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Fixed points in algebras of generalized functions and applications

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Abstract

I propose a self contained research paper. I hope it adds some news ideas and results to the fixed point theory in the framework of generalized functions algebras, with application to the Cauchy-Lipschitz problem in a generalized formulation including strongly irregular cases. This leads to the transport equation with distributions as coefficients we wish to treat later.

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1 Introduction

The paper is structured in three following sections (2, 3, 4).

Section 2 is devoted to the meaning of contraction in locally convex spaces or algebras. Fixed points of some operators F with a contraction property in some spaces (or algebras) E are involved to solve many problems in functional analysis. There are at least four journals on Fixed Point Theory, many publications on the subject (between [12] to [2]) and many books as the monograph [7]. However we are interested in the classical application to the Cauchy-Lipschitz theorem locally or globally formulated. Then the definition of a contraction we need is only a slight generalization (we suppose that E is locally convex) of the Frigon-Granás one ([6]) given when E is a Fréchet space. It leads to the expected result given in Theorem 1: *Any contraction $F : E \rightarrow E$ has a fixed point. If E is Hausdorff, this fixed point is unique.*

But the irregular cases of the Cauchy-Lipschitz theorem suggests a generalized formulation which is the subject of **Section 3** and invites to define some operator Φ in a factor algebra \mathcal{A} of generalized functions. \mathcal{A} is constructed ([10]) from a basic locally convex algebra (\mathcal{E}, τ) . The elements $x \in \mathcal{A}$ are classes $[x_\lambda]$ of some families $(x_\lambda)_{\lambda \in \Lambda}$ with "moderateness" linked to a factor ring \mathcal{C} of so-called generalized numbers. Under some hypotheses, Φ is well defined by

$$\mathcal{A} \ni [x_\lambda] = x \rightarrow \Phi(x) = [\Phi_\lambda(x_\lambda)] \in \mathcal{A}.$$

for some operator Φ_λ in \mathcal{E} . We suppose in addition that each Φ_λ is a contraction in $(\mathcal{E}, \tau_\lambda)$ endowed with a locally convex topology τ_λ depending on λ . Then Φ_λ has a fixed point z_λ in \mathcal{E} and unique if $(\mathcal{E}, \tau_\lambda)$ is Hausdorff. This leads to define Φ as a contraction in \mathcal{A} (Definition 4). I don't see any similar idea in the framework of generalized functions. I hope that it a good enough (or not too bad) one! Moreover with some additional hypotheses, we can prove the moderateness of $(z_\lambda)_\lambda$ and find (Theorem 6) a fixed point z of Φ through

$$(1) \quad \mathcal{A} \ni [z_\lambda] = z = \Phi(z) = [\Phi_\lambda(z_\lambda)] \in \mathcal{A}$$

But the uniqueness of z_λ is not sufficient to prove that $[z_\lambda] = z$ is the unique fixed point of Φ . Nevertheless we can obtain this uniqueness when taking, as in Theorem 10

$$\Phi_\lambda(x)(t) = x_0 + \int_0^t f_\lambda(s, x(s)) ds$$

for x_0 given in \mathbb{R} , $x \in C^0(\mathbb{R}_+, \mathbb{R})$, $t \in \mathbb{R}_+$ and $f_\lambda \in C^0(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ from what it is clear that Φ_λ is a map $\mathcal{E} \rightarrow \mathcal{E}$ with $\mathcal{E} = C^0(\mathbb{R}_+, \mathbb{R})$.

In a Subsection (of Section 3) we extend the results to the case where Φ is an operator in the product \mathcal{A}^m of algebras constructed on \mathcal{E}^m . The natural topology (denoted τ^m) on the product \mathcal{E}^m is defined by the family $(p_i^{(m)})_{i \in I}$ of seminorms such that $p_i^{(m)}(x) = p_i^{(m)}(x_1 \dots x_m) = \sum_{k=1}^{k=m} p_i(x_k)$. We denote by $(\mathcal{E}^m, \tau_\lambda^m)$ the topological space \mathcal{E}^m endowed by the family $(q_{\lambda,i}^{(m)})_{i \in I}$ with $q_{\lambda,i}^{(m)}(x) = \sum_{k=1}^{k=m} q_{\lambda,i}(x_k)$ for a given family $(q_{\lambda,i})_{i \in I}$ of seminorm on \mathcal{E} . The main result of that section is Theorem 8: *Any contraction $\Phi : \mathcal{A}^m \rightarrow \mathcal{A}^m$ has a fixed point in \mathcal{A}^m*

The expected application is the Cauchy-Lipschitz generalized problem studied in **Section 4**. Thanks to an embedding $(\mathfrak{C}_C^1(J, \mathbb{R}))^m \rightarrow (\mathfrak{C}_C^0(J, \mathbb{R}))^m$ with $\text{Im}(\mathfrak{C}_C^1(J, \mathbb{R}))^m$ as image, it is to solve

$$(2) \quad \begin{cases} \partial x = f(., x) \\ x(t_0) = \xi \end{cases}$$

with $x \in \text{Im}(\mathfrak{C}_C^1(J, \mathbb{R}))^m \subset (\mathfrak{C}_C^0(J, \mathbb{R}))^m$ and $f \in (\mathfrak{C}_{\tau, C}^0(J \times \mathbb{R}^m, \mathbb{R}))^m$ globally Lipschitz, for some ring of generalized numbers $C = A/I_A$, with $t_0 \in J$ and ξ is a given element $\in \widetilde{\mathbb{R}^m}$. The "derivation" ∂ is a map from $\text{Im}(\mathfrak{C}_C^1(J, \mathbb{R}))^m$ to $(\mathfrak{C}_C^0(J, \mathbb{R}))^m$. The algebra $(\mathfrak{C}_C^0(J, \mathbb{R}))^m$ (resp. $(\mathfrak{C}_C^1(J, \mathbb{R}))^m$) generalize $(C^0(J, \mathbb{R}))^m$ (resp. $(C^1(J, \mathbb{R}))^m$) and $(\mathfrak{C}_{\tau, C}^0(J \times \mathbb{R}^m, \mathbb{R}))^m$ is a generalization of $(C^0(J \times \mathbb{R}^m, \mathbb{R}))^m$ without use of derivatives.

The main result of that section (Theorem 10) is *that it exists a ring of generalized numbers $C = A/I_A$, such that $f \in (\mathfrak{C}_{\tau, C}^0(J \times \mathbb{R}^m, \mathbb{R}))^m$ and a map $\Phi : (\mathfrak{C}_C^0(J, \mathbb{R}))^m \rightarrow (\mathfrak{C}_C^0(J, \mathbb{R}))^m$ with an unique fixed point solving the Cauchy-Lipschitz problem (2) with $t_0 \in R_+$ and $\xi \in \widetilde{\mathbb{R}^m}$.*

All our ideas, technics and results are explicitly detailed and summarized in the final example (Example 2).

The last subsection shows a link between the Cauchy-Lipschitz theorem and the transport equation. We cite some results when the coefficients have a weak regularity of Sobolev type ([5]) or with controlled irregularities [3]. But it is not the case of distributions we wish to treat later with our generalized methods.

2 Contractions in locally convex and complete spaces

We suppose here tat the (seminormed with $\mathcal{P} = (p_i)_{i \in I}$) space E is sequentially complete. A basis of 0-neighbourhood is the set of all "balls" of the seminorms $(p_i)_{i \in I}$

$$\beta(i, r) = \{x \in E / p_i(x) < r\}$$

for all $i \in I$ and $r > 0$. Then, $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence iff

$$(\forall \varepsilon > 0) (\forall i \in I) (\exists n_0) (\forall n, p) (n > n_0, p > 0 \implies p_i(x_{n+p} - x_n) < \varepsilon)$$

and E is sequentially complete if any Cauchy sequence converges to an element e in E .

Definition 1 A map $F : E \rightarrow E$ is called a **contraction** if for all $i \in I$ it exists $k_i < 1$ such that

$$\forall (x, y) \in E \times E, p_i(F(x) - F(y)) \leq k_i p_i(x - y).$$

This definition is an obvious generalization of the Frigon-Granas one [6] given when \mathcal{P} is a countable family of semi norms $(p_i)_{i \in \mathbb{N}}$ rendering E a Fréchet space.

In this case, F is not necessary a contraction in the usual sense when E is endowed with the metric $d(x, y) = \sum_{i \in \mathbb{N}} p_i(x - y) / (1 + p_i(x - y))$.

Theorem 1 Any contraction $F : E \rightarrow E$ has a fixed point. If E is Hausdorff, this fixed point is unique.

Proof. Starting from $x_0 \in E$, define $x_{n+1} = F(x_n)$ by induction. It is easy to verify that x_n is a Cauchy sequence in the complete space E and converges to some $x \in E$. The contraction property of the map F implies obviously its continuity. Then, passing to the limit in $x_{n+1} = F(x_n)$, we obtain that x is a fixed point of E . If E is Hausdorff, for all $z \neq 0$ it exists $V \in \mathcal{V}(0)$ such that $z \notin V$. Then it exists i (depending on z) such that $p_i(z) > 0$. If x and y are two different fixed points of F , it exists j (depending on $x - y$) such that

$$0 < p_j(x - y) = p_j(F(x) - F(y)) \leq k_j p_j(x - y) < p_j(x - y)$$

which gives a contradiction ■

Notation 1 We denote by

- Λ a set of indices
- (E, τ) the space E endowed with the topology τ of the previous family $\mathcal{P} = (p_i)_{i \in I}$
- $(E, \tau_\lambda)_{\lambda \in \Lambda}$ the family of spaces (E, τ_λ) seminormed by the family $\mathcal{Q}_\lambda = (q_{\lambda, i})_{i \in I}$
- $(F_\lambda)_{\lambda \in \Lambda}$ a family of contractions $F_\lambda : (E, \tau_\lambda) \rightarrow (E, \tau_\lambda)$.

