(\mathcal{H}^0_{I_A}(J, \mathbb{R})\right)^m$,
- $\left(\mathfrak{C}^1_{\mathcal{C}}(J, \mathbb{R})\right)^m = \left(\mathcal{H}^1_A(J, \mathbb{R})\right)^m / \left(\mathcal{H}^1_A(J, \mathbb{R})\right)^m \cap \left(\mathcal{H}^0_{I_A}(J, \mathbb{R})\right)^m$

which leads to define as previously the map $\partial = \widetilde{\partial} \circ i^{-1}$ *from* $\mathrm{Im}\left(\mathfrak{C}^1_{\mathcal{C}}(J, \mathbb{R})\right)^m$ *to* $\left(\mathfrak{C}^0_{\mathcal{C}}(J, \mathbb{R})\right)^m$.

Theorem 9 *Let* $F = (F_1, ...F_m) \in \left(\mathfrak{C}^0_{\tau,\mathcal{C}}(J \times \mathbb{R}^m, \mathbb{R})\right)^m$ *and* $u = (u_1, u_2, ...u_m) \in \left(\mathfrak{C}^0_{\mathcal{C}}(J, \mathbb{R})\right)^m$. *Then,* $F(., u)$ *is a well defined element of* $\left(\mathfrak{C}^0_{\mathcal{C}}(J, \mathbb{R})\right)^m$.

Proof. F has for representatives $(F_\lambda)_\lambda$ with $F_\lambda : t \to F_\lambda(t, y)$. If u_λ is a representative of $u \in \left(\mathcal{A}^0(J)\right)^m$, we have to prove that the family $(t \to F_\lambda(t, u_\lambda(t))_\lambda$ lies in $\left(\mathcal{H}^0_{\tau,A}(J \times \mathbb{R}^m)\right)^m = \mathcal{H}^m_{(A, \mathrm{C}^0(J,\mathbb{R}), \mathcal{P}^0)}$. To simplify the proof, suppose that $m = 1$ for basic estimates.

With one hand, if $u = [u_\lambda] \in \mathcal{A}^0(J)$, we have: $\forall K \Subset J$, $\left(\sup_{t \in K} |u_\lambda(t)|\right)_\lambda \in |A|$, then

$$\forall t \in K, |u_\lambda(t)| \le \sup_{t \in K} |u_\lambda(t)| = |a_\lambda| \text{ with } a_\lambda \in A.$$

As A is solid, $(u_\lambda(t))_\lambda$ is in A (and $(|u_\lambda(t)|)_\lambda$ in $|A|$). Then, for any $p > 0$, $((1 + |u_\lambda(t)|)^p)_\lambda \in |A|$.

On the other hand, $F \in \mathcal{A}^0_{\tau,\mathcal{C}}(J \times \mathbb{R})$ has for representatives $(F_\lambda)_\lambda$ such that

$$\exists p > 0, \sup_{t \in K, y \in \mathbb{R}^m} (1 + |y|)^{-p} |F_\lambda(t, y)| = |b_\lambda| \text{ with } (b_\lambda)_\lambda \in |A|.$$

Then,

$$\exists p > 0, \forall t \in K, |F_\lambda(t, u_\lambda(t))| \le (1 + |u_\lambda(t)|)^p |b_\lambda| = |c_{p,\lambda}| \text{ with } (c_{p,\lambda})_\lambda \in A$$

and from solidness of A, $(|F_\lambda(t, u_\lambda(t))|)_\lambda \in |A|$. Then $(t \mapsto (F_\lambda(t, u_\lambda(t))_\lambda \in \mathcal{H}(A, \mathrm{C}^0(J, \mathbb{R}), \mathcal{P}^0)$.

If $(v_\lambda)_\lambda$is another representative of u and (G_λ) another representative of F, it is easy to show that $(t \mapsto (F_\lambda(t, u_\lambda(t))_\lambda - (t \mapsto (G_\lambda(t, v_\lambda(t))_\lambda \in \mathcal{H}(I_A, \mathrm{C}^0(J, \mathbb{R}), \mathcal{P}^0)$. Then $F(., u)$ is a well defined element of $\mathfrak{C}^0_{\mathcal{C}}(J)$. Similar estimates permits to replace $\mathbb{R}$ by $\mathbb{R}^m$, $\mathfrak{C}^0_{\mathcal{C}}(J)$ by $\left(\mathfrak{C}^0_{\mathcal{C}}(J)\right)^m$ and $\mathfrak{C}^0_{\tau,\mathcal{C}}(J \times \mathbb{R})$ by $\left(\mathfrak{C}^0_{\tau,\mathcal{C}}(J \times \mathbb{R}^m)\right)^m$. ■

