Null controllability of Salmonella spread within an industrial hen house

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Abstract: The main purpose of this paper is to propose a deterministic mathematical model with control variable which describes the spatiotemporal spread of \emph{Salmonella} within a laying flock. This model assume an indirect transmission of the disease within the flock through the bacteria density in the environment. We provide a mathematical analyze of the null controllability of the model: we begin to construct a family of solutions to the linear null-controllability problem using the the Carleman inequality. Then, by a generalized Leray-Shauder fixed point theorem, we show that one can bring the system to rest at final time T.

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Dear Editor,

We are pleased to submit our paper titled “Null controllability of Salmonella spread within an industrial hen house” for consideration in Applied Mathematical Modelling
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Best regards

Gisèle Mophou (corresponding author)

NULL CONTROLLABILITY OF SALMONELLA SPREAD WITHIN AN INDUSTRIAL HEN HOUSE

G. MOPHOU, P. ZONGO, AND R. DORVILLE

ABSTRACT. The main purpose of this paper is to propose a deterministic mathematical model with *control variable* which describes the spatiotemporal spread of *Salmonella* within a laying flock. This model assume an indirect transmission of the disease within the flock through the bacteria density in the environment. We provide a mathematical analyze of the null controllability of the model: we begin to construct a family of solutions to the linear null-controllability problem using the the Carleman inequality. Then, by a generalized Leray-Shauder fixed point theorem, we show that one can bring the system to rest at final time T.

1. INTRODUCTION

Salmonellosis is an infectious disease of humans and animals caused by Salmonella bacteria. The bacterium is commonly found in farm leading to contamination of poultry products, mostly eggs and egg products. Contamination of fruits and vegetables may occur when they have been fertilized or irrigated by faecal wastes. Here, we will focus on salmonella spread within industrial hen house.

Models have already been proposed to study Salmonella spread within industrial hen house [1, 10, 16, 17, 18, 22, 23]. But from the best of our knowledge, no model considered the null controllability of this food-borne illness. A biological system is said to be null controllable at time T if there exists a control such that the solution of system is null at time T. In the particular case of salmonellosis model, a solution of system is the density of infected hens and the density of bacteria; a control within hen house might be an antibacterial cleaning products or ventilation system or other control type. More precisely, let N, $M \in \mathbb{N}\backslash\{0\}$ and Ω be a bounded open subset of $\mathbb{R}^N$ with boundary Γ of class C^2. Let ω be an open non empty subset of Ω. For a time $T > 0$, we set $Q = \Omega \times (0, T)$, $\omega_T = \omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. We consider the following system modeling the salmonella spread within industrial hen house:

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$$\frac{\partial S(x,t)}{\partial t} = -\sigma S(x,t)C(x,t) \quad \text{in} \quad Q, \tag{1a}$$

$$\frac{\partial I(x,t)}{\partial t} = \sigma S(x,t)C(x,t) \quad \text{in} \quad Q, \tag{1b}$$

$$\frac{\partial C(x,t)}{\partial t} - D\Delta C(x,t) + \alpha C(x,t) - \beta(t)I(x,t) = v(x,t)\chi_\omega, \tag{1c}$$

$$\frac{\partial C}{\partial \nu} = 0 \quad \text{on} \quad \Sigma, \tag{1d}$$

$$C(x,0) = C_0(x) \quad \text{in} \quad \Omega, \tag{1e}$$

$$S(x,0) = S_0(x) \quad \text{in} \quad \Omega, \tag{1f}$$

$$I(x,0) = I_0(x) \quad \text{in} \quad \Omega. \tag{1g}$$

In equation (1), $C(x,t)$ denotes the density of bacteria in the environment at time t and position x, $S(x,t)$ denotes the density of susceptible hens at time t and position x, $I(x,t)$ denotes the density of infectious hens at time t and position x. $v(x,t)$ denotes the variable control on which one acts to reduce over time the density of bacteria within hen house. χ_ω is the characteristic function of the control set ω and ν denotes the unit outward normal vector to Γ. The term $\beta(t)I(x,t)$ represents the density of excreted bacteria by infectious hens at time t and position x. The reals σ, α and D denote respectively the transmission rate, the mortality rate of the bacteria and the diffusion coefficient for their dispersal in the environment. The domain Ω represents an industrial laying hens house in which the population of laying hens is confined and assumed motionless. In (1), the excretion rate of the bacteria by hens does not take into account the age of infection (the time lapsed since infection) as in [1], but only the time. The variable control is introduced in order to understand how to eliminate salmonella at a final time $t = T$ when disease was already introduced at a previous time $t = 0$. As in [1, 23], we assume that *Salmonella* disperses in the hens house via a diffusion process through dust particles, contaminated aerosols.

