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Abstract

In this paper, we supplement the results of Kamwa and Merlin (2017) for the selection of a subset of two alternatives out of four by computing the conditional probability of voting situations under which the Condorcet Committee à la Gehrlein (CCG) and the Condorcet Committee à la Fishburn (CCF) may both exist and coincide when voters' preferences on candidates are lexicographically extended on subsets. The CCG is a fixed-size subset of candidates such that each of its members defeats in a pairwise contest any candidate outside. The CCF is a fixed-size subset of candidates that is preferred to all other subsets of the same size by a majority of voters.

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1. Introduction

Consider an election where n ($n \geq 3$) voters have strict rankings (linear orders)[1] on $A = \{a, b, c, d\}$ a set of four candidates. The 24 possible linear orders on A are displayed in Table 1. In this

Table 1: Strict rankings on $A = \{a, b, c, d\}$

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}
a	a	a	a	a	a	b	b	b	b	b	b
b	b	c	c	d	d	a	a	c	c	d	d
c	d	b	d	b	c	c	d	a	d	a	c
d	c	d	b	c	b	d	c	d	a	c	a
n_{13}	n_{14}	n_{15}	n_{16}	n_{17}	n_{18}	n_{19}	n_{20}	n_{21}	n_{22}	n_{23}	n_{24}
c	c	c	c	c	c	d	d	d	d	d	d
a	a	b	b	d	d	a	a	b	b	c	c
b	d	a	d	a	b	b	c	a	c	a	b
d	b	d	a	b	a	c	b	c	a	b	a

table, n_i denotes the number of voters associated with the ranking of type i; voters of type 1 rank a before b, b before c and c before d. A voting situation $\tilde{n} = (n_1, n_2, ..., n_i, ..., n_{24})$ indicates the number of voters for each linear order such that $\sum_{i=1}^{24} n_i = n$. For $a, b \in A$, we denote by n_{ab} the number of voters who prefer a to b. Candidate a is majority preferred to b if $n_{ab} > n_{ba}$; we denote it by aMb.

We denote by $\mathcal{A}^g$ the set of all subsets of A of size g. Such a subset is called a *committee*. The set $\mathcal{A}^2$ of the possible two-member committees is : $\mathcal{A}^2 = \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}$.

Definition 1. (Gehrlein, 1985) A committee $C \in \mathcal{A}^g$ is the *Condorcet Committee à la Gehrlein* (CCG) if and only if $\forall a \in C$, we have aMb for all $b \in A \setminus C$.

For two committees $C, C' \in \mathcal{A}^g$, we denote by $n_{CC'}$ the number of voters who rank committee C before C'. Committee C is majority preferred to C' if $n_{CC'} > n_{C'C}$ and we denote it by CM^*C'. Notice that M stands for the majority relations on A and M^* stand for those on $\mathcal{A}^g$.

Definition 2. (Fishburn, 1981) A committee $C \in \mathcal{A}^g$ is the *Condorcet Committee à la Fishburn* (CCF) if and only if $\forall C' \in \mathcal{A}^g \setminus C$, CM^*C'.

To compute the CCF, we need voters' rankings over the set of the committees of size g. Most of the time, we only have voters' rankings on candidates. According to Kamwa and Merlin (2017), by selecting a committee that is the CCF when it exists, we are sure that we won't get a dominated committee, while selecting the CCG when it exists will prevent from selecting dominated candidates. The ideal would be to select a committee that is both a CCG and a CCF. So, we have to connect voters's rankings on candidates to their rankings of the committees. That is what Kamwa and Merlin (2017) did by assuming two kinds of preference extension techniques: the *Leximax* and the *Leximin*. We also assume these techniques here[2].

The *Leximax* extension ranks subsets of candidates according to their best elements. If two subsets have the same best element, the ranking will depend upon the second best element and so on. At some point, for $X \subseteq Y$, if X and Y are still equivalent according to the *Leximax* while there is no more alternative left in X for further comparison, X is declared better than Y. The Leximax extended rankings on $\mathcal{A}^2$ are displayed in Table 2.

[1] Indifference, intransitive or cyclic rankings are not allowed.

[2] We refer to Barberà *et al.* (2001) for a review of preference extension methods and their normative properties.

Table 2: The Leximax extended rankings on $\mathcal{A}^2$

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}
(a,b)	(a,b)	(a,c)	(a,c)	(a,d)	(a,d)	(a,b)	(a,b)	(b,c)	(b,c)	(b,d)	(b,d)
(a,c)	(a,d)	(a,b)	(a,d)	(a,b)	(a,c)	(b,c)	(b,d)	(a,b)	(b,d)	(a,b)	(b,c)
(a,d)	(a,c)	(a,d)	(a,b)	(a,c)	(a,b)	(b,d)	(b,c)	(b,d)	(a,b)	(b,c)	(a,b)
(b,c)	(b,d)	(b,c)	(c,d)	(b,d)	(c,d)	(a,c)	(a,d)	(a,c)	(c,d)	(a,d)	(c,d)
(b,d)	(b,c)	(c,d)	(b,c)	(c,d)	(b,d)	(a,d)	(a,c)	(c,d)	(a,c)	(c,d)	(a,d)
(c,d)	(c,d)	(b,d)	(b,d)	(b,c)	(b,c)	(c,d)	(c,d)	(a,d)	(a,d)	(a,c)	(a,c)