Theorem 2 Each F_λ has a fixed point $z_\lambda \in E$. If in addition we suppose that (E, τ) is Hausdorff and for each $i \in I$ and $\lambda \in \Lambda$ it exists a strictly positive constant $a_{\lambda, i}$ such that

$$a_{\lambda, i} p_i \leq q_{\lambda, i}$$

then z_λ is unique

Proof. From Theorem 1 we know that each F_λ has a fixed point $z_\lambda \in E$. If (E, τ) is Hausdorff, for each $x \neq 0$ in E , it exists $i(x) \in I$ such that $p_{i(x)}(x) > 0$. Then we have $q_{\lambda, i(x)}(x) > 0$ which implies that (E, τ_λ) is Hausdorff. As $F_\lambda : (E, \tau_\lambda) \rightarrow (E, \tau_\lambda)$ is a contraction, z_λ is unique. ■

3 Contractions in generalized spaces or algebras

3.1 The $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ setting

We consider the setting of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebras which is an extension of Colombeau's one. It allows to construct multiparametrized generalized spaces or algebras where the "asymptotic \mathcal{C} " is given independantly from the basis topological space or algebra \mathcal{C} . To summarize the definitions and results given in [10, 11] the asymptotics is given by

- (1) Λ : a set of indices;
- (2) A : a solid subring of the ring \mathbb{K}^Λ ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}); this means that whenever $(|s_\lambda|)_\lambda \leq (|r_\lambda|)_\lambda$ for some $((s_\lambda)_\lambda, (r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times A$, that is, $|s_\lambda| \leq |r_\lambda|$ for all λ , it follows that $(s_\lambda)_\lambda \in A$;

(3) I_A : a solid ideal of A .

Then \mathcal{C} is defined as the factor ring A/I_A . On the other hand we give

(4) \mathcal{E} : a \mathbb{K} -topological space endowed with a family $\mathcal{P} = (p_i)_{i \in I}$ of semi-norms.

Define $|B| = \{(|r_\lambda|)_\lambda, (r_\lambda)_\lambda \in B\}$, $B = A$ or I_A , and set

$$\begin{aligned}\mathcal{H}_{(A, \mathcal{E}, \mathcal{P})} &= \left\{ (u_\lambda)_\lambda \in [\mathcal{E}]^\Lambda \mid \forall i \in I, ((p_i(u_\lambda))_\lambda \in |A| \right\} \\ \mathcal{H}_{(I_A, \mathcal{E}, \mathcal{P})} &= \left\{ (u_\lambda)_\lambda \in [\mathcal{E}]^\Lambda \mid \forall i \in I, (p_i(u_\lambda))_\lambda \in |I_A| \right\}\end{aligned}$$

The following result summarize some results recalled in [9].

Theorem 3 *If A is a solid subring of \mathbb{K}^Λ , then $\mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}$ is an A -module, and an A -algebra if \mathcal{E} is a topological algebra.*

Moreover, if I_A is an ideal of A sharing the solidness property, then $\mathcal{H}_{(I_A, \mathcal{E}, \mathcal{P})}$ is an A -linear subspace of $\mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}$, and an ideal of $\mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}$ if \mathcal{E} is a topological algebra.

As a consequence, the factor space $\mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}/\mathcal{H}_{(I_A, \mathcal{E}, \mathcal{P})}$ is again an A -module, but also an A/I_A -module (and of course an algebra, if \mathcal{E} is a topological algebra).

For $(\mathcal{E}, \mathcal{P}) = (\mathbb{K}, \{|\cdot|\})$, we get $\mathcal{H}_{(A, \mathbb{K}, |\cdot|)}/\mathcal{H}_{(I_A, \mathbb{K}, |\cdot|)} = A/I_A$.

Remark 1 *If we require \mathcal{E} to be a topological algebra, this means that multiplication in \mathcal{E} is continuous for the topology defined by the family of seminorms.*

Definition 2 *The factor ring $\mathcal{C} = A/I_A$ is called the ring of generalized numbers (associated to A and I_A), and the \mathcal{C} -algebra*

$$\mathcal{A}_{\mathcal{C}}(\mathcal{E}, \mathcal{P}) := \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}/\mathcal{H}_{(I_A, \mathcal{E}, \mathcal{P})}$$

is called the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebra of \mathcal{C} -generalized functions. We denote by $[x_\lambda]$ the class in $\mathcal{A}_{\mathcal{C}}(\mathcal{E}, \mathcal{P})$ of the family $(x_\lambda)_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}$.

Example 1 *Let us define*

$$\begin{aligned}A &= \left\{ (r_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]} \mid \exists m \in \mathbb{N} : |u_\varepsilon| = o(\varepsilon^{-m}), \text{ as } \varepsilon \rightarrow 0 \right\} \\ I_A &= \left\{ (r_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]} \mid \forall q \in \mathbb{N} : |u_\varepsilon| = o(\varepsilon^q), \text{ as } \varepsilon \rightarrow 0 \right\}.\end{aligned}$$

In this case (with $\mathcal{E} = C^\infty(\mathbb{R}^n)$ and $\mathcal{P} = \{p_{K, \alpha} : f \mapsto \|\partial^\alpha f\|_{K \in \mathbb{R}^n, \alpha \in \mathbb{N}^n}\}$), the algebra $\mathcal{A} = \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}/\mathcal{H}_{(I_A, \mathcal{E}, \mathcal{P})}$ is exactly the so-called special Colombeau algebra $\mathcal{G}(\mathbb{R}^n)$.

Remark 2 *If \mathcal{E} is a sheaf of \mathbb{K} -topological algebras over a topological space X , we can prove that the factor $\mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}/\mathcal{H}_{(I_A, \mathcal{E}, \mathcal{P})}$ is at least a presheaf satisfying the localization principle. But we don't need the sheaf structure in the sequel.*

We are going now to the the concept of "Overgenerated algebras". It is useful when constructing a $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -structure to solve some problems with irregular data or coefficients.

Choose B_p a finite family of p nets in $(\mathbb{R}_+^*)^\Lambda$ (usually given by the asymptotic structure of the problem.) Consider B the subset of elements in $(\mathbb{R}_+^*)^\Lambda$ obtained as **rational fractions** with coefficients in \mathbb{R}_+^* , of elements in B_p as variables.

Definition 3 *Define*

$$A = \left\{ (a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \exists (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda \right\}.$$

We say that A **is overgenerated** by B_p (and it is easy to see that A is a solid subring of \mathbb{K}^Λ). If I_A is some solid ideal of A , we also say that $\mathcal{C} = A/I_A$ **is overgenerated** by B_p . For example, as a “canonical” ideal of A , we can take

$$I_A = \{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \forall (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda\}.$$

This definition implies that B is stable by inverse.

3.2 Contraction operator in $\mathcal{A}_C(\mathcal{E}, \mathcal{P})$

First, we are looking if it is possible to define a map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ by means of a given family $(\Phi_\lambda)_{\lambda \in \Lambda}$ of maps $\Phi_\lambda : \mathcal{E} \rightarrow \mathcal{E}$. The general requirement is given in the following

Lemma 4 *Let $(\Phi_\lambda)_{\lambda \in \Lambda}$ be a given family of maps $\mathcal{E} \rightarrow \mathcal{E}$.*

Suppose that for each $(x_\lambda)_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}$ and $(i_\lambda)_\lambda \in \mathcal{H}_{(I_A, \mathcal{E}, \mathcal{P})}$ we have

- (i) $(\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}$,
 - (ii) $(\Phi_\lambda(x_\lambda + i_\lambda))_\lambda - (\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{I}_{(A, \mathcal{E}, \mathcal{P})}$.
- Then, $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is well defined by*

$$\mathcal{A} \ni [x_\lambda] = x \rightarrow \Phi(x) = [\Phi_\lambda(x_\lambda)] \in \mathcal{A}.$$

Proof. From (i) we see that the class $[\Phi_\lambda(x_\lambda)]$ lies in \mathcal{A} . Let $x_\lambda + i_\lambda$ be another representative of $x = [x_\lambda]$. From (ii) we have $[\Phi_\lambda(x_\lambda + i_\lambda)] = [\Phi_\lambda(x_\lambda)]$. Then, Φ is well defined. ■

Theorem 5 *Let be fulfilled the following hypotheses: It exists a family $(a_{n,\lambda})_{n,\lambda \in \mathbb{N} \times \Lambda}$ of positive numbers with $(a_{n,\lambda})_\lambda \in A$, verifying: for each $i \in I$ there exists $N(i)$ and $j(i) \in I$ such that, for each $\lambda \in \Lambda$ and $e \in \mathcal{E}$*

$$p_i(\Phi_\lambda(e)) \leq \sum_{n=0}^{N(i)} a_{n,\lambda} p_{j(i)}^n(e).$$

Then for each $(x_\lambda)_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}$ we have $(\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}$.

Proof. $(x_\lambda)_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}$ implies $\forall i \in I, (p_i(x_\lambda))_\lambda \in |A|$. Then, $\sum_{n=0}^{N(i)} a_{n,\lambda} p_{j(i)}^n(x_\lambda) \in |A|$. As A is a solid subring of \mathbb{K}^Λ , it follows that $(p_i(\Phi_\lambda(x_\lambda)))_\lambda \in |A|$ ■

Definition 4 *The following hypotheses permit to well define a map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ from the family $(\Phi_\lambda)_{\lambda \in \Lambda}$ and to call it a **contraction**.*

- (a) *for each $(x_\lambda)_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}$, $(\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}$,*
- (b) *Each Φ_λ is a contraction in $(\mathcal{E}, \tau_\lambda)$ endowed with the family $\mathcal{Q}_\lambda = (q_{\lambda,i})_{i \in I}$, and the corresponding contraction constants are denoted by $k_{\lambda,i} < 1$,*
- (c) *For each $i \in I$ and $\lambda \in \Lambda$ it exist some strictly positive constants $\alpha_{\lambda,i}$ and $\beta_{\lambda,i}$ such that*

$$\alpha_{\lambda,i} p_i \leq q_{\lambda,i} \leq \beta_{\lambda,i} p_i,$$

- (d) *For each $i \in I$, the families $\left(\frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}}\right)_\lambda$ and $\left(\frac{1}{1 - k_{\lambda,i}}\right)_\lambda$ lies in $|A|$.*

Theorem 6 *Any contraction $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ has a fixed point in \mathcal{A} .*

Proof. Remark that condition (a) which is (i) in Lemma 4 can be fulfilled when giving the hypotheses in Theorem 5, that is to say $(a_{n,\lambda})_\lambda \in A$. Now we have from (c)

$$p_i(\Phi_\lambda(x_\lambda + i_\lambda) - (\Phi_\lambda(x_\lambda))_\lambda) \leq \frac{1}{\alpha_{\lambda,i}} k_{i,\lambda} q_{\lambda,i}(i_\lambda) \leq \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} p_i(i_\lambda)$$

from what we deduce that $(p_i(\Phi_\lambda(x_\lambda + i_\lambda) - (\Phi_\lambda(x_\lambda))_\lambda))_\lambda \in A$ and the condition (ii) in Lemma 4 is verified. Then Φ is well defined. From Theorem 2 we know that each Φ_λ has a fixed point z_λ obtained as limit of the Cauchy sequence $z_{n,\lambda}$ defined by induction by $z_{n+1,\lambda} = \Phi_\lambda(z_{n,\lambda})$. Starting from $z_0 = [z_{0,\lambda}] \in \mathcal{A}$, we deduce that $z_1 = [\Phi_\lambda(z_{0,\lambda})] \in \mathcal{A}$, and $z_1 - z_0 \in \mathcal{A}$. That is to say $p_i(z_{1,\lambda} - z_{0,\lambda})_\lambda \in |A|$.