2. Preliminary

Before going further, we need to reformulate the problem (1) in terms of a scalar non-local parabolic equation. We proceed as in [1]. Thus, coming back to (1), one obtains that the quantity

$$S(x,t) + I(x,t) \equiv S_0(x) + I_0(x) \quad \text{in} \quad Q.$$

The integration of equations (1a) and (1b) allows us to reduce the system (1) to the following problem

$$(2)\qquad \begin{cases} S(x,t) = S_0(x)e^{-\sigma\int_0^t C(s,x)ds}, \\ I(x,t) = I_0(x) + S_0(x)(1-e^{-\sigma\int_0^t C(s,x)ds}), \\ \frac{\partial C(x,t)}{\partial t} - D\Delta C(x,t) + \alpha C(x,t) - \beta(t)I(x,t) = v(x,t)\chi_\omega \ \text{ in } Q \\ \dfrac{\partial C}{\partial \nu} = 0 \quad \text{on} \quad \Sigma, \\ C(x,0) = C_0(x) \quad \text{in} \quad \Omega. \end{cases}$$

This allows us to reduce the problem to the following simplified scalar equation for C in which the state variable S and I has been eliminated

$$(3)\qquad \begin{cases} \dfrac{\partial C}{\partial t} - D\Delta C + \alpha C - \beta(t)\left(I_0(x) + S_0(x)(1-e^{-\sigma\int_0^t C(s,x)ds})\right) &= v\chi_\omega & \text{in} & Q, \\ \dfrac{\partial C}{\partial \nu} &= 0 & \text{on} & \Sigma, \\ C(x,0) &= C_0(x) & \text{in} & \Omega. \end{cases}$$

In what follows, we normalize the diffusion coefficient D to 1 by using the rescaling $x := x/\sqrt{D}$ and we focus on the situation when the initial distribution of infectious hens, namely $I_0(x) \equiv 0$. Thus, system (3) can be rewritten as:

$$(4)\qquad \begin{cases} \dfrac{\partial C}{\partial t} - \Delta C + \alpha C - \beta(t)S_0(x)F\left(\displaystyle\int_0^t C(x,s)ds\right) &= v\chi_\omega & \text{in} & Q, \\ \dfrac{\partial C}{\partial \nu} &= 0 & \text{on} & \Sigma, \\ C(x,0) &= C_0(x) & \text{in} & \Omega \end{cases}$$

where $\alpha > 0$ and $\sigma > 0$. The real function F is given by $F(z) = 1 - e^{-\sigma z}$ for any $z \in I\!R^+$. Note that the function F is of class $\mathcal{C}^1$ on $I\!R^+$ and globally Lipschitz. Moreover F is such that $|F(z)| < \sigma|z|$.

We are now interested by the following null controllability problem: Given $\alpha > 0$, $\sigma > 0$, $\beta \in L^\infty(0,T)$, $C_0 \in L^2(\Omega)$ and $S_0 \in L^\infty(\Omega)$, find a control $v \in L^2(\omega_T)$ such that if C is solution to (4) then C satisfies

$$(5)\qquad C(T) = C(x,T) = 0 \quad \text{in } \Omega.$$

There are many literature on null controllability for parabolic equation. Indeed, consider the following parabolic equation:

$$(6)\qquad \frac{\partial y}{\partial t} - \Delta y + f(y) = Bv \text{ with } y(0) = y^0,$$

where v is the control, B is a linear continuous operator defined on the control space and f is a suitable function. The function y^0 is given.