Definition 12 *Let* $f \in \left(\mathfrak{C}^0_{\tau,\mathcal{C}}(\mathbb{R}_+ \times \mathbb{R}^m)\right)^m$. *We tell that* f *is globally Lipschiz if for any representative* $(f_\varepsilon)_\varepsilon$ *of* f *we have (with* $||x|| = \sum_{k=1}^{k=m} |x_k|$ *when* $x = (x_1, ...x_m) \in \mathbb{R}^m$*)*

$$\forall t \in \mathbb{R}_+, \forall \varepsilon \in]0, 1], \exists k_\varepsilon(t) > 0, \forall (v, w) \in \mathbb{R}^m \times \mathbb{R}^m, ||f_\varepsilon((t, v) - f_\varepsilon((t, w)|| \le k_\varepsilon(t) ||v - w||$$

with

$$\forall T \in \mathbb{R}_+, \sup_{t \in [0,T]} k_\varepsilon(t) = M_{T,\varepsilon} < +\infty.$$

4.2 The generalized Cauchy-Lipschitz problem

In terms of generalized functions the Cauchy-Lipschitz problem (4) can be defined as

Definition 13 *The Cauchy-Lipschitz generalized problem is to solve* (2) *that is*

$$\left\{\begin{array}{c} \partial x = f(.,x) \\ x(t_0) = \xi \end{array}\right.$$

with $x \in \operatorname{Im}\left(\mathfrak{C}^1_{\mathcal{C}}(J)\right)^m \subset \left(\mathfrak{C}^0_{\mathcal{C}}(J)\right)^m$ *and* $f \in \left(\mathfrak{C}^0_{\tau,\mathcal{C}}(J \times \mathbb{R}^m)\right)^m$ *gobally Lipschitz, for some ring of generalized numbers* $\mathcal{C} = A/I_A$*, with* $t_0 \in J$ *and* ξ *is a given element* $\in \widetilde{\mathbb{R}^m}$

It is classical to choose $J = \mathbb{R}_+$ when t is the time parameter, then we do that, without loss of generality. But we have exactly the same result when J is an open subset of $\mathbb{R}$ and taking $J = \mathbb{R}$ simplifies some estimates.

Theorem 10 *It exists a ring of generalized numbers* $\mathcal{C} = A/I_A$*, such that* $f \in \left(\mathfrak{C}^0_{\tau,\mathcal{C}}(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R})\right)^m$ *and a map* $\Phi : \left(\mathfrak{C}^0_{\mathcal{C}}(\mathbb{R}_+, \mathbb{R})\right)^m \to \left(\mathfrak{C}^0_{\mathcal{C}}(\mathbb{R}_+, \mathbb{R})\right)^m$ *with a fixed point solving the Cauchy-Lipschitz problem* (2) *with* $t_0 \in \mathbb{R}_+$ *and* $\xi \in \widetilde{\mathbb{R}^m}$.

Proof. To construct the convenient ring of generalized numbers $\mathcal{C} = A/I_A$ interacting with the consrtruction of Φ and its fixed point, we follow four steps to verify the hypotheses of Definition of a generalized contraction

• (a) For $x_0 \in \mathbb{R}^m, x \in \left(\mathrm{C}^0(\mathbb{R}_+, \mathbb{R})\right)^m, t \in \mathbb{R}_+$ and $f_\varepsilon \in \left(\mathrm{C}^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R})\right)^m$ we pose

$$\Phi_\varepsilon(x)(t) = x_0 + \textstyle\int_0^t f_\varepsilon(s, x(s))ds$$

from what it is clear that Φ_ε is a map $\mathcal{E}^m \to \mathcal{E}^m$ with $\mathcal{E} = \mathrm{C}^0(\mathbb{R}_+, \mathbb{R})$.