n_{13}	n_{14}	n_{15}	n_{16}	n_{17}	n_{18}	n_{19}	n_{20}	n_{21}	n_{22}	n_{23}	n_{24}
(a,c)	(a,c)	(b,c)	(b,c)	(c,d)	(c,d)	(a,d)	(a,d)	(b,d)	(b,d)	(c,d)	(c,d)
(b,c)	(c,d)	(a,c)	(c,d)	(a,c)	(b,c)	(b,d)	(c,d)	(a,d)	(c,d)	(a,d)	(b,d)
(c,d)	(b,c)	(c,d)	(a,c)	(b,c)	(a,c)	(c,d)	(b,d)	(c,d)	(a,d)	(b,d)	(a,d)
(a,b)	(a,d)	(a,b)	(b,d)	(a,d)	(b,d)	(a,b)	(a,c)	(a,b)	(b,c)	(a,c)	(b,c)
(a,d)	(a,b)	(b,d)	(a,b)	(b,d)	(a,d)	(a,c)	(a,b)	(b,c)	(a,b)	(b,c)	(a,c)
(b,d)	(b,d)	(a,d)	(a,d)	(a,b)	(a,b)	(b,c)	(b,c)	(a,c)	(a,c)	(a,b)	(a,b)

The *Leximin* extension is dual of the *Leximax*. It compares subsets by their worst elements. If two subsets have the same worst element, the ranking depends upon the next worst elements and so on. At some point, for $X \subseteq Y$, if X and Y are still equivalent according to the *Leximin* while there is no more alternative left in X for further comparison, Y is declared better than X. The Leximin extended rankings on $\mathcal{A}^2$ are displayed in Table 3.

Table 3: The Leximin extended rankings on $\mathcal{A}^2$

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}
(a,b)	(a,b)	(a,c)	(a,c)	(a,d)	(a,d)	(a,b)	(a,b)	(b,c)	(b,c)	(b,d)	(b,d)
(a,c)	(a,d)	(a,b)	(a,d)	(a,b)	(a,c)	(b,c)	(b,d)	(a,b)	(b,d)	(a,b)	(b,c)
(b,c)	(b,d)	(b,c)	(c,d)	(b,d)	(c,d)	(a,c)	(a,d)	(a,c)	(c,d)	(a,d)	(c,d)
(a,d)	(a,c)	(a,d)	(a,b)	(a,c)	(a,b)	(b,d)	(b,c)	(b,d)	(a,b)	(b,c)	(a,b)
(b,d)	(b,c)	(c,d)	(b,c)	(c,d)	(b,d)	(a,d)	(a,c)	(c,d)	(a,c)	(c,d)	(a,d)
(c,d)	(c,d)	(b,d)	(b,d)	(b,c)	(b,c)	(c,d)	(c,d)	(a,d)	(a,d)	(a,c)	(a,c)

n_{13}	n_{14}	n_{15}	n_{16}	n_{17}	n_{18}	n_{19}	n_{20}	n_{21}	n_{22}	n_{23}	n_{24}
(a,c)	(a,c)	(b,c)	(b,c)	(c,d)	(c,d)	(a,d)	(a,d)	(b,d)	(b,d)	(c,d)	(c,d)
(b,c)	(c,d)	(a,c)	(c,d)	(a,c)	(b,c)	(b,d)	(c,d)	(a,d)	(c,d)	(a,d)	(b,d)
(a,b)	(a,d)	(a,b)	(b,d)	(a,d)	(b,d)	(a,b)	(a,c)	(a,b)	(b,c)	(a,c)	(b,c)
(c,d)	(b,c)	(c,d)	(a,c)	(b,c)	(a,c)	(c,d)	(b,d)	(c,d)	(a,d)	(b,d)	(a,d)
(a,d)	(a,b)	(b,d)	(a,b)	(b,d)	(a,d)	(a,c)	(a,b)	(b,c)	(a,b)	(b,c)	(a,c)
(b,d)	(b,d)	(a,d)	(a,d)	(a,b)	(a,b)	(b,c)	(b,c)	(a,c)	(a,c)	(a,b)	(a,b)

As one can notice, the Leximax and the Leximin rankings on the two-member committees are not the same. Kamwa and Merlin (2017) claimed that based on the Leximax and the Leximin, we can derive the majority relations among committees from those among candidates; they showed that $\forall a, b \in A$ and $\forall Z \subseteq A \setminus \{a, b\}$, $xMy \Leftrightarrow \{a\}M^*\{b\}$ and $\{a\}M^*\{b\} \Leftrightarrow \{a\} \cup ZM^*\{b\} \cup Z$. They were able to connect the CCG to the CCF and they characterized the voting situations where it is possible to have a committee that is both a CCG and CCF. What remains is to see whether such a coincidence is just a rare oddity or is a common occurrence. We focus on this question for the selection of a subset two alternatives out of four by computing the conditional probability of voting situations under which the *Condorcet Committee à la Gehrlein* (CCG) and the *Condorcet Committee à la Fishburn* (CCF) may both exist and coincide when voters' preferences on candidates are extended on subsets lexicographically.