By induction we can compute for all $n, p \in \mathbb{N}$

$$q_{\lambda,i}(z_{n+p,\lambda} - z_{n,\lambda}) \leq \frac{k_{\lambda,i}^n}{1 - k_{\lambda,i}} q_{\lambda,i}(z_{1,\lambda} - z_{0,\lambda}) \text{ giving}$$

$$q_{\lambda,i}(z_{p,\lambda} - z_{0,\lambda}) \leq \frac{1}{1 - k_{\lambda,i}} q_{i,\lambda}(z_{1,\lambda} - z_{0,\lambda}).$$

When taking the limit z_λ of $z_{p,\lambda}$ in $(\mathcal{E}, \tau_\lambda)$ when $p \rightarrow \infty$, we get

$$q_{\lambda,i}(z_\lambda - z_{0,\lambda}) \leq \frac{1}{1 - k_{\lambda,i}} q_{\lambda,i}(z_{1,\lambda} - z_{0,\lambda}).$$

Writing now $q_{\lambda,i}(z_\lambda) \leq q_{\lambda,i}(z_\lambda - z_{0,\lambda}) + q_{\lambda,i}(z_{0,\lambda})$ we have

$$p_i(z_\lambda) \leq \frac{1}{\alpha_{\lambda,i}} q_{i,\lambda}(z_\lambda) \leq \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} \left[\frac{1}{1 - k_{\lambda,i}} (p_i(z_{1,\lambda} - z_{0,\lambda}) + p_i(z_{0,\lambda})) \right].$$

Then, from the hypotheses $(p_i(z_\lambda))_\lambda \in |A|$, that is to say $((p_i(z_\lambda)))_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$. If $z = [z_\lambda]$, then we have $\Phi(z) = [\Phi_\lambda(z_\lambda)] = [z_\lambda] = z$. Then z is a fixed point of Φ .

However following the under Remark 3 we cannot prove the uniqueness of z without other hypotheses than uniqueness of fixed points of Φ_λ . ■

3.3 Contractions in product of algebras

Definition 5 For $m \in \mathbb{N}$, we define

$$\begin{aligned} \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}^m &= \left\{ (u_\lambda)_\lambda = (u_{1,\lambda}, \dots, u_{m,\lambda}) \in [\mathcal{E}^m]^\Lambda \mid \forall i \in I, \forall k = 1, \dots, m, ((p_i(u_{k,\lambda}))_\lambda \in |A|) \right\}, \\ \mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}^m &= \left\{ (u_\lambda)_\lambda = (u_{1,\lambda}, \dots, u_{m,\lambda}) \in [\mathcal{E}^m]^\Lambda \mid \forall i \in I, \forall k = 1, \dots, m, ((p_i(u_{k,\lambda}))_\lambda \in |I_A|) \right\}. \end{aligned}$$

According to the results and definitions in Subsection 3.1, if A is a solid subring of \mathbb{K}^Λ , then $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}^m$ is an A -module, and an A -algebra if \mathcal{E} is a topological algebra. Moreover, if I_A is an ideal of A sharing the solidness property, then $\mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}^m$ is an A -linear subspace of $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}^m$, and an ideal of $\mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}^m$ if \mathcal{E} is a topological algebra. As a consequence, the factor space $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}^m / \mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}^m$ is again an A -module, but also an A/I_A -module (and of course an algebra, if \mathcal{E} is a topological algebra).

Definition 6 We pose $\mathcal{A}_C^m(\mathcal{E}, \mathcal{P}) := \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}^m / \mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}^m$ and denote by $[x_\lambda]$ the class in $\mathcal{A}_C^m(\mathcal{E}, \mathcal{P})$ of the family $(x_\lambda)_\lambda = (x_{1,\lambda}, \dots, x_{m,\lambda})_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}^m$.

First, it is possible as previously, to define a map $\Phi : \mathcal{A}^m \rightarrow \mathcal{A}^m$ by means of a given family $(\Phi_\lambda)_{\lambda \in \Lambda}$ of maps $\Phi_\lambda : \mathcal{E}^m \rightarrow \mathcal{E}^m$. The general requirement is similar to the previous one given in Lemma 4.

Suppose that for each $(x_\lambda)_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}^m$ and $(i_\lambda)_\lambda \in \mathcal{H}_{(I, \mathcal{E}, \mathcal{P})}^m$ we have

- (i) $(\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}^m$,
- (ii) $(\Phi_\lambda(x_\lambda + i_\lambda))_\lambda - (\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{H}_{(I, \mathcal{E}, \mathcal{P})}^m$.

Then, $\Phi : \mathcal{A}^m \rightarrow \mathcal{A}^m$ is well defined by

$$\mathcal{A}^m \ni [x_\lambda] = x \rightarrow \Phi(x) = [\Phi_\lambda(x_\lambda)] \in \mathcal{A}^m.$$

Now, we are defining a contraction property for the given family $(\Phi_\lambda)_{\lambda \in \Lambda}$ of maps $\Phi_\lambda : \mathcal{E}^m \rightarrow \mathcal{E}^m$.

Notation 2 The natural topology (denoted τ^m) on the product \mathcal{E}^m is defined by the family $(p_i^{(m)})_{i \in I}$ of seminorms such that $p_i^{(m)}(x) = p_i^{(m)}(x_1 \dots x_m) = N(p_i(x_1), \dots, p_i(x_m))$ where N is

any norm on \mathbb{R}^n . For example, we can choose $p_i^{(m)}(x) = \sum_{k=1}^{k=m} p_i(x_k)$. Denote by $(\mathcal{E}^m, \tau_\lambda^m)$ the

topological space \mathcal{E}^m endowed by the family $(q_{\lambda, i}^{(m)})_{i \in I}$ with $q_{\lambda, i}^{(m)}(x) = \sum_{k=1}^{k=m} q_{\lambda, i}(x_k)$ for a given family $(q_{\lambda, i})_{i \in I}$ of seminorm on \mathcal{E}

Definition 7 $(\Phi_\lambda)_{\lambda \in \Lambda}$ is called a family of **contractions in \mathcal{E}^m** if for each $(i, \lambda) \in I \times \Lambda$ it exists a semi norm $q_{i, \lambda}$ and a constant $k_{\lambda, i} < 1$ such that for all $(x, y) \in \mathcal{E}^m \times \mathcal{E}^m$

$$(3) \quad q_{\lambda, i}^{(m)}(\Phi_\lambda(x) - \Phi_\lambda(y)) \leq k_{\lambda, i} \sum_{k=1}^{k=m} q_{\lambda, i}(x_k - y_k).$$

Proposition 7 Each contraction Φ_λ in \mathcal{E}^m has a fixed point z_λ . Moreover if (\mathcal{E}, τ) is Hausdorff and for each $(i, \lambda) \in I \times \Lambda$ it exists a strictly positive constant $a_{i, \lambda}$ such that $a_{i, \lambda} p_i \leq q_{i, \lambda}$, then z_λ is unique.

Proof. We deduce from (3) that $q_{\lambda, i}^{(m)}(\Phi_\lambda(x) - \Phi_\lambda(y)) \leq k_{\lambda, i} q_{\lambda, i}^{(m)}(x - y)$. From Theorem 2 we know that each Φ_λ has a fixed point $z_\lambda \in \mathcal{E}^m$. If (\mathcal{E}, τ) is Hausdorff, for each nonnull $r \in \mathcal{E}$, it exists $i \in I$ such that $p_i(r) > 0$. Then $q_{\lambda, i}(r) > 0$ which implies that $(\mathcal{E}, \tau_\lambda)$ is Hausdorff. If $y_\lambda \neq z_\lambda$ is another fixed point of Φ_λ , it exists at least $h \in \mathbb{N}$ with $1 \leq h \leq m$ such that $y_{\lambda, h} - z_{\lambda, h} \neq 0$. Therefore, exists $j \in I$ such that

$$0 < q_{\lambda, j}(y_{\lambda, h} - z_{\lambda, h}) \leq q_{\lambda, j}^{(m)}(y_\lambda - z_\lambda) = q_{\lambda, j}^{(m)}(\Phi_\lambda(y_\lambda) - \Phi_\lambda(z_\lambda)) \leq k_{\lambda, j} q_{\lambda, j}^{(m)}(y_\lambda - z_\lambda) < q_{\lambda, j}^{(m)}(y_\lambda - z_\lambda)$$

which leads to a contradiction ■

And now, to achieve the construction of $\Phi : \mathcal{A}^m \rightarrow \mathcal{A}^m$ and prove the existence of a fixed point in the same way as in Theorem 6, we propose the following

Definition 8 The following hypotheses permit to well define a map $\Phi : \mathcal{A}^m \rightarrow \mathcal{A}^m$ from the family $(\Phi_\lambda)_{\lambda \in \Lambda}$ and to call it a **contraction**.

- (a) for each $(x_\lambda)_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}^m$, $(\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{H}_{(A, \mathcal{E}, \mathcal{P})}^m$,
- (b) The map $\Phi_\lambda : \mathcal{E}^m \rightarrow \mathcal{E}^m$ is a contraction of \mathcal{E}^m following Definition 7,
- (c) For each $i \in I$ and $\lambda \in \Lambda$ it exist some strictly positive constants $\alpha_{\lambda, i}$ and $\beta_{\lambda, i}$ such that

$$\alpha_{\lambda, i} p_i \leq q_{\lambda, i} \leq \beta_{\lambda, i} p_i,$$

- (d) For each $i \in I$, the families $\left(\frac{\beta_{\lambda, i}}{\alpha_{\lambda, i}}\right)_\lambda$ and $\left(\frac{1}{1 - k_{\lambda, i}}\right)_\lambda$ lies in $|A|$.