$(\mathcal{E}^m, \tau^m)$ is here a topological space where τ^m is given by the family of norms $\left(p_T^{(m)}\right)_{T \in \mathbb{R}_+}$ such that $p_T^{(m)}(x) = \sup\limits_{t \in [0,T]} ||x(t)||$. We suppose that it exists a family $(a_\varepsilon)_\varepsilon \in \mathcal{E}^{]0'1]} = \left(\mathrm{C}^0(\mathbb{R}_+, \mathbb{R})\right)^{]0,1]}$ such that

$$\left\{\begin{array}{c} \forall T \in \mathbb{R}_+, (p_T(a_\varepsilon))_\varepsilon \in |A| \\ \forall t \in \mathbb{R}_+, ||f_\varepsilon(t, x(t))|| \leq |a_\varepsilon(t)|\, ||x(t)||\,. \end{array}\right.$$

Then, we have

$$||\Phi_\varepsilon(x)(t)|| \leq ||x_0|| + \textstyle\int_0^t |a_\varepsilon(s)|\, ||x(s)||\, ds \leq ||x_0|| + |a_\varepsilon(r)| \int_0^t ||x(s)||\, ds, r \in [0,t]$$

which leads to $p_T^{(m)}(\Phi_\varepsilon(x)) \leq ||x_0|| + T p_T(a_\varepsilon)\, p_T^{(m)}(x)$.

Then if $(x_\varepsilon)_\varepsilon \in \mathcal{H}^m_{(A,\mathrm{C}^0(J,\mathbb{R}),\mathcal{P}^0)}$, that is to say $\left(p_T^{(m)}(x_\varepsilon)\right)_\varepsilon \in |A|$, we have

$$\left(p_T^{(m)}(\Phi_\varepsilon(x_\varepsilon))\right)_\varepsilon \leq (||x_{0,\varepsilon}||)_\varepsilon + T\,(p_T(a_\varepsilon))_\varepsilon \left(p_T^{(m)}(x_\varepsilon)\right)_\varepsilon$$

which leads (from $\exists n \in \mathbb{N}$ such that $T\,(p_T(a_\varepsilon))_\varepsilon \leq n\,(p_T(a_\varepsilon))_\varepsilon$) to $\left(p_T^{(m)}(\Phi_\varepsilon(x_\varepsilon))\right)_\varepsilon \in |A|$ that is to say $(\Phi_\varepsilon(x_\varepsilon)_\varepsilon \in \mathcal{H}^m_{(A,\mathrm{C}^0(J,\mathbb{R}),\mathcal{P}^0)}$.

• (b) Putting $\Lambda =]0,1]$, $t_0 = 0$ and $\lambda = \varepsilon$, we first have to write (2) in term of representatives $k_\varepsilon(t)$

(5)
$$\left\{\begin{array}{c} x'_\varepsilon(t) = f_\varepsilon(t, x_\varepsilon(t)) \\ x_\varepsilon(0) = \xi_\varepsilon \end{array}\right.$$

where $(||\xi_\varepsilon||)_\varepsilon \in |A|$.

We recall that if $(f_\varepsilon)_\varepsilon$ is a representative of $f \in \left(\mathfrak{C}^0_{\tau,\mathcal{C}}(\mathbb{R}_+ \times \mathbb{R}^m)\right)^m$, we have

$f_\varepsilon \in \left(\mathrm{C}^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R})\right)^m$ and $\forall K \Subset J, \exists p > 0, \left(q^{(m)}_{K,-p}(f_\varepsilon)\right)_\varepsilon \in |A|$ and independantly of this estimate we have to add the following one

$$\forall t \in \mathbb{R}_+, \forall \varepsilon \in \left]0,1\right], \exists k_\varepsilon(t) > 0, \forall (v,w) \in \mathbb{R}^m \times \mathbb{R}^m, ||f_\varepsilon((t,v) - f_\varepsilon((t,w)|| \leq k_\varepsilon(t)\, ||v-w||$$

with

$$\forall T \in \mathbb{R}_+, \sup_{t\in[0,T]} k_\varepsilon(t) = M_{T,\varepsilon} < +\infty.$$

For x_0 given in $\mathbb{R}^m, x \in \left(\mathrm{C}^0(\mathbb{R}_+, \mathbb{R})\right)^m, t \in \mathbb{R}_+$ and $f_\varepsilon \in \left(\mathrm{C}^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R})\right)^m$ we pose

$$\Phi_\varepsilon(x)(t) = x_0 + \textstyle\int_0^t f_\varepsilon(s, x(s))ds$$

from what it is clear that Φ_ε is a map $\mathcal{E}^m \to \mathcal{E}^m$ with $\mathcal{E} = \mathrm{C}^0(\mathbb{R}_+, \mathbb{R})$.