2. Coincidence probabilities

First of all, we define the conditions on $\mathcal{A}^2$ such that the CCG and the CCF coincide. Let us recall that $(a, b) = CCG$ only if candidates a and b beat c and d in pairwise majority: aMc, aMd, bMc and bMd; using the labels of Table 1, this is respectively equivalent to

$$n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8+n_{11}+n_{19}+n_{20}+n_{21} > n_9+n_{10}+n_{12}+n_{13}+n_{14}+n_{15}+n_{16}+n_{17}+n_{18}+n_{22}+n_{23}+n_{24} \quad (1)$$
$$n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8+n_9+n_{13}+n_{14}+n_{15} > n_{10}+n_{11}+n_{12}+n_{16}+n_{17}+n_{18}+n_{19}+n_{20}+n_{21}+n_{22}+n_{23}+n_{24} \quad (2)$$
$$n_1+n_2+n_5+n_7+n_8+n_9+n_{10}+n_{11}+n_{12}+n_{19}+n_{21}+n_{22} > n_3+n_4+n_6+n_{13}+n_{14}+n_{15}+n_{16}+n_{17}+n_{18}+n_{20}+n_{23}+n_{24} \quad (3)$$
$$n_1+n_2+n_3+n_7+n_8+n_9+n_{10}+n_{11}+n_{12}+n_{13}+n_{15}+n_{16} > n_4+n_5+n_6+n_{14}+n_{17}+n_{18}+n_{19}+n_{20}+n_{21}+n_{22}+n_{23}+n_{24} \quad (4)$$

For $(a, b) = CCF$ it implies that $(a, b)M^*(a, c)$, $(a, b)M^*(a, d)$, $(a, b)M^*(b, c)$, $(a, b)M^*(b, d)$ and $(a, b)M^*(c, d)$. Using the labels of Table 2 under the Leximax, this is respectively equivalent to

$$n_1+n_2+n_5+n_7+n_8+n_9+n_{10}+n_{11}+n_{12}+n_{19}+n_{21}+n_{22} > n_3+n_4+n_6+n_{13}+n_{14}+n_{15}+n_{16}+n_{17}+n_{18}+n_{20}+n_{23}+n_{24} \quad (5)$$
$$n_1+n_2+n_3+n_7+n_8+n_9+n_{10}+n_{11}+n_{12}+n_{13}+n_{15}+n_{16} > n_4+n_5+n_6+n_{14}+n_{17}+n_{18}+n_{19}+n_{20}+n_{21}+n_{22}+n_{23}+n_{24} \quad (6)$$
$$n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8+n_{11}+n_{19}+n_{20}+n_{21} > n_9+n_{10}+n_{12}+n_{13}+n_{14}+n_{15}+n_{16}+n_{17}+n_{18}+n_{22}+n_{23}+n_{24} \quad (7)$$
$$n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8+n_9+n_{13}+n_{14}+n_{15} > n_{10}+n_{11}+n_{12}+n_{16}+n_{17}+n_{18}+n_{19}+n_{20}+n_{21}+n_{22}+n_{23}+n_{24} \quad (8)$$
$$n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8+n_9+n_{10}+n_{11}+n_{12} > n_{13}+n_{14}+n_{15}+n_{16}+n_{17}+n_{18}+n_{19}+n_{20}+n_{21}+n_{22}+n_{23}+n_{24} \quad (9)$$

One can easily notice that Equations (1) to (4) are exactly the same as (5) to (8). We get the same conclusion with the Leximin. Thus, for the CCG and the CCF to coincide, it is necessary for Equations (5) to (9) to be satisfied. The equations will enable us to compute the probabilities of coincidence between the CCG and the CCF for $m = 4$ and $g = 2$.

The Impartial Culture assumption (IC) is one of the most used assumptions in the social choice literature when computing the likelihood of given events. Under the IC assumption, it is assumed that each voter chooses her preference following a uniform probability distribution which gives probability $\frac{1}{m!}$ to each ranking to be chosen independently. The likelihood of the voting situation $\tilde{n} = (n_1, n_2, ..., n_t, ..., n_{24})$ is

$$Prob(\tilde{n}) = \frac{n!}{\prod_{i=1}^{24} n_i!} \times (24)^{-n}$$

For more about the IC and other probabilistic assumptions, see among others, Gehrlein and Fishburn (1976), Gehrlein and Lepelley (2010), Tataru and Merlin (1997).

Gehrlein (1985) showed under the IC that the limit probability that there is a CCG for $m = 3$ and $g = 2$ is 0.916. For $m = 4$ the probability is 0.739 for $g = 2$ and 0.824 for $g = 3$ and $g = 1$.