In the linear case, D. Russell in [19] has proved that exact controllability for the wave equation implies exact controllability for the heat equation. Later on Lebeau and Robbiano in [9] solved null boundary controllability of (6) in the case $f \equiv 0$ using observability inequalities deriving from Carleman inequalities. At the same time, O. Y. Imanuvilov and A. Fursikov in [2] obtained the same result for more general operators including variable coefficients and nonzero potentials using more directly global Carleman inequalities for the evolution operator. In [7], D. Tataru showed that for linear equations, local and global controllability are equivalent and hold for any time $T > 0$. Considering a linear Fourier boundary condition, E. Fernandez-Cara, M. González-Burgos, S. Guerrero and J.P. Puel in [5] use the Carleman estimate for the weak solution of heat equation with non-homogeneous Neumann boundary conditions to prove the null controllability of (6). In [13], O. Nakoulima gives a result of null controllability for of (6) with constraint on a distributed control. His results is based on an observability inequality adapted to the constraint.

In the nonlinear case, the problem of finite dimensional null controllability is studied by E. Zuazua in [8]. The author proved that for a rather general and natural class of non-linearities, the problem is solvable if the initial data are small enough. In [2] A. Fursikov and O. Yu. Imanuvilov showed that, when the control acts on the boundary, null controllability holds for bounded continuous and sufficiently small initial data. Let us also mention results in [29, 25], where the methods in [2] have been combined with the variational approach to controllability in [30] to prove null controllability results for (6) with nonlinearities that grow at infinity in a super linear way.

When nonlinear term contains gradient terms, O. Yu. Imanuvilov and M. Yamamoto showed in [26], by using Carleman inequalities in Sobolev spaces of negative order, that global exact zero controllability of (6) holds when the semilinear term has sublinear growth at infinity. Later on Doubova et al.[24] proved that system (6) is null controllable at any time if the nonlinear term $f(y, \nabla y)$ grows slower than $|y| \log^{3/2}(1+|y|+|\nabla y|)+|\nabla y| \log^{1/2}(1+|y|+|\nabla y|)$ at infinity. In [27], G. Mophou proved that the null controllability problem with a finite number of constraints on the state for (6) involving gradient terms holds. Her results is based on an observability inequality adapted to the constraint.

In the case null controllability of parabolic equation with nonlocal linearities, few result are known. We refer for instance to the work of E. Fernandez-Cara et al. [6] where the authors proved a local null controllability for the following parabolic equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + A(t)y = v\chi_\omega & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(0) = y^0 & \text{in} \quad \Omega, \end{cases}$$

where

$$A(t)y = \sum_{i,j=1}^{N} A_{ij}(y(.,t),t)\frac{\partial^2 y}{\partial x_i \partial x_j},$$

the functions A_{ij} are given in a suitable space and can have the form

$$A_{ij}(y(.,t),t) = c_{ij}\left(\int_\Omega y(x,t)\,dx\right)$$

with c_{ij} which are positive reals. Also in [31], O. Traoré establishes a null controllability result for a nonlinear population dynamics model in which the birth term is nonlocal.

In this paper we consider the nonlocal parabolic system (4). Then assuming that

$$\frac{2}{\alpha}|\beta|^4_{L^\infty(0,T)}T^3\sigma^2|S_0|^4_{L^\infty(\Omega)} < 1, \tag{7}$$

we prove that one can bring the system to rest at time T. More precisely, we show the following results:

Theorem 2.1. *Let Ω be a bounded open subset of $I\!R^N$ with boundary Γ of class C^2 and ω a non-empty open subset of Γ. Assume that (7) holds. Then there exists a positive real weight function θ (a precise definition of θ will be given later on) such that, for any $\alpha > 0$ and $\sigma > 0$, for any function $S_0 \in L^\infty(\Omega)$, $C_0 \in L^2(\Omega)$, and any function $\beta \in L^\infty(0,T)$ with $\theta\beta \in L^\infty(Q)$, there exists a control $v \in L^2(\omega_T)$ such that (v, C) with $C = C(v)$ is solution of the null controllability problem (4)-(5).*

The proof of the null controllability problem (4)-(5) lies on the existence of a function θ, a Carleman inequality (see Subsection 3) and a generalized Leray-Shauder fixed point theorem.

The paper is organized as follows : Section 3 is devoted to resolution of a linear null-controllability problem. In this section, we construct by means of Carleman inequality a family of controls that can bring the associate states to rest at time T. Then we prove that among these controls, there exists one of minimal norm in the control space. The proof of Theorem 2.1 is given in Section 4.