The natural topology (denoted τ^m) on the product $\mathcal{E}^m$ is here defined by the family $\left(p^{(m)}_T\right)_{T\in\mathbb{R}_+}$ of seminorms such that for $x = (x_{1,\ldots}x_m), p^{(m)}_T(x) = \sum\limits_{k=1}^{k=m} p_T(x_k)$ with $p_T(x_k) = \sup\limits_{t\in[0,T]} |x_k(t)|$. Denote by $(\mathcal{E}^m, \tau^m_\varepsilon)$ the topological space $\mathcal{E}^m$ endowed by the family $\left(q^{(m)}_{T,\varepsilon}\right)_{T\in\mathbb{R}_+}$ with $q^{(m)}_{T,\varepsilon}(x) = \sum\limits_{k=1}^{k=m} q_{T,\varepsilon}(x_k)$ for a given family $(q_{T,\varepsilon})_{T\in\mathbb{R}_+}$ of seminorm on $\mathcal{E} = \mathrm{C}^0(\mathbb{R}_+, \mathbb{R})$ such that for each $\varepsilon \in \left]0,1\right]$ and $y \in \mathrm{C}^0(\mathbb{R}_+, \mathbb{R})$ we have $q_{T,\varepsilon}(y) = \sup\limits_{t\in[0,T]} \left(|y(t)|\, e^{-tM_{T,\varepsilon}}\right)$. Then $q^{(m)}_{T,\varepsilon}(x) = \sup\limits_{t\in[0,T]} \left(||x(t)||\, e^{-tM_{T,\varepsilon}}\right)$

We claim that the map Φ_ε is a contraction in $(\mathcal{E}^m, \tau^m_\varepsilon)$. From Definition 12, we have to prove that that for each $(T,\varepsilon) \in \mathbb{R}_+ \times \left]0,1\right]$ it exists a constant $k_{T,\varepsilon} < 1$ such that for all $(x,y) \in \mathcal{E}^m \times \mathcal{E}^m$

$$q^{(m)}_{T,\varepsilon}(\Phi_\varepsilon(x) - \Phi_\varepsilon(y)) \leq k_{T,\varepsilon} q^{(m)}_{T,\varepsilon}(x-y).$$

We have, for each $t \in \mathbb{R}_+$

$$\Phi_\varepsilon(x)(t) - \Phi_\varepsilon(y)(t) = x_0 + \textstyle\int_0^t (f_\varepsilon(s, x(s) - f_\varepsilon(s, y(s)))\, ds$$

from what we deduce

$$e^{-tM_{T,\varepsilon}}\, ||\Phi_\varepsilon(x)(t) - \Phi_\varepsilon(y)(t)|| \leq e^{-tM_{T,\varepsilon}} \textstyle\int_0^t ||f_\varepsilon(s, x(s) - f_\varepsilon(s, y(s))||\, ds$$

and, for $t \in [0,T]$

$$e^{-tM_{T,\varepsilon}}\, ||\Phi_\varepsilon(x)(t) - \Phi_\varepsilon(y)(t)|| \leq e^{-tM_{T,\varepsilon}} \textstyle\int_0^t M_{T,\varepsilon}\, ||x(s) - y(s)||\, ds.$$

Writing now

$$e^{-tM_{T,\varepsilon}} \textstyle\int_0^t M_{T,\varepsilon}\, ||x(s) - y(s)||\, ds = e^{-tM_{T,\varepsilon}} \textstyle\int_0^t M_{T,\varepsilon} e^{sM_{T,\varepsilon}} \left(e^{-sM_{T,\varepsilon}}\, ||x(s) - y(s)||\right) ds$$

we obtain

$$e^{-tM_{T,\varepsilon}} \textstyle\int_0^t M_{T,\varepsilon} e^{sM_{T,\varepsilon}} \left(e^{-sM_{T,\varepsilon}}\, ||x(s) - y(s)||\right) ds \leq e^{-tM_{T,\varepsilon}} \textstyle\int_0^t M_{T,\varepsilon} e^{sM_{T,\varepsilon}} q^{(m)}_{T,\varepsilon}(x-y)\, ds$$

which leads to

$e^{-tM_{T,\varepsilon}}\left\|\Phi_\varepsilon(x)(t)-\Phi_\varepsilon(y)(t)\right\| \leq e^{-tM_{T,\varepsilon}} q_{T,\varepsilon}^{(m)}(x-y)\left(e^{tM_{T,\varepsilon}}-1\right)=\left(1-e^{-tM_{T,\varepsilon}}\right) q_{T,\varepsilon}^{(m)}(x-y)$.