Theorem 8 Any contraction $\Phi : \mathcal{A}^m \rightarrow \mathcal{A}^m$ has a fixed point in \mathcal{A}^m .

Proof. First we can re-write $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}^m$ and $\mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}^m$ as

$$\begin{aligned}\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}^m &= \left\{ (u_\lambda)_\lambda \in [\mathcal{E}^m]^\Lambda \mid \forall i \in I, \left((p_i^{(m)}(u_\lambda))_\lambda \in |A| \right) \right\}, \\ \mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}^m &= \left\{ (u_\lambda)_\lambda \in [\mathcal{E}^m]^\Lambda \mid \forall i \in I, \left((p_i^{(m)}(u_\lambda))_\lambda \in |I_A| \right) \right\}.\end{aligned}$$

We deduce from (c) that

$$\alpha_{\lambda,i} p_i^{(m)} \leq q_{\lambda,i}^{(m)} \leq \beta_{\lambda,i} p_i^{(m)}.$$

For $(x_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}^m$ and $(i_\lambda)_\lambda \in \mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}^m$ we have

$$p_i^{(m)} (\Phi_\lambda(x_\lambda + i_\lambda) - \Phi_\lambda(x_\lambda)) \leq \frac{1}{\alpha_{\lambda,i}} k_{\lambda,i} q_{\lambda,i}^{(m)}(i_\lambda) \leq \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} p_i^{(m)}(i_\lambda) = \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} \sum_{k=1}^{k=m} p_i(i_k).$$

Then, $\left(p_i^{(m)} (\Phi_\lambda(x_\lambda + i_\lambda) - \Phi_\lambda(x_\lambda)) \right)_\lambda \in |I_A|$ and $(\Phi_\lambda(x_\lambda + i_\lambda))_\lambda - (\Phi_\lambda(x_\lambda))_\lambda \in \mathcal{H}_{(I_A,\mathcal{E},\mathcal{P})}^m$. It follows that $\Phi : \mathcal{A}^m \rightarrow \mathcal{A}^m$ is well defined by

$$\mathcal{A}^m \ni [x_\lambda] = x \rightarrow \Phi(x) = [\Phi_\lambda(x_\lambda)] \in \mathcal{A}^m.$$

From (b) and Theorem 2 we know that each Φ_λ has a fixed point $z_\lambda = (z_{1,\lambda}, \dots, z_{m,\lambda}) \in \mathcal{E}^m$ and from Theorem 1 we know that z_λ is obtained as limit of the Cauchy sequence $z_{n,\lambda}$ defined by induction by $z_{n+1,\lambda} = \Phi_\lambda(z_{n,\lambda})$. Starting from $z_0 = [z_{0,\lambda}] \in \mathcal{A}^m$, we deduce that $z_1 = [\Phi_\lambda(z_{0,\lambda})] \in \mathcal{A}^m$, and $z_1 - z_0 \in \mathcal{A}^m$. That is to say $p_i^{(m)}(z_{1,\lambda} - z_{0,\lambda})_\lambda \in |A|$.

By induction we can compute for all $n, p \in \mathbb{N}$

$$q_{\lambda,i}^{(m)}(z_{n+p,\lambda} - z_{n,\lambda}) \leq \frac{k_{\lambda,i}^n}{1 - k_{\lambda,i}} q_{\lambda,i}^{(m)}(z_{1,\lambda} - z_{0,\lambda}) \text{ giving}$$

$$q_{\lambda,i}^{(m)}(z_{p,\lambda} - z_{0,\lambda}) \leq \frac{1}{1 - k_{\lambda,i}} q_{\lambda,i}^{(m)}(z_{1,\lambda} - z_{0,\lambda}).$$

When taking the limit z_λ of $z_{p,\lambda}$ in $(\mathcal{E}^m, \tau_\lambda^m)$ when $p \rightarrow \infty$, we get

$$q_{\lambda,i}^{(m)}(z_\lambda - z_{0,\lambda}) \leq \frac{1}{1 - k_{\lambda,i}} q_{\lambda,i}^{(m)}(z_{1,\lambda} - z_{0,\lambda}).$$

Writing now $q_{\lambda,i}^{(m)}(z_\lambda) \leq q_{\lambda,i}^{(m)}(z_\lambda - z_{0,\lambda}) + q_{\lambda,i}^{(m)}(z_{0,\lambda})$ we have

$$p_i^{(m)}(z_\lambda) \leq \frac{1}{\alpha_{\lambda,i}} q_{\lambda,i}^{(m)}(z_\lambda) \leq \frac{\beta_{\lambda,i}}{\alpha_{\lambda,i}} \left[\frac{1}{1 - k_{\lambda,i}} (p_i^{(m)}(z_{1,\lambda} - z_{0,\lambda}) + p_i^{(m)}(z_{0,\lambda})) \right].$$

From the hypotheses of Definition 8, we have $\left(p_i^{(m)}(z_\lambda) \right)_\lambda \in |A|$, then $((z_\lambda))_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}^m$. Therefore, if $z = [z_\lambda]$, we have $\Phi(z) = [\Phi_\lambda(z_\lambda)] = [z_\lambda] = z$. Then z is a fixed point of Φ . ■

Remark 3 If $y = [y_\lambda]$ is any fixed point of Φ , that point verifies $y_\lambda = \Phi_\lambda(y_\lambda) + i_\lambda$ for some $(i_\lambda)_\lambda \in \mathcal{H}_{(I_A, \mathcal{E}, \mathcal{P})}^m$. We try prove that $[y_\lambda] = [z_\lambda]$ that is to say $(y_\lambda - z_\lambda)_\lambda \in \mathcal{H}_{(I_A, \mathcal{E}, \mathcal{P})}^m$. Then, it exists at least $\mu \in \Lambda$ such that $y_\mu - z_\mu \neq 0$. As $(\mathcal{E}^m, \tau_\mu^m)$ is Hausdorff it exists $j \in I$ (depending on $y_\mu - z_\mu$) such that $q_{\mu, j}^{(m)}(y_\mu - z_\mu) > 0$. Writing

$$\begin{aligned} 0 < q_{j, \mu}^{(m)}(y_\mu - z_\mu) &\leq q_{j, \mu}^{(m)}(\Phi_\mu(y_\mu) - \Phi_\mu(z_\mu) + i_\mu) \leq k_{j, \mu} q_{j, \mu}^{(m)}(y_\mu - z_\mu) + q_{j, \mu}^{(m)}(i_\mu) \\ &< q_{j, \mu}^{(m)}(y_\mu - z_\mu) + q_{j, \mu}^{(m)}(i_\mu). \end{aligned}$$

We don't see a contadiction and uniqueness cannot be proved without other hypotheses.

4 The Cauchy-Lipschitz theorem

We try to give a generalized formulation of the Cauchy-Lipschitz theorem close to the classical one. We can limit the order of derivatives to one or even zero, as in the globally Lipschitz problem.

Let J be an intervall of \mathbb{R} and f a continuous function $\in C^0(J \times \mathbb{R}^m, \mathbb{R}^m) = (C^0(J \times \mathbb{R}^m, \mathbb{R}))^m$ satisfying a global Lipschitz condition as

$$\forall K \in J, \exists k > 0, \forall t \in J, \forall y, z \in \mathbb{R}^m, \|f(t, y) - f(t, z)\| \leq k \|y - z\|.$$

Then, the Cauchy problem

$$(4) \quad \begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

with $x_0 \in \mathbb{R}^m$ and $t_0 \in J$ admits one unique global solution $x \in C^1(J, \mathbb{R}^m) = (C^1(J, \mathbb{R}))^m$.

The problem reduces to finding a fixed point of the map $F \in C^0(J, \mathbb{R}^m) \rightarrow C^0(J, \mathbb{R}^m)$ such that

$$\forall t \in J, F(x)(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds$$

that is to say, for each $k = 1, 2 \dots m$ and $x = (x_1, \dots, x_m)$, $F = (F_1, \dots, F_m)$, $f = (f_1, \dots, f_m) \in C^0(J \times \mathbb{R}^m, \mathbb{R})^m$

$$\forall t \in J, k = 1, 2 \dots m : F_k(x)(t) = x_{0, k} + \int_{t_0}^t f_k(s, x(s)) ds.$$

The standard proof gives a convenient norm on $C^0(J, \mathbb{R}^m)$ for which F is a contraction, and the Picard procedure gives its fixed point.

To formulate (4) in a generalized setting we have to introduce some convenient algebras.