Let us denote by $P_{IC}^{C_{\text{Max}}}(4, g, n)$ and $P_{IC}^{C_{\text{Min}}}(4, g, n)$ the probability under IC that given 4 candidates, a CCG of size g exists and coincides with the CCF by Leximax and by Leximin extension respectively. Similarly, we denote by $\overline{P}_{IC}^{C_{\text{Max}}}(4, g, n)$ and $\overline{P}_{IC}^{C_{\text{Min}}}(4, g, n)$ the probability under IC that a CCG of size g exists and there is no CCF. Proposition 1 gives the limit probabilities for $m = 4$ and $g = 2$ as n grows to infinity.

Proposition 1.

$$P_{IC}^{C_{Max}}(4, 2, \infty) = P_{IC}^{C_{Min}}(4, 2, \infty) = 0.739 + \frac{9}{4\pi^2}\int_0^2 I(t)dt = 0.581$$

$$\overline{P}_{IC}^{C_{Max}}(4, 2, \infty) = \overline{P}_{IC}^{C_{Min}}(4, 2, \infty) = -\frac{9}{4\pi^2}\int_0^2 I(t)dt = 0.158$$

where

$$I(t) = \frac{(4-t)\left(\pi - 2\arccos(\frac{3\,t-4}{4\sqrt{14-8t+3t^2}}) + \arccos(\frac{32-8t+3t^2}{112-64t+24t^2})\right)}{(4+t^2-2t)\sqrt{16+3t^2-8t}}$$
$$- \frac{2(-8+5t)}{(4+t^2-2t)\sqrt{128+35t^2-80t}}\Bigg[2\,\pi - \arccos\left(\frac{3\,(-8+5t)\,\sqrt{10}}{20\sqrt{56-44t+17t^2}}\right)$$
$$- \arccos\left(\frac{(-24+5t)\,\sqrt{10}}{40\sqrt{14-8t+3t^2}}\right)$$
$$- \arccos\left(\frac{3(16-12t+3t^2)}{4\sqrt{(14-8t+3t^2)\,(56-44t+17t^2)}}\right)\Bigg]$$
$$- \frac{\sqrt{3}\left(\pi + \arccos\left(\frac{7}{20}\right) - 2\arccos\left(\frac{\sqrt{10}}{20}\right)\right)}{4+t^2-2t}$$

The details of the probability computations are provided in the Appendix. We learn from Proposition 1 that, in a four-candidate elections, a CCG and a CCF of two members coincide with probability 0.581 while the probability that the CCG fails to coincide with the CCF (*i.e* there is a cycle on committees) is equal to 0.158. The conditional probability for $m = 4$ and $g = 2$ that there is no CCF given that a CCG exists is equal to $\frac{0.158}{0.739} = 0.214$.

Table 4: Limiting probabilities

$m = 4$			
	$g = 1$	$g = 2$	$g = 3$
$\widehat{P}^C_{IC}(4, g, \infty)$	0.824	0.739	0.824
$P^C_{IC}(4, g, \infty)$	0.824	0.581	0.824
$\overline{P}^C_{IC}(4, g, \infty)$	0	0.158	0

Table 4 summarizes our results and those from Gehrlein (1985). In this table the values, $\widehat{P}^C_{IC}(m, g, \infty)$ is the limit probability under IC that given m, a CCG of size g exists. For simplicity, we write $P^C_{IC}(4, 2, \infty) = P^{C_{\text{Max}}}_{IC}(4, 2, \infty) = P^{C_{\text{Min}}}_{IC}(4, 2, \infty)$ and $\overline{P}^C_{IC}(4, 2, \infty) = \overline{P}^{C_{\text{Min}}}_{IC}(4, 2, \infty) = \overline{P}^{C_{\text{Min}}}_{IC}(4, 2, \infty)$.

Table 5: Existence probabilities for the CCG and the CCF for $m = 4$ and $g = 2$

	$\exists CCF$	$\nexists CCF$	Total
$\exists CCG$	0.581	0.158	0.739
$\nexists CCG$	0	0.261	0.261
Total	0.581	0.419	1

Table 5 provides probabilities of existence of the CCG and that of the CCF obtained by lexicographic extension for two-member committees. When preferences are lexicographically extended, there is a 58.1% of chance that a CCF exists. At this point, we can also compare our approach via lexicographic extension of preferences over the subsets of two elements to a more direct approach where all the voters form preferences on the six subsets of size two independently from their preferences on A. Hence, the existence of a CCF of size two is equivalent to the search of a subset that dominates the five others via M^*. That is, the probability of existence of a CCF of size two is equivalent to the probability of existence of a Condorcet winner among six elements (here, subsets). Let us denote by $P_{IC}^{CW}(m, \infty)$, the probability under IC that a Condorcet winner exists with m candidates when the electorate is infinite. From Gehrlein (2006) or Gehrlein and Fishburn (1976), we have :

$$P_{IC}^{CW}(6, \infty) = 3 - 5P_{IC}^{CW}(3, \infty) + 3P_{IC}^{CW}(5, \infty)$$

Hence, $P_{IC}^{CW}(6, \infty) = 3 - 5(0.916) + 3(0.749) = 0.667$

Imposing a restriction via the Leximax or Leximin extension, reduces the probability existence of a CCF (0.581 vs 0.667). In other words, restricting preferences via lexicographic extension is more susceptible to lead to cycle among committees than when voters are are free to have any type of rankings they want.