When taking $\sup_{t\in[0,T]}$ we finally obtain a constant $k_{T,\varepsilon}=\left(1-e^{-TM_{T,\varepsilon}}\right)<1$ such that

for all $(x,y)\in\mathcal{E}^m\times\mathcal{E}^m$

$$q_{T,\varepsilon}^{(m)}\left(\Phi_\varepsilon(x)-\Phi_\varepsilon(y)\right)\leq k_{T,\varepsilon}q_{T,\varepsilon}^{(m)}(x-y).$$

Then according to Definition 12, Φ_ε is a contraction in $\mathcal{E}^m$ and has an unique fixed point z_ε from Theorem 6.

• (c) We can write for $x\in\mathcal{E}^m$

$$e^{-TM_{T,\varepsilon}}\sup_{t\in[0,T]}\|x(t)\|\leq\sup_{t\in[0,T]}\left(\|x(t)\|\,e^{-tM_{T,\varepsilon}}\right)\leq\sup_{t\in[0,T]}\|x(t)\|$$

then

$$e^{-TM_{T,\varepsilon}}p_T^{(m)}\leq q_{T,\varepsilon}^{(m)}\leq p_T^{(m)}.$$

• (d). Assume now that for each $T\in\mathbb{R}_+$ the family $\left(e^{TM_{T,\varepsilon}}\right)_\varepsilon$ lies in $|A|$ and recall that we have asked in (a) that $\left(p_T(a_\varepsilon)\right)_\varepsilon\in|A|$. As $k_{T,\varepsilon}=\left(1-e^{-TM_{T,\varepsilon}}\right)$ we have $\left(\dfrac{1}{1-k_{T,\varepsilon}}\right)_\varepsilon=\left(e^{TM_{T,\varepsilon}}\right)_\varepsilon$.

Define now

$$\Phi_\varepsilon(x_\varepsilon)(t)=\xi_\varepsilon+\textstyle\int_0^t f_\varepsilon(s,x_\varepsilon(s))ds$$

where $(\xi_\varepsilon)_\varepsilon$is a given representative of the given element $\xi\in\widetilde{\mathbb{R}^m}$

Finally, from Definition 8, the map $\Phi:\left(\mathfrak{C}_{\mathcal{C}}^0(\mathbb{R}_+,\mathbb{R})\right)^m\to\left(\mathfrak{C}_{\mathcal{C}}^0(\mathbb{R}_+,\mathbb{R})\right)^m$ such that

$$[x_\varepsilon]\mapsto[\Phi_\varepsilon(x_\varepsilon)]$$

is a contraction, with $z=[z_\varepsilon]$ as fixed point from Theorem 8, z_ε being the unique fixed point of Φ_ε verifying

$$z_\varepsilon(t)=\xi_\varepsilon+\textstyle\int_0^t f_\varepsilon(s,z_\varepsilon(s))ds.$$

Then, $z_\varepsilon'(t)=f_\varepsilon(t,z_\varepsilon(t))$. and $z_\varepsilon(0)=\xi_\varepsilon$. Moreover z is solution to the Cauchy-Lipschitz problem (2) given in Definition 13.

We are going to prove that z is the unique fixed point of Φ, and therefore the unique solution of ().

If $y=[y_\varepsilon]$ is another fixed point of Φ, we have $y_\varepsilon=\Phi_\varepsilon(y_\varepsilon)+i_\varepsilon$ with $(i_\varepsilon)_\varepsilon\in\mathcal{H}^m_{(I_A,\mathrm{C}^0(\mathbb{R}_+,\mathbb{R}),\mathcal{P}^0)}$, that is to say $\left(p_T^{(m)}(i_\varepsilon)\right)_\varepsilon\in I_A$. Writing now

$$z_\varepsilon(t)-y_\varepsilon(t)=\textstyle\int_0^t\left(f_\varepsilon(s,z_\varepsilon(s)-f_\varepsilon(s,y_\varepsilon(s))\right)ds+i_\varepsilon(t)$$