4.1 The generalized framework

Definition 9 When $l = 0$ or 1 , with $\mathcal{E} = C^l(J, \mathbb{R})$ and $\mathcal{P}^l = (p_{K, l})_K$ such that

$p_{K, l}(u) = \sup_{t \in K, 0 \leq j \leq l} |u^{(j)}(t)|$, we can write $\mathcal{H}_{(A, \mathcal{E}, \mathcal{P}^l)}$ and $\mathcal{H}_{(I_A, \mathcal{E}, \mathcal{P}^l)}$ as

$$\begin{aligned} \mathcal{H}_A^l(J, \mathbb{R}) &= \mathcal{H}_{(A, C^l(J, \mathbb{R}), \mathcal{P}^l)} = \left\{ (u_\lambda)_\lambda \in [\mathcal{E}]^\Lambda \mid \forall K \in J, ((p_{K, l}(u_\lambda))_\lambda) \in |A| \right\} \\ \mathcal{H}_{I_A}^l(J, \mathbb{R}) &= \mathcal{H}_{(I_A, C^l(J, \mathbb{R}), \mathcal{P}^l)} = \left\{ (u_\lambda)_\lambda \in [\mathcal{E}]^\Lambda \mid \forall K \in J, ((p_{K, l}(u_\lambda))_\lambda) \in |I_A| \right\}. \end{aligned}$$

and define $\mathcal{H}_A^\infty(J, \mathbb{R})$ or $\mathcal{H}_{I_A}^\infty(J, \mathbb{R})$ by replacing l by ∞ in the previous definitions, with

$\mathcal{P}^\infty = (p_{K,l})_{K \in J, l \in \mathbb{N}}$. We begin to pose:

$$\begin{aligned}\mathfrak{C}_C^0(J, \mathbb{R}) &= \mathcal{H}_A^0(J, \mathbb{R}) / \mathcal{H}_{I_A}^0(J, \mathbb{R}), \\ \mathfrak{C}_C^1(J, \mathbb{R}) &= \mathcal{H}_A^1(J, \mathbb{R}) / \mathcal{H}_A^1(J, \mathbb{R}) \cap \mathcal{H}_{I_A}^0(J, \mathbb{R}).\end{aligned}$$

Remark 4 We have the classical embedding: $C^1(J, \mathbb{R}) \rightarrow C^0(J, \mathbb{R})$ which inspires our generalized requirements. But the only way to embed the factor algebra $\mathfrak{C}_C^1(J, \mathbb{R})$ (defined from $\mathcal{H}_A^1(J, \mathbb{R})$) into $\mathfrak{C}_C^0(J, \mathbb{R})$ is to define $\mathfrak{C}_C^1(J, \mathbb{R})$ as $\mathcal{H}_A^1(J, \mathbb{R}) / \mathcal{H}_A^1(J, \mathbb{R}) \cap \mathcal{H}_{I_A}^0(J, \mathbb{R})$, following a well-know result on the embedding of factor algebras. We would like to define $\mathfrak{C}_C^1(J, \mathbb{R}) = \mathcal{H}_A^1(J, \mathbb{R}) / \mathcal{H}_{I_A}^1(J, \mathbb{R})$. Unfortunately, in this case the natural mapping $\mathfrak{C}_C^1(J, \mathbb{R}) \rightarrow \mathfrak{C}_C^0(J, \mathbb{R})$ is neither injective nor surjective. Indeed we cannot prove here that $\mathcal{H}_A^1(J, \mathbb{R}) \cap \mathcal{H}_{I_A}^0(J, \mathbb{R}) = \mathcal{H}_{I_A}^1(J, \mathbb{R})$, in contrary to the "well known"equality $\mathcal{H}_A^\infty(J, \mathbb{R}) \cap \mathcal{H}_{I_A}^0(J, \mathbb{R}) = \mathcal{H}_{I_A}^\infty(J, \mathbb{R})$, proved in Lemma 4.4 in [9] which generalize 1.2.3 Theorem in [8]. However, we can define a map ∂ from $\text{Im } \mathfrak{C}_C^1(J, \mathbb{R})$ to $\mathfrak{C}_C^0(J, \mathbb{R})$ which looks like a derivation.

Definition 10 If $i(x) \in \text{Im } \mathfrak{C}_C^1(J, \mathbb{R})$ the embedding i verifies

$$i : (x_\varepsilon)_\varepsilon + \mathcal{H}_A^1(J) \cap \mathcal{H}_{I_A}^0(J) \rightarrow (x_\varepsilon)_\varepsilon + \mathcal{H}_{I_A}^0(J)$$

which implies that $(x'_\varepsilon)_\varepsilon \in \mathcal{H}_A^0(J)$ and allows to define

$$\tilde{\partial} : \mathfrak{C}_C^1(J, \mathbb{R}) \ni i^{-1}(x) = (x_\varepsilon)_\varepsilon + \mathcal{H}_A^1(J) \cap \mathcal{H}_{I_A}^0(J) \rightarrow (x'_\varepsilon)_\varepsilon + \mathcal{H}_{I_A}^0(J) \in \mathfrak{C}_C^0(J, \mathbb{R})$$

which leads to define the map $\partial = \tilde{\partial} \circ i^{-1}$ from $\text{Im } \mathfrak{C}_C^1(J, \mathbb{R})$ to $\mathfrak{C}_C^0(J, \mathbb{R})$.

For $f \in C^0(J \times \mathbb{R}, \mathbb{R})$, $K \in J, p > 0$, we pose

$$q_{K,-p}(f) = \sup_{t \in K, y \in \mathbb{R}} (1 + |y|)^{-p} |f(t, y)|,$$

then for any solid unitary subring A with ideal I_A of \mathbb{K}^Λ , we define

$$\begin{aligned}\mathcal{H}_{\tau,A}^0(J \times \mathbb{R}, \mathbb{R}) &= \left\{ (f_\lambda)_\lambda \in [C^0(J \times \mathbb{R}, \mathbb{R})]^\Lambda \mid \forall K \in J, \exists p > 0, (q_{K,-p}(f_\lambda))_\lambda \in |A| \right\}, \\ \mathcal{H}_{\tau,I_A}^0(J \times \mathbb{R}, \mathbb{R}) &= \left\{ (f_\lambda)_\lambda \in [C^0(J \times \mathbb{R}, \mathbb{R})]^\Lambda \mid \forall K \in J, \exists p > 0, (q_{K,-p}(f_\lambda))_\lambda \in |I_A| \right\}.\end{aligned}$$

For $f = (f_1, \dots, f_m) \in (C^0(J \times \mathbb{R}^m, \mathbb{R}))^m$, $K \in J, p > 0$, we pose

$$q_{K,-p}^{(m)}(f) = \sup_{t \in K, y \in \mathbb{R}^m} (1 + |y|)^{-p} |f(t, y)|^{(m)} \quad \text{with } |f(t, y)|^{(m)} = \sum_{k=1}^{k=m} |f_k(t, y)|,$$

and define, for $f_\lambda = (f_{1,\lambda}, \dots, f_{m,\lambda})$

$$\begin{aligned}(\mathcal{H}_{\tau,A}^0(J \times \mathbb{R}^m, \mathbb{R}))^m &= \left\{ (f_\lambda)_\lambda \in [(C^0(J \times \mathbb{R}^m, \mathbb{R}))^m]^\Lambda \mid \forall K \in J, \exists p > 0, \left(q_{K,-p}^{(m)}(f_\lambda) \right)_\lambda \in |A| \right\}, \\ (\mathcal{H}_{\tau,I_A}^0(J \times \mathbb{R}^m, \mathbb{R}))^m &= \left\{ (f_\lambda)_\lambda \in [(C^0(J \times \mathbb{R}^m, \mathbb{R}))^m]^\Lambda \mid \forall K \in J, \exists p > 0, \left(q_{K,-p}^{(m)}(f_\lambda) \right)_\lambda \in |I_A| \right\}.\end{aligned}$$

In the same way when $l = 0$ or 1 , with $\mathcal{E} = C^l(J, \mathbb{R})$ and $\mathcal{P}^l = (p_{K,l}^{(m)})_{K \in J}$ such that $p_{K,l}^{(m)}(u) = \sup_{t \in K, 0 \leq j \leq l} \|u^{(j)}(t)\|$, we can re-write $\mathcal{H}_{(A,\mathcal{E},\mathcal{P}^l)}^m$ and $\mathcal{H}_{(I_A,\mathcal{E},\mathcal{P}^l)}^m$ as

$$\begin{aligned}(\mathcal{H}_A^l(J, \mathbb{R}))^m &= \mathcal{H}_{(A,C^l(J,\mathbb{R}),\mathcal{P}^l)}^m = \left\{ (u_\lambda)_\lambda \in [\mathcal{E}^m]^\Lambda \mid \forall K \in J, \left(p_{K,l}^{(m)}(u_\lambda) \right)_\lambda \in |A| \right\}, \\ (\mathcal{H}_{I_A}^l(J, \mathbb{R}))^m &= \mathcal{H}_{(I_A,C^l(J,\mathbb{R}),\mathcal{P}^l)}^m = \left\{ (u_\lambda)_\lambda \in [\mathcal{E}^m]^\Lambda \mid \forall K \in J, \left(p_{K,l}^{(m)}(u_\lambda) \right)_\lambda \in |I_A| \right\}.\end{aligned}$$

Definition 11 Summarizing, we finally define for any $m \in \mathbb{N}$

- $\left(\mathfrak{C}_{\tau, \mathcal{C}}^0(J \times \mathbb{R}^m, \mathbb{R})\right)^m = \left(\mathcal{H}_{\tau, A}^0(J \times \mathbb{R}^m, \mathbb{R})\right)^m / \left(\mathcal{H}_{\tau, I_A}^0(J \times \mathbb{R}^m, \mathbb{R})\right)^m$,
- $\left(\mathfrak{C}_{\mathcal{C}}^0(J, \mathbb{R})\right)^m = \left(\mathcal{H}_A^0(J, \mathbb{R})\right)^m / \left(\mathcal{H}_{I_A}^0(J, \mathbb{R})\right)^m$,
- $\left(\mathfrak{C}_{\mathcal{C}}^1(J, \mathbb{R})\right)^m = \left(\mathcal{H}_A^1(J, \mathbb{R})\right)^m / \left(\mathcal{H}_{I_A}^1(J, \mathbb{R})\right)^m \cap \left(\mathcal{H}_{I_A}^0(J, \mathbb{R})\right)^m$

which leads to define as previously the map $\partial = \tilde{\partial} \circ i^{-1}$ from $\text{Im} \left(\mathfrak{C}_{\mathcal{C}}^1(J, \mathbb{R})\right)^m$ to $\left(\mathfrak{C}_{\mathcal{C}}^0(J, \mathbb{R})\right)^m$.

Theorem 9 Let $F = (F_1, \dots, F_m) \in \left(\mathfrak{C}_{\tau, \mathcal{C}}^0(J \times \mathbb{R}^m, \mathbb{R})\right)^m$ and $u = (u_1, u_2, \dots, u_m) \in \left(\mathfrak{C}_{\mathcal{C}}^0(J, \mathbb{R})\right)^m$. Then, $F(\cdot, u)$ is a well defined element of $\left(\mathfrak{C}_{\mathcal{C}}^0(J, \mathbb{R})\right)^m$.

Proof. F has for representatives $(F_\lambda)_\lambda$ with $F_\lambda : t \rightarrow F_\lambda(t, y)$. If u_λ is a representative of $u \in \left(\mathcal{A}^0(J)\right)^m$, we have to prove that the family $(t \rightarrow F_\lambda(t, u_\lambda(t)))_\lambda$ lies in $\left(\mathcal{H}_{\tau, A}^0(J \times \mathbb{R}^m)\right)^m = \mathcal{H}_{(A, \mathcal{C}^0(J, \mathbb{R}), \mathcal{P}^0)}^m$. To simplify the proof, suppose that $m = 1$ for basic estimates.