3. Conclusion

We found that, with four candidates and committees of size two, there is a 21.4% of chance that, given a CCG exists, there is no CCF. In this respect, the CCG is a voting concept that is more likely to exist than the CCF. Also, we found that, imposing a restriction via lexicographic preferences reduces the probability existence of a CCF. However, it is not clear whether our conclusions can be generalized. First, computations with more than four candidates would also allow us to have a complete picture of the coincidence of the Condorcet committees. But, this remains a hard and cumbersome task under the Impartial Culture using the same method as in Saari and Tataru (1999) or Gehrlein and Fishburn (1976). To circumvent this, the only solution would be to rely on Monte-Carlo simulations.

Appendix

Our objective is to evaluate the probability of the event described by Equations (5) to (9) under the IC assumption, for $n \to \infty$. The first four equations describe the fact that $\{a, b\}$ is the CCG. Gehrlein (1985) already evaluated the probability of this event under IC for n large at 0.739. We rewrite Equation (9) by using a parameter t.

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_9 + (t-1)n_{10} + (t-1)n_{11} + (t-1)n_{12} &> \\ (t-1)n_{13} + (t-1)n_{14} + (t-1)n_{15} + n_{16} + n_{17} + n_{18} + n_{19} + n_{20} + n_{21} + n_{22} + n_{23} + n_{24} & \end{aligned} \quad (10)$$

When $t = 0$, Equation (10) is equivalent to Equation (8). At $t = 2$, it fully describes Equation (9). Our proof technique will in fact evaluate the probability that Equations (5) to (10) are satisfied under IC for n large. At $t = 0$, we recover the value 0.739 while at $t = 2$, we will derive the probability of disagreement between the CCG and the CCF.

With four candidates, it is assumed under the Impartial Culture assumption that each voter is equally likely to have one of the 24 preference types. Let x_i be the random variable that associates to each voter i a 24-component vector with probability $\frac{1}{24}$ of having 1 in each position. The expectation of x_i is

$$E(x_i) = \left(\frac{1}{24}, \frac{1}{24}, \ldots, \frac{1}{24}\right)$$

and the covariance matrix is a diagonal 24×24 matrix with the common entry σ given by

$$\sigma^2 = E(x_i^2) - E(x_i)^2$$

Let

$$m^T = (m_1, m_2, \ldots m_{24})^T = \frac{1}{\sigma\sqrt{n}}\left[\begin{pmatrix} n_1 \\ \vdots \\ n_{24}\end{pmatrix} - \begin{pmatrix} \frac{n}{24} \\ \vdots \\ \frac{n}{24}\end{pmatrix}\right]$$

The Central Limit Theorem in $\mathbb{R}^{23}$ implies

$$\mu\left[m^T\right] \mapsto \frac{1}{(\sqrt{2}\pi)^{23}} e^{\frac{-|t|^2}{2}}\lambda$$

as $n \to \infty$ where $t = (t_1, t_2, \ldots, t_{24}) \in \mathbb{R}^{24}$, $|t|^2 = t_1^2 + \cdots + t_{24}^2$ and λ is the Lebesgue measure on the 23-dimensional hyperplane $t_1 + \cdots + t_{24} = 0$. Note that since m^T has the measure supported on the hyperplane $m_1 + \cdots + m_{24} = 0$, the limit of m^T as $n \to \infty$ is also a measure supported on $t_1 + \cdots + t_{24} = 0$. To compute the probability that given a CCG exists, it is also the CCF, we need to evaluate the probability that a voting situation is characterized by the five inequalities (5) to (10); m satisfies inequalities (5) to (10) if and only if $\tilde{n} = (n_1, n_2, \ldots, n_{24})$ also satisfies them. Then, by the Central Limit Theorem, we write

$$Pr\left(m^T \quad \text{satisfies} \quad (5) \ \text{ to } \ (10)\right) \mapsto \frac{1}{(\sqrt{2}\pi)^{23}} \int_C e^{\frac{-|t|^2}{2}} d\lambda$$

where $C = \{t \in \mathbb{R}^{24}, t \text{ satisfies } (5) \ \text{ to } \ (10); \text{and } \sum_{i=1}^{24}(t_i) = 0\}$.

As the measure

$$\bar{\mu} \equiv \frac{1}{(\sqrt{2}\pi)^{23}} e^{\frac{-|t|^2}{2}}\lambda$$

is absolutely continuous and radially symmetric, computing

$$\frac{1}{(\sqrt{2}\pi)^{23}} \int_C e^{\frac{-|t|^2}{2}} d\lambda$$

reduces to finding the measure $\bar{\mu}$ of the cone C, when the measure is invariant to rotations. The measure $\bar{\mu}$ of such a cone is proportional to the Euclidean measure of the cone, that is, the measure on the sphere.