we have

$$\|z_\varepsilon(t)-y_\varepsilon(t)\|\leq\textstyle\int_0^t\|(f_\varepsilon(s,z_\varepsilon(s)-f_\varepsilon(s,y_\varepsilon(s))\|\,ds+\|i_\varepsilon(t)\|$$

$$\|z_\varepsilon(t)-y_\varepsilon(t)\|\leq\|i_\varepsilon(t)\|+\textstyle\int_0^t M_{T,\varepsilon}\|z_\varepsilon(s)-y_\varepsilon(s)\|\,ds$$

and, from Gronwall lemma, for $t\in[0,T]$

$$\|z_\varepsilon(t)-y_\varepsilon(t)\|\leq\|i_\varepsilon(t)\|\,e^{TM_{T,\varepsilon}}$$

$$p_T^{(m)}(z_\varepsilon-y_\varepsilon)\leq p_T^{(m)}(i_\varepsilon)e^{TM_{T,\varepsilon}}.$$

As $\left(e^{TM_{T,\varepsilon}}\right)_\varepsilon \in |A|$ and $\left(p_T^{(m)}(i_\varepsilon)\right)_\varepsilon \in |I_A|$, we have $\left(p_T^{(m)}(z_\varepsilon - y_\varepsilon)\right)_\varepsilon \in |I_A|$, which finish the proof. ■

All our ideas, technics and results can be explicitely detailed and summarized in the following example

Example 2 *The Cauchy-Lipschitz generalized problem to solve is given as*

$$(6) \qquad \left\{ \begin{array}{c} \partial x = \Delta(., x) \\ x(0) = \xi \end{array} \right.$$

with $x \in \mathfrak{C}^0_{\mathcal{C}}(\mathbb{R}, \mathbb{R})$, *with* $t_0 \in \mathbb{R}$ *and* ξ *is a given element* $\in \widetilde{\mathbb{R}}$. *We define* Δ *from* $\Delta_\varepsilon(t, x) = \varphi_\varepsilon(t)\varphi_\varepsilon(x)$ *when* φ_ε *is the "standard mollifier"* $\varphi_\varepsilon(.) = \varphi\left(\frac{.}{\varepsilon}\right)$, $\varphi \in \mathrm{C}^1(\mathbb{R})$, *supp*$\varphi = [-1, 1]$, $\int \varphi(s)\, ds = 1$.

- *Existence of* Δ.

We begin to take A *as containing some elements such that* Δ *lies in* $\mathfrak{C}^0_{\tau,\mathcal{C}}(\mathbb{R} \times \mathbb{R})$ *and is globally Lipschitz for* $\mathcal{C} = A/I_A$. *From* $\Delta_\varepsilon(t, x) = \varphi_\varepsilon(t)\varphi_\varepsilon(x)$ *we have, for any* $p \geq 0$

$$|\Delta_\varepsilon(t, x)| \leq \frac{1}{\varepsilon^2}\left|\varphi\left(\frac{t}{\varepsilon}\right)\varphi\left(\frac{x}{\varepsilon}\right)\right| \leq \frac{M^2}{\varepsilon^2} \leq \frac{M^2}{\varepsilon^2}(1 + |x|)^p$$

with $M = \sup_{s \in \mathbb{R}} |\varphi(s)|$. *Then*

$$\forall T \geq 0, \forall p \geq 0, q_{T,-p}(\Delta_\varepsilon) \leq \frac{M^2}{\varepsilon^2}$$

with $q_{T,-p}(\Delta_\varepsilon) = \sup_{t \in [-T,T], x \in \mathbb{R}} (1 + |x|)^{-p} |\Delta_\varepsilon(t, x)|$.

We constuct here A *as a set of families of real elements, being a solid unitary subring of* $\mathbb{R}^{]0,1]}$, *and such that the family* $(\varepsilon)_\varepsilon \in A$ *(or* $|A|$*). As exists* $n \in \mathbb{N}$ *such that* $M^2 \leq n$ *and* $\left(\frac{n}{\varepsilon^2}\right)_\varepsilon \in |A|$, *then we have:* $(q_{T,-p}(\Delta_\varepsilon))_\varepsilon \leq |A|$ *that is to say* Δ *lies in* $\mathfrak{C}^0_{\tau,\mathcal{C}}(\mathbb{R} \times \mathbb{R})$.