With one hand, if $u = [u_\lambda] \in \mathcal{A}^0(J)$, we have: $\forall K \Subset J$, $\left(\sup_{t \in K} |u_\lambda(t)|\right)_\lambda \in |A|$, then

$$\forall t \in K, |u_\lambda(t)| \leq \sup_{t \in K} |u_\lambda(t)| = |a_\lambda| \text{ with } a_\lambda \in A.$$

As A is solid, $(u_\lambda(t))_\lambda$ is in A (and $(|u_\lambda(t)|)_\lambda$ in $|A|$). Then, for any $p > 0$, $((1 + |u_\lambda(t)|)^p)_\lambda \in |A|$.

On the other hand, $F \in \mathcal{A}_{\tau, \mathcal{C}}^0(J \times \mathbb{R})$ has for representatives $(F_\lambda)_\lambda$ such that

$$\exists p > 0, \sup_{t \in K, y \in \mathbb{R}^m} (1 + |y|)^{-p} |F_\lambda(t, y)| = |b_\lambda| \text{ with } (b_\lambda)_\lambda \in |A|.$$

Then,

$$\exists p > 0, \forall t \in K, |F_\lambda(t, u_\lambda(t))| \leq (1 + |u_\lambda(t)|)^p |b_\lambda| = |c_{p, \lambda}| \text{ with } (c_{p, \lambda})_\lambda \in A$$

and from solidness of A , $(|F_\lambda(t, u_\lambda(t))|)_\lambda \in |A|$. Then $(t \mapsto (F_\lambda(t, u_\lambda(t)))_\lambda) \in \mathcal{H}(A, \mathcal{C}^0(J, \mathbb{R}), \mathcal{P}^0)$.

If $(v_\lambda)_\lambda$ is another representative of u and (G_λ) another representative of F , it is easy to show that $(t \mapsto (F_\lambda(t, u_\lambda(t)))_\lambda - (t \mapsto (G_\lambda(t, v_\lambda(t)))_\lambda) \in \mathcal{H}(I_A, \mathcal{C}^0(J, \mathbb{R}), \mathcal{P}^0)$. Then $F(\cdot, u)$ is a well defined element of $\mathfrak{C}_{\mathcal{C}}^0(J)$. Similar estimates permits to replace \mathbb{R} by \mathbb{R}^m , $\mathfrak{C}_{\mathcal{C}}^0(J)$ by $\left(\mathfrak{C}_{\mathcal{C}}^0(J)\right)^m$ and $\mathfrak{C}_{\tau, \mathcal{C}}^0(J \times \mathbb{R})$ by $\left(\mathfrak{C}_{\tau, \mathcal{C}}^0(J \times \mathbb{R}^m)\right)^m$. ■

Definition 12 Let $f \in \left(\mathfrak{C}_{\tau, \mathcal{C}}^0(\mathbb{R}_+ \times \mathbb{R}^m)\right)^m$. We tell that f is globally Lipschitz if for any representative $(f_\varepsilon)_\varepsilon$ of f we have (with $\|x\| = \sum_{k=1}^{k=m} |x_k|$ when $x = (x_1, \dots, x_m) \in \mathbb{R}^m$)

$$\forall t \in \mathbb{R}_+, \forall \varepsilon \in]0, 1], \exists k_\varepsilon(t) > 0, \forall (v, w) \in \mathbb{R}^m \times \mathbb{R}^m, \|f_\varepsilon((t, v)) - f_\varepsilon((t, w))\| \leq k_\varepsilon(t) \|v - w\|$$

with

$$\forall T \in \mathbb{R}_+, \sup_{t \in [0, T]} k_\varepsilon(t) = M_{T, \varepsilon} < +\infty.$$

4.2 The generalized Cauchy-Lipschitz problem

In terms of generalized functions the Cauchy-Lipschitz problem (4) can be defined as

Definition 13 *The Cauchy-Lipschitz generalized problem is to solve (2) that is*

$$\begin{cases} \partial x = f(., x) \\ x(t_0) = \xi \end{cases}$$

with $x \in \text{Im}(\mathfrak{C}_{\mathcal{C}}^1(J))^m \subset (\mathfrak{C}_{\mathcal{C}}^0(J))^m$ and $f \in (\mathfrak{C}_{\tau, \mathcal{C}}^0(J \times \mathbb{R}^m))^m$ globally Lipschitz, for some ring of generalized numbers $\mathcal{C} = A/I_A$, with $t_0 \in J$ and ξ is a given element $\in \widetilde{\mathbb{R}^m}$

It is classical to choose $J = \mathbb{R}_+$ when t is the time parameter, then we do that, without loss of generality. But we have exactly the same result when J is an open subset of \mathbb{R} and taking $J = \mathbb{R}$ simplifies some estimates.

Theorem 10 *It exists a ring of generalized numbers $\mathcal{C} = A/I_A$, such that $f \in (\mathfrak{C}_{\tau, \mathcal{C}}^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}))^m$ and a map $\Phi : (\mathfrak{C}_{\mathcal{C}}^0(\mathbb{R}_+, \mathbb{R}))^m \rightarrow (\mathfrak{C}_{\mathcal{C}}^0(\mathbb{R}_+, \mathbb{R}))^m$ with a fixed point solving the Cauchy-Lipschitz problem (2) with $t_0 \in \mathbb{R}_+$ and $\xi \in \widetilde{\mathbb{R}^m}$.*

Proof. To construct the convenient ring of generalized numbers $\mathcal{C} = A/I_A$ interacting with the construction of Φ and its fixed point, we follow four steps to verify the hypotheses of Definition of a generalized contraction

- (a) For $x_0 \in \mathbb{R}^m, x \in (C^0(\mathbb{R}_+, \mathbb{R}))^m, t \in \mathbb{R}_+$ and $f_\varepsilon \in (C^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}))^m$ we pose

$$\Phi_\varepsilon(x)(t) = x_0 + \int_0^t f_\varepsilon(s, x(s)) ds$$

from what it is clear that Φ_ε is a map $\mathcal{E}^m \rightarrow \mathcal{E}^m$ with $\mathcal{E} = C^0(\mathbb{R}_+, \mathbb{R})$.

(\mathcal{E}^m, τ^m) is here a topological space where τ^m is given by the family of norms $(p_T^{(m)})_{T \in \mathbb{R}_+}$ such that $p_T^{(m)}(x) = \sup_{t \in [0, T]} \|x(t)\|$. We suppose that it exists a family $(a_\varepsilon)_\varepsilon \in \mathcal{E}^{[0,1]} = (C^0(\mathbb{R}_+, \mathbb{R}))^{[0,1]}$ such that

$$\begin{cases} \forall T \in \mathbb{R}_+, (p_T(a_\varepsilon))_\varepsilon \in |A| \\ \forall t \in \mathbb{R}_+, \|f_\varepsilon(t, x(t))\| \leq |a_\varepsilon(t)| \|x(t)\|. \end{cases}$$

Then, we have

$$\|\Phi_\varepsilon(x)(t)\| \leq \|x_0\| + \int_0^t |a_\varepsilon(s)| \|x(s)\| ds \leq \|x_0\| + |a_\varepsilon(r)| \int_0^t \|x(s)\| ds, r \in [0, t]$$

which leads to $p_T^{(m)}(\Phi_\varepsilon(x)) \leq \|x_0\| + T p_T(a_\varepsilon) p_T^{(m)}(x)$.

Then if $(x_\varepsilon)_\varepsilon \in \mathcal{H}_{(A, C^0(J, \mathbb{R}), \mathcal{P}^0)}^m$, that is to say $(p_T^{(m)}(x_\varepsilon))_\varepsilon \in |A|$, we have

$$(p_T^{(m)}(\Phi_\varepsilon(x_\varepsilon)))_\varepsilon \leq (\|x_{0, \varepsilon}\|)_\varepsilon + T (p_T(a_\varepsilon))_\varepsilon (p_T^{(m)}(x_\varepsilon))_\varepsilon$$

which leads (from $\exists n \in \mathbb{N}$ such that $T (p_T(a_\varepsilon))_\varepsilon \leq n (p_T(a_\varepsilon))_\varepsilon$) to $(p_T^{(m)}(\Phi_\varepsilon(x_\varepsilon)))_\varepsilon \in |A|$ that is to say $(\Phi_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{H}_{(A, C^0(J, \mathbb{R}), \mathcal{P}^0)}^m$.

- (b) Putting $\Lambda =]0, 1]$, $t_0 = 0$ and $\lambda = \varepsilon$, we first have to write (2) in term of representatives $k_\varepsilon(t)$

$$(5) \quad \begin{cases} x'_\varepsilon(t) = f_\varepsilon(t, x_\varepsilon(t)) \\ x_\varepsilon(0) = \xi_\varepsilon \end{cases}$$

where $(\|\xi_\varepsilon\|)_\varepsilon \in |A|$.

We recall that if $(f_\varepsilon)_\varepsilon$ is a representative of $f \in \left(\mathfrak{C}_{\tau, \mathcal{C}}^0(\mathbb{R}_+ \times \mathbb{R}^m)\right)^m$, we have

$f_\varepsilon \in (C^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}))^m$ and $\forall K \in J, \exists p > 0, \left(q_{K, -p}^{(m)}(f_\varepsilon)\right)_\varepsilon \in |A|$ and independantly of this estimate we have to add the following one

$$\forall t \in \mathbb{R}_+, \forall \varepsilon \in]0, 1], \exists k_\varepsilon(t) > 0, \forall (v, w) \in \mathbb{R}^m \times \mathbb{R}^m, \|f_\varepsilon((t, v)) - f_\varepsilon((t, w))\| \leq k_\varepsilon(t) \|v - w\|$$

with

$$\forall T \in \mathbb{R}_+, \sup_{t \in [0, T]} k_\varepsilon(t) = M_{T, \varepsilon} < +\infty.$$

For x_0 given in $\mathbb{R}^m, x \in (C^0(\mathbb{R}_+, \mathbb{R}))^m, t \in \mathbb{R}_+$ and $f_\varepsilon \in (C^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}))^m$ we pose

$$\Phi_\varepsilon(x)(t) = x_0 + \int_0^t f_\varepsilon(s, x(s)) ds$$

from what it is clear that Φ_ε is a map $\mathcal{E}^m \rightarrow \mathcal{E}^m$ with $\mathcal{E} = C^0(\mathbb{R}_+, \mathbb{R})$.