Saari and Tataru (1999) have developed a method of computing the probabilities of voting events under the Impartial culture. Some refinements of this method are done in Merlin *et al.* (2000), Merlin and Valognes (2004). This method is mainly based on linear algebra and the calculation of a differential volume in a spherical simplex of dimension ν using the Schläfli's formula (See Coxeter (1935), Schläfli (1950), Milnor (1982), Kellerhals (1989)). This formula is given by:

$$dvol_\nu(C) = \frac{1}{(\nu-1)} \sum_{0 \le j < k \le \nu} vol_{\nu-2}(S_j \cap S_k) d\alpha_{jk}; \quad vol_0 = 1$$

with α_{jk} the dihedral angle formed by the facets S_j and S_k of the cone C. Following the arguments given by Saari and Tataru (1999), the probability that these inequalities are met simultaneously for a voting situation when $p_i = \frac{1}{24}$, $i = 1, \ldots, 24$ for n large is equal to the surface of the spherical simplex T described by equations (5),(6),(7), (8), (10) on the surface of the unit sphere in $\mathbb{R}^5$, divided by the surface of this sphere. More precisely, we will derive

$$P_{IC}^{C_{\text{Max}}}(4, 2, \infty) = 0.73946 + \frac{6}{\omega^5} \int_0^t dvol_\nu(C)$$

where $\omega^5 = \frac{8\pi^2}{3}$ is the volume of the surface of the unit sphere in $\mathbb{R}^5$.

Given the cone C, let S_1 be the facet defined by the equation (5), S_2 the facet defined by the equation (6), so on for S_3, S_4 and S_5.

Let $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}$ be the normal vectors to the hyperplanes S_1, S_2, S_3, S_4, S_5.

$$\begin{aligned}
\vec{v_1} &= (1,1,-1,-1,1,-1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,1,-1,1,1,-1,-1) \\
\vec{v_2} &= (1,1,1,-1,-1,-1,1,1,1,1,1,1,1,-1,1,1,-1,-1,-1,-1,-1,-1,-1,-1) \\
\vec{v_3} &= (1,1,1,1,1,1,1,1,-1,-1,1,-1,-1,-1,-1,-1,-1,-1,1,1,1,-1,-1,-1) \\
\vec{v_4} &= (1,1,1,1,1,1,1,1,1,-1,-1,-1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,-1) \\
\vec{v_5} &= (1,1,1,1,1,1,1,1,1,t-1,t-1,t-1,1-t,1-t,1-t,-1,-1,-1,-1,-1,-1,-1,-1,-1)
\end{aligned}$$

Since $\vec{v_j}$ and $\vec{v_k}$ are respectively normal to S_j and S_k, we can use the relationship

$$cos(\alpha_{jk}) = \frac{-v_j.v_k}{||v_j||.||v_k||}$$

to derive the value of the dihedral angle α_{jk} between vectors $\vec{v_j}$ and $\vec{v_k}$.

$$\begin{aligned}
&\alpha_{12} = \alpha_{13} = \alpha_{24} = \alpha_{34} = \pi - \arccos\left(\tfrac{1}{3}\right) \\
&\alpha_{14} = \alpha_{23} = \tfrac{\pi}{2} \\
&\alpha_{15} = \pi - \arccos\left(\tfrac{t\sqrt{6}}{2\sqrt{24+6t^2-12t}}\right) \\
&\alpha_{25} = \alpha_{35} = \pi - \arccos\left(\tfrac{(4+t)\sqrt{6}}{6\sqrt{24+6t^2-12t}}\right) \\
&\alpha_{45} = \arccos\left(\tfrac{(-4+t)\sqrt{6}}{2\sqrt{24+6t^2-12t}}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&d\alpha_{12} = d\alpha_{13} = d\alpha_{14} = d\alpha_{23} = d\alpha_{24} = d\alpha_{34} = 0 \\
&d\alpha_{15} = \tfrac{-t+4}{(4+t^2-2t)\sqrt{16+3t^2-8t}} \\
&d\alpha_{25} = d\alpha_{35} = \tfrac{8-5t}{(4+t^2-2t)\sqrt{128+35t^2-80t}} \\
&d\alpha_{45} = \tfrac{-\sqrt{3}}{4+t^2-2t}
\end{aligned}$$

The vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}$ lie in a 5-dimension space. Vectors v_6 to v_{24} form a basis for the orthogonal subspace:

$$\begin{array}{rcl}
\vec{v}_6 & = & (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1) \\
\vec{v}_7 & = & (-2,0,2,-1,0,0,0,0,2,-1,0,0,-1,0,0,0,0,0,0,0,1,0,0,0) \\
\vec{v}_8 & = & (-2,0,2,-1,0,0,0,0,2,-1,0,0,-1,0,0,0,0,0,1,0,0,0,0,0) \\
\vec{v}_9 & = & (-1,0,0,0,0,0,0,0,1,-1,1,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{v}_{10} & = & (-1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{v}_{11} & = & (-1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{v}_{12} & = & (-1,0,1,-1,0,0,0,0,2,-1,0,0,-1,0,0,0,0,0,0,1,0,0,0,0) \\
\vec{v}_{13} & = & (-1,0,1,-1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{v}_{14} & = & (-1,0,2,-1,0,0,0,0,1,-1,0,0,-1,0,0,0,0,0,0,0,0,1,0,0) \\
\vec{v}_{15} & = & (-1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{v}_{16} & = & (0,0,0,-1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{v}_{17} & = & (0,0,0,0,0,0,0,0,0,-1,0,1,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{v}_{18} & = & (0,0,0,0,0,0,0,0,0,0,0,0,-1,0,1,0,0,0,0,0,0,0,0,0) \\
\vec{v}_{19} & = & (0,0,0,0,0,0,0,0,1,-1,0,0,-1,0,0,1,0,0,0,0,0,0,0,0) \\
\vec{v}_{20} & = & (0,0,1,-1,0,0,0,0,0,0,0,0,-1,1,0,0,0,0,0,0,0,0,0,0) \\
\vec{v}_{21} & = & (0,0,1,-1,0,0,0,0,1,-1,0,0,-1,0,0,0,0,0,0,0,0,0,0,1) \\
\vec{v}_{22} & = & (0,0,1,-1,0,0,0,0,1,-1,0,0,-1,0,0,0,0,0,0,0,0,0,1,0) \\
\vec{v}_{23} & = & (0,0,1,-1,0,0,0,0,1,-1,0,0,-1,0,0,0,0,1,0,0,0,0,0,0) \\
\vec{v}_{24} & = & (0,0,1,-1,0,0,0,0,1,-1,0,0,-1,0,0,0,1,0,0,0,0,0,0,0)
\end{array}$$

Then, we can calculate the vertexes $P_{1234} = S_1 \cap S_2 \cap S_3 \cap S_4$, $P_{1235} = S_1 \cap S_2 \cap S_3 \cap S_5$, $P_{1245} = S_1 \cap S_2 \cap S_4 \cap S_5$, $P_{1345} = S_1 \cap S_3 \cap S_4 \cap S_5$ and $P_{2345} = S_2 \cap S_3 \cap S_4 \cap S_5$ by solving the following systems

$$P_{1234} : \begin{cases} S_1 = 0 \\ S_2 = 0 \\ S_3 = 0 \\ S_4 = 0 \\ S_5 > 0 \\ S_6 = 0 \\ S_7 = 0 \\ \vdots \\ \vdots \\ S_{23} = 0 \\ S_{24} = 0 \end{cases} \quad P_{1235} : \begin{cases} S_1 = 0 \\ S_2 = 0 \\ S_3 = 0 \\ S_4 > 0 \\ S_5 = 0 \\ S_6 = 0 \\ S_7 = 0 \\ \vdots \\ \vdots \\ S_{23} = 0 \\ S_{24} = 0 \end{cases} \quad P_{1245} : \begin{cases} S_1 = 0 \\ S_2 = 0 \\ S_3 > 0 \\ S_4 = 0 \\ S_5 = 0 \\ S_6 = 0 \\ S_7 = 0 \\ \vdots \\ \vdots \\ S_{23} = 0 \\ S_{24} = 0 \end{cases} \quad P_{1345} : \begin{cases} S_1 = 0 \\ S_2 > 0 \\ S_3 = 0 \\ S_4 = 0 \\ S_5 = 0 \\ S_6 = 0 \\ S_7 = 0 \\ \vdots \\ \vdots \\ S_{23} = 0 \\ S_{24} = 0 \end{cases} \quad P_{2345} : \begin{cases} S_1 > 0 \\ S_2 = 0 \\ S_3 = 0 \\ S_4 = 0 \\ S_5 = 0 \\ S_6 = 0 \\ S_7 = 0 \\ \vdots \\ \vdots \\ S_{23} = 0 \\ S_{24} = 0 \end{cases}$$

The solutions of theses systems are:

$$\begin{array}{rcl}
P_{1234} & = & (-1,-1,2,5,2,5,-1,-1,2,5,2,5,-5,-2,-5,-2,1,1,-5,-2,-5,-2,1,1) \\
P_{1235} & = & (4+t,4+t,2t-8,-20+11t,-8+10t,-20+11t,4+t,4+t,-8+10t, \\
 & & \quad -20+3t,-8-6t,-20+3t,20-3t,8+6t,20-3t,8-10t,-4-t, \\
 & & \quad -4-t,20-11t,8-10t,20-11t,-2t+8,-4-t,-4-t) \\
P_{1245} & = & (3,3,6,1,-2,1,3,3,-10,-7,6,-7,-1,-6,-1,2,-3,-3,7,10,7,-6,-3,-3) \\
P_{1345} & = & (3,3,6,-7,-10,-7,3,3,-2,1,6,1,7,-6,7,10,-3,-3,-1,2,-1,-6,-3,-3) \\
P_{2345} & = & (3,3,-10,-7,6,-7,3,3,6,1,-2,1,-1,2,-1,-6,-3,-3,7,-6,7,10,-3,-3)
\end{array}$$