- *The Cauchy-Lipschitz condition.*

We have to prove that

$$\forall \varepsilon > 0, \forall T \in \mathbb{R}, \exists M_{T,\varepsilon} > 0, \sup_{t \in [-T,T], (v,w) \in \mathbb{R}^2} |\Delta_\varepsilon(t, v) - \Delta_\varepsilon(t, w)| \leq M_{T,\varepsilon} |v - w| .$$

Writing

$$\Delta_\varepsilon(t, v) - \Delta_\varepsilon(t, w) = \frac{1}{\varepsilon^2}\varphi\left(\frac{t}{\varepsilon}\right)\left[\varphi\left(\frac{v}{\varepsilon}\right) - \varphi\left(\frac{w}{\varepsilon}\right)\right] = \frac{1}{\varepsilon^2}\varphi\left(\frac{t}{\varepsilon}\right)\varphi'(\frac{r}{\varepsilon})\left[\frac{v}{\varepsilon} - \frac{w}{\varepsilon}\right]$$

with $\frac{r}{\varepsilon} \in (\frac{v}{\varepsilon}, \frac{w}{\varepsilon})$ *we obtain*

$$\sup_{t \in [-T,T], (v,w) \in \mathbb{R}^2} |\Delta_\varepsilon(t, v) - \Delta_\varepsilon(t, w)| \leq \frac{MM'}{\varepsilon^3} |v - w|$$

with $M') = \sup_{s \in \mathbb{R}} |\varphi'(s)|$.

- *Construction of the factor ring* $\mathcal{C} = A/I_A$.

From the (d) point of Theorem 10, we have to choose A containing $\left(e^{\frac{TMM'}{\varepsilon^3}}\right)_\varepsilon$. *As it exists* $n \in \mathbb{N}$ *such that* $TMM' \leq n$, *we have* $e^{\frac{TMM'}{\varepsilon^3}} = \left(e^{\frac{1}{\varepsilon^3}}\right)^{TMM'} \leq \left(e^{\frac{1}{\varepsilon^3}}\right)^n$. *As A is a solid subring, we only have to require that* $\left(e^{\frac{1}{\varepsilon^3}}\right)^n_\varepsilon \in |A|$, *that is to say* $\left(e^{\frac{1}{\varepsilon^3}}\right)_\varepsilon \in |A|$, *and as we have* $\frac{1}{\varepsilon} < e^{\frac{1}{\varepsilon^3}}$, *then* $\left(\frac{1}{\varepsilon}\right)_\varepsilon$ *and* $(\varepsilon)_\varepsilon$ *also are in* $|A|$. *But* $e^{-\frac{1}{\varepsilon^3}} \leq e^{-\frac{1}{\varepsilon}}$ *and the previous statements are obviously verified if we require only that* $\left(e^{\frac{1}{\varepsilon}}\right)_\varepsilon \in |A|$ *.Then it suffice to take* $\mathcal{C}$ *"**overgenerated** " by the family* $\left(e^{\frac{1}{\varepsilon}}\right)_\varepsilon$ *in the meaning of Definition 3.*

- *Construction of the algebra* $\mathfrak{C}^0_{\mathcal{C}}(\mathbb{R},\mathbb{R}) = \mathcal{H}^0_A(\mathbb{R},\mathbb{R})/\mathcal{H}^0_{I_A}(\mathbb{R},\mathbb{R})$.
  We recall that

$$\mathcal{H}^0_A(\mathbb{R},\mathbb{R}) = \left\{(u_\varepsilon)_\varepsilon \in \left(\mathrm{C}^0(\mathbb{R},\mathbb{R})\right)^{]0,1]} \mid \forall K \Subset \mathbb{R}, \left((p_{K,0}(u_\varepsilon)\right)_\varepsilon \in |A|\right\},$$
$$\mathcal{H}^0_{I_A}(\mathbb{R},\mathbb{R}) = \left\{(u_\varepsilon)_\varepsilon \in \left(\mathrm{C}^0(\mathbb{R},\mathbb{R})\right)^{]0,1]} \mid \forall K \Subset \mathbb{R}, \left((p_{K,0}(u_\varepsilon)\right)_\varepsilon \in |I_A|\right\}.$$