The natural topology (denoted τ^m) on the product \mathcal{E}^m is here defined by the family $\left(p_T^{(m)}\right)_{T \in \mathbb{R}_+}$ of seminorms such that for $x = (x_1, \dots, x_m), p_T^{(m)}(x) = \sum_{k=1}^{k=m} p_T(x_k)$ with $p_T(x_k) = \sup_{t \in [0, T]} |x_k(t)|$.

Denote by $(\mathcal{E}^m, \tau_\varepsilon^m)$ the topological space \mathcal{E}^m endowed by the family $\left(q_{T, \varepsilon}^{(m)}\right)_{T \in \mathbb{R}_+}$ with $q_{T, \varepsilon}^{(m)}(x) =$

$\sum_{k=1}^{k=m} q_{T, \varepsilon}(x_k)$ for a given family $(q_{T, \varepsilon})_{T \in \mathbb{R}_+}$ of seminorm on $\mathcal{E} = C^0(\mathbb{R}_+, \mathbb{R})$ such that for each

$\varepsilon \in]0, 1]$ and $y \in C^0(\mathbb{R}_+, \mathbb{R})$ we have $q_{T, \varepsilon}(y) = \sup_{t \in [0, T]} (|y(t)| e^{-tM_{T, \varepsilon}})$. Then $q_{T, \varepsilon}^{(m)}(x) = \sup_{t \in [0, T]} (\|x(t)\| e^{-tM_{T, \varepsilon}})$

We claim that the map Φ_ε is a contraction in $(\mathcal{E}^m, \tau_\varepsilon^m)$. From Definition 12, we have to prove that that for each $(T, \varepsilon) \in \mathbb{R}_+ \times]0, 1]$ it exists a constant $k_{T, \varepsilon} < 1$ such that for all $(x, y) \in \mathcal{E}^m \times \mathcal{E}^m$

$$q_{T, \varepsilon}^{(m)}(\Phi_\varepsilon(x) - \Phi_\varepsilon(y)) \leq k_{T, \varepsilon} q_{T, \varepsilon}^{(m)}(x - y).$$

We have, for each $t \in \mathbb{R}_+$

$$\Phi_\varepsilon(x)(t) - \Phi_\varepsilon(y)(t) = x_0 + \int_0^t (f_\varepsilon(s, x(s)) - f_\varepsilon(s, y(s))) ds$$

from what we deduce

$$e^{-tM_{T, \varepsilon}} \|\Phi_\varepsilon(x)(t) - \Phi_\varepsilon(y)(t)\| \leq e^{-tM_{T, \varepsilon}} \int_0^t \|f_\varepsilon(s, x(s)) - f_\varepsilon(s, y(s))\| ds$$

and, for $t \in [0, T]$

$$e^{-tM_{T, \varepsilon}} \|\Phi_\varepsilon(x)(t) - \Phi_\varepsilon(y)(t)\| \leq e^{-tM_{T, \varepsilon}} \int_0^t M_{T, \varepsilon} \|x(s) - y(s)\| ds.$$

Writing now

$$e^{-tM_{T, \varepsilon}} \int_0^t M_{T, \varepsilon} \|x(s) - y(s)\| ds = e^{-tM_{T, \varepsilon}} \int_0^t M_{T, \varepsilon} e^{sM_{T, \varepsilon}} (e^{-sM_{T, \varepsilon}} \|x(s) - y(s)\|) ds$$

we obtain

$$e^{-tM_{T, \varepsilon}} \int_0^t M_{T, \varepsilon} e^{sM_{T, \varepsilon}} (e^{-sM_{T, \varepsilon}} \|x(s) - y(s)\|) ds \leq e^{-tM_{T, \varepsilon}} \int_0^t M_{T, \varepsilon} e^{sM_{T, \varepsilon}} q_{T, \varepsilon}^{(m)}(x - y) ds$$

which leads to

$$e^{-tM_{T,\varepsilon}} \|\Phi_\varepsilon(x)(t) - \Phi_\varepsilon(y)(t)\| \leq e^{-tM_{T,\varepsilon}} q_{T,\varepsilon}^{(m)}(x-y) (e^{tM_{T,\varepsilon}} - 1) = (1 - e^{-tM_{T,\varepsilon}}) q_{T,\varepsilon}^{(m)}(x-y).$$

When taking $\sup_{t \in [0, T]}$ we finally obtain a constant $k_{T,\varepsilon} = (1 - e^{-TM_{T,\varepsilon}}) < 1$ such that

for all $(x, y) \in \mathcal{E}^m \times \mathcal{E}^m$

$$q_{T,\varepsilon}^{(m)}(\Phi_\varepsilon(x) - \Phi_\varepsilon(y)) \leq k_{T,\varepsilon} q_{T,\varepsilon}^{(m)}(x-y).$$

Then according to Definition 12, Φ_ε is a contraction in \mathcal{E}^m and has an unique fixed point z_ε from Theorem 6.

• (c) We can write for $x \in \mathcal{E}^m$

$$e^{-TM_{T,\varepsilon}} \sup_{t \in [0, T]} \|x(t)\| \leq \sup_{t \in [0, T]} (\|x(t)\| e^{-tM_{T,\varepsilon}}) \leq \sup_{t \in [0, T]} \|x(t)\|$$

then

$$e^{-TM_{T,\varepsilon}} p_T^{(m)} \leq q_{T,\varepsilon}^{(m)} \leq p_T^{(m)}.$$

• (d). Assume now that for each $T \in \mathbb{R}_+$ the family $(e^{TM_{T,\varepsilon}})_\varepsilon$ lies in $|A|$ and recall that we have asked in (a) that $(p_T(a_\varepsilon))_\varepsilon \in |A|$. As $k_{T,\varepsilon} = (1 - e^{-TM_{T,\varepsilon}})$ we have $\left(\frac{1}{1 - k_{T,\varepsilon}}\right)_\varepsilon = (e^{TM_{T,\varepsilon}})_\varepsilon$.

Define now

$$\Phi_\varepsilon(x_\varepsilon)(t) = \xi_\varepsilon + \int_0^t f_\varepsilon(s, x_\varepsilon(s)) ds$$

where $(\xi_\varepsilon)_\varepsilon$ is a given representative of the given element $\xi \in \widetilde{\mathbb{R}^m}$

Finally, from Definition 8, the map $\Phi : (\mathfrak{C}_C^0(\mathbb{R}_+, \mathbb{R}))^m \rightarrow (\mathfrak{C}_C^0(\mathbb{R}_+, \mathbb{R}))^m$ such that

$$[x_\varepsilon] \mapsto [\Phi_\varepsilon(x_\varepsilon)]$$

is a contraction, with $z = [z_\varepsilon]$ as fixed point from Theorem 8, z_ε being the unique fixed point of Φ_ε verifying

$$z_\varepsilon(t) = \xi_\varepsilon + \int_0^t f_\varepsilon(s, z_\varepsilon(s)) ds.$$

Then, $z'_\varepsilon(t) = f_\varepsilon(t, z_\varepsilon(t))$. and $z_\varepsilon(0) = \xi_\varepsilon$. Moreover z is solution to the Cauchy-Lipschitz problem (2) given in Definition 13.

We are going to prove that z is the unique fixed point of Φ , and therefore the unique solution of ().

If $y = [y_\varepsilon]$ is another fixed point of Φ , we have $y_\varepsilon = \Phi_\varepsilon(y_\varepsilon) + i_\varepsilon$ with $(i_\varepsilon)_\varepsilon \in \mathcal{H}_{(I_A, C^0(\mathbb{R}_+, \mathbb{R}), \mathcal{P}^0)}^m$, that is to say $(p_T^{(m)}(i_\varepsilon))_\varepsilon \in I_A$. Writing now

$$z_\varepsilon(t) - y_\varepsilon(t) = \int_0^t (f_\varepsilon(s, z_\varepsilon(s)) - f_\varepsilon(s, y_\varepsilon(s))) ds + i_\varepsilon(t)$$

we have

$$\|z_\varepsilon(t) - y_\varepsilon(t)\| \leq \int_0^t \|(f_\varepsilon(s, z_\varepsilon(s)) - f_\varepsilon(s, y_\varepsilon(s)))\| ds + \|i_\varepsilon(t)\|$$

$$\|z_\varepsilon(t) - y_\varepsilon(t)\| \leq \|i_\varepsilon(t)\| + \int_0^t M_{T,\varepsilon} \|z_\varepsilon(s) - y_\varepsilon(s)\| ds$$

and, from Gronwall lemma, for $t \in [0, T]$

$$\|z_\varepsilon(t) - y_\varepsilon(t)\| \leq \|i_\varepsilon(t)\| e^{TM_{T,\varepsilon}}$$

$$p_T^{(m)}(z_\varepsilon - y_\varepsilon) \leq p_T^{(m)}(i_\varepsilon) e^{TM_{T,\varepsilon}}.$$

As $(e^{TM_{T,\varepsilon}})_\varepsilon \in |A|$ and $(p_T^{(m)}(i_\varepsilon))_\varepsilon \in |I_A|$, we have $(p_T^{(m)}(z_\varepsilon - y_\varepsilon))_\varepsilon \in |I_A|$, which finish the proof. ■

All our ideas, technics and results can be explicitly detailed and summarized in the following example

Example 2 *The Cauchy-Lipschitz generalized problem to solve is given as*

$$(6) \quad \begin{cases} \partial x = \Delta(\cdot, x) \\ x(0) = \xi \end{cases}$$

with $x \in \mathfrak{C}_C^0(\mathbb{R}, \mathbb{R})$, with $t_0 \in \mathbb{R}$ and ξ is a given element $\in \tilde{\mathbb{R}}$. We define Δ from $\Delta_\varepsilon(t, x) = \varphi_\varepsilon(t)\varphi_\varepsilon(x)$ when φ_ε is the "standard mollifier" $\varphi_\varepsilon(\cdot) = \varphi\left(\frac{\cdot}{\varepsilon}\right)$, $\varphi \in C^1(\mathbb{R})$, $\text{supp}\varphi = [-1, 1]$, $\int \varphi(s) ds = 1$.