Knowing these vertices, we are able to compute the volumes $(S_1 \cap S_5)$, $(S_2 \cap S_5)$, $(S_3 \cap S_5)$ and $(S_4 \cap S_5)$. Each of these volumes is the area of a triangle on the sphere in $\mathbb{R}^3$ defined by

Table 6: volumes and directions

volumes	Directions
$(S_1 \cap S_5)$	$P_{1235},\ P_{1245},\ P_{1345}$
$(S_2 \cap S_5)$	$P_{1235},\ P_{1245},\ P_{2345}$
$(S_3 \cap S_5)$	$P_{1235},\ P_{1345},\ P_{2345}$
$(S_4 \cap S_5)$	$P_{1245},\ P_{1345},\ P_{2345}$

some directions. Table 6 gives the direction for each of these volumes. Let us consider the volume $(S_1 \cap S_5)$. By the Gauss-Bonnet theorem, the area of the triangle on the sphere in R^3 defined by directions P_{1235}, P_{1245} and P_{1345} is equal to the sum of the angles on the surface of the triangle minus π. We denote by γ_{1235}, γ_{1245} and γ_{1345} the angles on the surface of the triangle respectively defined at the vertexes P_{1235}, P_{1245} and P_{1345}. Also, we define the angles $\delta_1 = \widehat{P_{1235}, P_{1245}}$, $\delta_2 = \widehat{P_{1235}, P_{1345}}$ and $\delta_3 = \widehat{P_{1245}, P_{1345}}$. By applying the the Gauss-Bonnet formula, we have

$$\begin{aligned}
\cos(\gamma_{1345}) &= \frac{\cos(\delta_1) - \cos(\delta_2)\cos(\delta_3)}{\sin(\delta_2)\sin(\delta_3)} \\
\cos(\gamma_{1245}) &= \frac{\cos(\delta_2) - \cos(\delta_1)\cos(\delta_3)}{\sin(\delta_1)\sin(\delta_3)} \\
\cos(\gamma_{1235}) &= \frac{\cos(\delta_3) - \cos(\delta_1)\cos(\delta_2)}{\sin(\delta_1)\sin(\delta_2)}
\end{aligned}$$

Where

$$\begin{aligned}
\cos(\delta_1) = \cos(\delta_2) &= -3/13\,\frac{(3\,t-4)\sqrt{39}}{\sqrt{240-168\,t+63\,t^2}} \\
\cos(\delta_3) &= \frac{5}{13}
\end{aligned}$$

It comes by the Gauss-Bonnet theorem that

$$\begin{aligned}
vol(S_1 \cap S_5) &= \gamma_{1235} + \gamma_{1245} + \gamma_{1345} - \pi \\
&= \pi - 2\ \arccos\left(\frac{3\,t-4}{4\sqrt{14-8\,t+3\,t^2}}\right) + \arccos\left(\frac{32-8\,t+3\,t^2}{8(14-8\,t+3\,t^2)}\right)
\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}
vol(S_2 \cap S_5) = vol(S_3 \cap S_5) &= 2\pi - \arccos\left(\frac{3(-8+5t)\sqrt{10}}{20\sqrt{56-44t+17t^2}}\right) \\
&\quad - \arccos\left(\frac{(-24+5\,t)\sqrt{10}}{40\sqrt{14-8\,t+3\,t^2}}\right) \\
&\quad - \arccos\left(\frac{3(16-12\,t+3\,t^2)}{4\sqrt{56-44\,t+17\,t^2}\sqrt{14-8\,t+3\,t^2}}\right)
\end{aligned}$$

$$vol(S_4 \cap S_5) = \pi + \arccos\left(\frac{7}{20}\right) - 2\ \arccos\left(\frac{\sqrt{10}}{20}\right)$$

It comes from the Schläfli's formula that,

$$I(t) = vol(S_1 \cap S_5)d\alpha_{15} + vol(S_2 \cap S_5)d\alpha_{25} + vol(S_3 \cap S_5)d\alpha_{35} + vol(S_4 \cap S_5)d\alpha_{45}$$

We have to multiply $I(t)$ by six and divide it by $\frac{8\pi^2}{3}$ the volume of the hypersphere in $\mathbb{R}^5$ to obtain the final differential volume $-\frac{9}{4\pi^2}\int_0^t I(t)dt$. At $t = 2$, the value of this differential volume is equivalent to the probability $\overline{P}_{IC}^{C_{\text{Max}}}(4, 2, \infty)$ that the CCG fails to meet the CCF. To obtain the probability $P_{IC}^{C_{\text{Max}}}(4, 2, \infty)$ that the CCG and the CCF coincide, we just subtract $\overline{P}_{IC}^{C_{\text{Max}}}(4, 2, \infty)$ from 0.739 the probability obtained by Gehrlein (1985).

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