We explicit the construction of A and I_A *"overgenerated " by the family* $\left(e^{\frac{1}{\varepsilon}}\right)_\varepsilon$. *First, consider B the subset of elements in* $(\mathbb{R}^*_+)^{]0,1]}$ *obtained as rational fractions with coefficients in* $\mathbb{R}^*_+$, *of* $e^{\frac{1}{\varepsilon}}$ *as variable. It follows that*

$$A = \left\{(a_\varepsilon)_\varepsilon \in \mathbb{R}^{]0,1]} \mid \exists\, (b_\varepsilon)_\varepsilon \in B, \exists \varepsilon_0 \in \left]0,1\right], \forall \varepsilon \prec \varepsilon_0 : |a_\varepsilon| \leq b_\varepsilon\right\}$$
$$I_A = \left\{(a_\varepsilon)_\varepsilon \in \mathbb{R}^{]0,1]} \mid \forall\, (b_\varepsilon)_\varepsilon \in B, \exists \varepsilon_0 \in \left]0,1\right], \forall \varepsilon \prec \varepsilon_0 : |a_\varepsilon| \leq b_\varepsilon\right\}$$

- *Fixed point of the map* Φ *and solution to (6).*
  We know that the map $\Phi : \mathfrak{C}^0_{\mathcal{C}}(\mathbb{R}_+,\mathbb{R}) \to \mathfrak{C}^0_{\mathcal{C}}(\mathbb{R}_+,\mathbb{R})$ *such that*

$$[x_\varepsilon] \mapsto [\Phi_\varepsilon(x_\varepsilon)]$$

is a contraction, with $z = [z_\varepsilon]$ *as fixed point,* z_ε *being the unique fixed point of* Φ_ε *verifying*

$$z_\varepsilon(t) = \xi_\varepsilon + \textstyle\int_0^t f_\varepsilon(s, z_\varepsilon(s))ds$$

where $(\xi_\varepsilon)_\varepsilon$ *is a given representative of the given element* $\xi \in \widetilde{\mathbb{R}}$. *Moreover z is the unique solution to the Cauchy-Lipschitz problem (6).*

4.3 Towards the transport equation with irregular coefficients

We consider the following Cauchy problem for the transport equation in (t,x)-variables

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = f, \quad u(0,x) = u_0(x) \tag{7}$$

In a general case (typically the distribution one), the nonlinear term needs some regularity to be defined. And it is not the case if α is a distribution in (t,x)-variables. Under some hypotheses on α: weak regularity (of Sobolev type), control of its uniform divergency in space and some space increasing condition, Di Perna and Lions [5] have obtained some results with uniqueness and stability. More recently the paper of L. Ambrosio [1] studies the same subject.

However this context don't permits to define and solve (7) when α is a distribution in (t, x)-variables.

In a simplified case where $f = 0$ and $\alpha = a_t \otimes 1_x$ where $a_t \in \mathcal{D}'(\mathbb{R})$, the problem is posed an solved in [4]. When $\alpha \in \mathcal{D}'(\mathbb{R}^2)$ we turn back to the regular case as the starting point of our generalized methods. It is well known that when α, f and u_0 are of class C^1, the problem (7) admits a unique solution of class C^1 given by integrating along the characteristics

$$u(t,x) = u_0(X(0,t,x) + \int_0^t f(s, X(s,t,x)ds. \tag{8}$$

We can see that the regular solution of (7) is linked to the following Cauchy-Lipschitz problem of which $X(s,t,x)$ is the unique solution

$$\begin{cases} \dfrac{dX}{ds}(s,t,x) = \alpha(s, X(s,t,x)) \\ \qquad X(t,t,x) = x. \end{cases}$$

When α and f are not continuous, N. Caroff [3] propose an approach based on the approximation of discontinuous data by C^1 function α_n and f_n and an Egorov theorem. She gives a result similar to (8) when the irregularities are controled by: $u_0 \in Lip_{loc}(\mathbb{R}, \mathbb{R}); \alpha \in L^\infty(\mathbb{R}^2, \mathbb{R})$ and for some $\delta > 0, \delta^{-1} \leq \alpha(t,x) \leq \delta; f \in L^\infty(\mathbb{R}^2, \mathbb{R}); \forall x \in \mathbb{R}, \alpha(., x)$ and $f(., x)$ are locally Lipschitz uniformly in x.

But as in the previous case, this context don't permit to define and solve (7) when α is a distribution in (t, x)-variables.

With the generalized methods over exposed we are trying to generalize that result to the distributional case in consideration.

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