• *Existence of Δ .*

We begin to take A as containing some elements such that Δ lies in $\mathfrak{C}_{\tau, \mathcal{C}}^0(\mathbb{R} \times \mathbb{R})$ and is globally Lipschitz for $\mathcal{C} = A/I_A$. From $\Delta_\varepsilon(t, x) = \varphi_\varepsilon(t)\varphi_\varepsilon(x)$ we have, for any $p \geq 0$

$$|\Delta_\varepsilon(t, x)| \leq \frac{1}{\varepsilon^2} \left| \varphi\left(\frac{t}{\varepsilon}\right) \varphi\left(\frac{x}{\varepsilon}\right) \right| \leq \frac{M^2}{\varepsilon^2} \leq \frac{M^2}{\varepsilon^2} (1 + |x|)^p$$

with $M = \sup_{s \in \mathbb{R}} |\varphi(s)|$. Then

$$\forall T \geq 0, \forall p \geq 0, q_{T, -p}(\Delta_\varepsilon) \leq \frac{M^2}{\varepsilon^2}$$

with $q_{T, -p}(\Delta_\varepsilon) = \sup_{t \in [-T, T], x \in \mathbb{R}} (1 + |x|)^{-p} |\Delta_\varepsilon(t, x)|$.

We constuct here A as a set of families of real elements, being a solid unitary subring of $\mathbb{R}^{[0,1]}$, and such that the family $(\varepsilon)_\varepsilon \in A$ (or $|A|$). As exists $n \in \mathbb{N}$ such that $M^2 \leq n$ and $\left(\frac{n}{\varepsilon^2}\right)_\varepsilon \in |A|$, then we have: $(q_{T, -p}(\Delta_\varepsilon))_\varepsilon \in |A|$ that is to say Δ lies in $\mathfrak{C}_{\tau, \mathcal{C}}^0(\mathbb{R} \times \mathbb{R})$.

• *The Cauchy-Lipschitz condition.*

We have to prove that

$$\forall \varepsilon > 0, \forall T \in \mathbb{R}, \exists M_{T, \varepsilon} > 0, \sup_{t \in [-T, T], (v, w) \in \mathbb{R}^2} |\Delta_\varepsilon(t, v) - \Delta_\varepsilon(t, w)| \leq M_{T, \varepsilon} |v - w|.$$

Writing

$$\Delta_\varepsilon(t, v) - \Delta_\varepsilon(t, w) = \frac{1}{\varepsilon^2} \varphi\left(\frac{t}{\varepsilon}\right) \left[\varphi\left(\frac{v}{\varepsilon}\right) - \varphi\left(\frac{w}{\varepsilon}\right) \right] = \frac{1}{\varepsilon^2} \varphi\left(\frac{t}{\varepsilon}\right) \varphi'\left(\frac{r}{\varepsilon}\right) \left[\frac{v}{\varepsilon} - \frac{w}{\varepsilon} \right]$$

with $\frac{r}{\varepsilon} \in \left(\frac{v}{\varepsilon}, \frac{w}{\varepsilon}\right)$ we obtain

$$\sup_{t \in [-T, T], (v, w) \in \mathbb{R}^2} |\Delta_\varepsilon(t, v) - \Delta_\varepsilon(t, w)| \leq \frac{MM'}{\varepsilon^3} |v - w|$$

with $M' = \sup_{s \in \mathbb{R}} |\varphi'(s)|$.

• *Construction of the factor ring $\mathcal{C} = A/I_A$.*

From the (d) point of Theorem 10, we have to choose A containing $\left(e^{\frac{TMM'}{\varepsilon^3}}\right)_\varepsilon$. As it exists $n \in \mathbb{N}$ such that $TMM' \leq n$, we have $e^{\frac{TMM'}{\varepsilon^3}} = \left(e^{\frac{1}{\varepsilon^3}}\right)^{TMM'} \leq \left(e^{\frac{1}{\varepsilon^3}}\right)^n$. As A is a solid subring, we only have to require that $\left(e^{\frac{1}{\varepsilon^3}}\right)_\varepsilon \in |A|$, that is to say $\left(e^{\frac{1}{\varepsilon^3}}\right)_\varepsilon \in |A|$, and as we have $\frac{1}{\varepsilon} < e^{\frac{1}{\varepsilon^3}}$, then $\left(\frac{1}{\varepsilon}\right)_\varepsilon$ and $(\varepsilon)_\varepsilon$ also are in $|A|$. But $e^{-\frac{1}{\varepsilon^3}} \leq e^{-\frac{1}{\varepsilon}}$ and the previous statements are obviously verified if we require only that $\left(e^{\frac{1}{\varepsilon}}\right)_\varepsilon \in |A|$. Then it suffice to take \mathcal{C} "**overgenerated**" by the family $\left(e^{\frac{1}{\varepsilon}}\right)_\varepsilon$ in the meaning of Definition 3.

- Construction of the algebra $\mathfrak{C}_\mathcal{C}^0(\mathbb{R}, \mathbb{R}) = \mathcal{H}_A^0(\mathbb{R}, \mathbb{R}) / \mathcal{H}_{I_A}^0(\mathbb{R}, \mathbb{R})$.

We recall that

$$\begin{aligned}\mathcal{H}_A^0(\mathbb{R}, \mathbb{R}) &= \left\{ (u_\varepsilon)_\varepsilon \in (C^0(\mathbb{R}, \mathbb{R}))^{[0,1]} \mid \forall K \in \mathbb{R}, ((p_{K,0}(u_\varepsilon))_\varepsilon \in |A|) \right\}, \\ \mathcal{H}_{I_A}^0(\mathbb{R}, \mathbb{R}) &= \left\{ (u_\varepsilon)_\varepsilon \in (C^0(\mathbb{R}, \mathbb{R}))^{[0,1]} \mid \forall K \in \mathbb{R}, ((p_{K,0}(u_\varepsilon))_\varepsilon \in |I_A|) \right\}.\end{aligned}$$

We explicit the construction of A and I_A "overgenerated" by the family $\left(e^{\frac{1}{\varepsilon}}\right)_\varepsilon$. First, consider B the subset of elements in $(\mathbb{R}_+^*)^{[0,1]}$ obtained as rational fractions with coefficients in \mathbb{R}_+^* , of $e^{\frac{1}{\varepsilon}}$ as variable. It follows that

$$\begin{aligned}A &= \left\{ (a_\varepsilon)_\varepsilon \in \mathbb{R}^{[0,1]} \mid \exists (b_\varepsilon)_\varepsilon \in B, \exists \varepsilon_0 \in]0, 1], \forall \varepsilon \prec \varepsilon_0 : |a_\varepsilon| \leq b_\varepsilon \right\} \\ I_A &= \left\{ (a_\varepsilon)_\varepsilon \in \mathbb{R}^{[0,1]} \mid \forall (b_\varepsilon)_\varepsilon \in B, \exists \varepsilon_0 \in]0, 1], \forall \varepsilon \prec \varepsilon_0 : |a_\varepsilon| \leq b_\varepsilon \right\}\end{aligned}$$

- Fixed point of the map Φ and solution to (6).

We know that the map $\Phi : \mathfrak{C}_\mathcal{C}^0(\mathbb{R}_+, \mathbb{R}) \rightarrow \mathfrak{C}_\mathcal{C}^0(\mathbb{R}_+, \mathbb{R})$ such that

$$[x_\varepsilon] \mapsto [\Phi_\varepsilon(x_\varepsilon)]$$

is a contraction, with $z = [z_\varepsilon]$ as fixed point, z_ε being the unique fixed point of Φ_ε verifying

$$z_\varepsilon(t) = \xi_\varepsilon + \int_0^t f_\varepsilon(s, z_\varepsilon(s)) ds$$

where $(\xi_\varepsilon)_\varepsilon$ is a given representative of the given element $\xi \in \widetilde{\mathbb{R}}$. Moreover z is the unique solution to the Cauchy-Lipschitz problem (6).

4.3 Towards the transport equation with irregular coefficients

We consider the following Cauchy problem for the transport equation in (t, x) -variables

$$(7) \quad \frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = f, \quad u(0, x) = u_0(x)$$

In a general case (typically the distribution one), the nonlinear term needs some regularity to be defined. And it is not the case if α is a distribution in (t, x) -variables. Under some hypotheses on α : weak regularity (of Sobolev type), control of its uniform divergency in space and some

space increasing condition, Di Perna and Lions [5] have obtained some results with uniqueness and stability. More recently the paper of L. Ambrosio [1] studies the same subject.

However this context don't permits to define and solve (7) when α is a distribution in (t, x) -variables.

In a simplified case where $f = 0$ and $\alpha = a_t \otimes 1_x$ where $a_t \in \mathcal{D}'(\mathbb{R})$, the problem is posed and solved in [4]. When $\alpha \in \mathcal{D}'(\mathbb{R}^2)$ we turn back to the regular case as the starting point of our generalized methods. It is well known that when α , f and u_0 are of class C^1 , the problem (7) admits a unique solution of class C^1 given by integrating along the characteristics

$$(8) \quad u(t, x) = u_0(X(0, t, x)) + \int_0^t f(s, X(s, t, x)) ds.$$

We can see that the regular solution of (7) is linked to the following Cauchy-Lipschitz problem of which $X(s, t, x)$ is the unique solution

$$\begin{cases} \frac{dX}{ds}(s, t, x) = \alpha(s, X(s, t, x)) \\ X(t, t, x) = x. \end{cases}$$

When α and f are not continuous, N. Caroff [3] propose an approach based on the approximation of discontinuous data by C^1 function α_n and f_n and an Egorov theorem. She gives a result similar to (8) when the irregularities are controled by: $u_0 \in Lip_{loc}(\mathbb{R}, \mathbb{R})$; $\alpha \in L^\infty(\mathbb{R}^2, \mathbb{R})$ and for some $\delta > 0$, $\delta^{-1} \leq \alpha(t, x) \leq \delta$; $f \in L^\infty(\mathbb{R}^2, \mathbb{R})$; $\forall x \in \mathbb{R}$, $\alpha(\cdot, x)$ and $f(\cdot, x)$ are locally Lipschitz uniformly in x .

But as in the previous case, this context don't permit to define and solve (7) when α is a distribution in (t, x) -variables.

With the generalized methods over exposed we are trying to generalize that result to the distributional case in consideration.

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