3. Study of a linear null-controllability problem

In this section, we construct a solution to the following linear null controllability: Given $\alpha > 0$, $\sigma > 0$, $S_0 \in L^\infty(\Omega))$, $\beta \in L^\infty(0,T)$, $C_0 \in L^2(\Omega)$ and $b \in L^2(Q)$, find a control $v \in L^2(\omega_T)$ such that, if C is solution of

$$\left\{\begin{array}{lcll}\dfrac{\partial C}{\partial t}-\Delta C-\alpha C-\beta(t)S_0(x)b(x,t) & =v\chi_\omega & \text{in} & Q,\\ \dfrac{\partial C}{\partial \nu} & = \quad 0 & \text{on} & \Sigma,\\ C(x,0) & = \quad C_0(x) & \text{in} & \Omega\end{array}\right. \tag{8}$$

then

$$C(T)=0 \text{ in } \Omega. \tag{9}$$

First of all, observe that if $\alpha>0$ and $\sigma>0$, functions b, C_0, and S_0 belong to $L^2(Q)$, $L^2(\Omega)$ and $L^\infty(\Omega)$ respectively, and the control function v is in $L^2(Q)$ then the system (8) has a weak solution $C\in L^2((0,T);H^1(\Omega))\cap\mathcal{C}([0,T],L^2(\Omega))$ in the sense that C satisfies (see [12])

$$\left\{\begin{array}{l}C\in L^2((0,T);H^1(\Omega))\cap\mathcal{C}([0,T],L^2(\Omega)),\\ \langle\dfrac{\partial C}{\partial t}(x,t),\phi(x,t)\rangle_{(H^1(\Omega))',(H^1(\Omega))}+\displaystyle\int_\Omega\nabla C(x,t).\nabla\phi(x,t)dx+\\ \alpha\displaystyle\int_\Omega C(x,t)\phi(x,t)dx+\int_\Omega\beta(t)S_0(x)b(x,t)\,\phi(x,t)dx=\\ \displaystyle\int_\omega v(x,t)\phi(x,t)dx,\ \text{ a.e. in } (0,T),\ \forall\phi\in H^1(\Omega)\\ \text{with the initial condition:}C(x,0)=C_0(x)\end{array}\right. \tag{10}$$

where $(H^1(\Omega))'$ is the dual of $H^1(\Omega)$.

To prove the null controllability of (8) and (9), we use an observability inequality which is consequence of the global Carleman inequality [3]. So, let us consider an auxiliary function $\psi\in C^2(\overline{\Omega})$ which satisfies the following conditions:

$$\left\{\begin{array}{ll}\Psi(x)>0 & \forall x\in\Omega\\ |\nabla\Psi|>0 & \forall x\in\overline{\Omega\backslash\omega'}\\ \Psi(x)=0 & \text{on }\Gamma\end{array}\right. \tag{11}$$

where $\omega'\subset\subset\omega$ is an nonempty open set. Such a function ψ exists according to A. Fursikov and O. Yu. Imanuvilov [2].

For $(x,t)\in Q$, we define for any positive parameter value $\lambda\geq1$ the following weight functions :

$$\varphi(x,t)=\frac{\mathrm{e}^{\lambda\Psi(x)}}{t(T-t)}, \tag{12}$$

$$\eta(x,t)=\frac{\mathrm{e}^{2\lambda|\Psi|_\infty}-\mathrm{e}^{\lambda\Psi(x)}}{t(T-t)}. \tag{13}$$

From now on, we adopt the following notations

$$(14)\qquad \begin{cases} L &= \dfrac{\partial}{\partial t} - \Delta + \alpha I, \\ L^* &= -\dfrac{\partial}{\partial t} - \Delta + \alpha I, \\ L_0^* &= -\dfrac{\partial}{\partial t} - \Delta, \\ \mathcal{V} &= \{\rho \in C^\infty(\overline{Q}),\ \dfrac{\partial \rho}{\partial \nu} = 0 \text{ on } \Sigma\} \end{cases}$$

where $\alpha > 0$ and I is the operator identity.

For any $f \in L^2(Q)$, let ρ be a solution to

$$(15)\qquad \begin{cases} L_0^* \rho &= f(x,t) & \text{in} & Q, \\ \dfrac{\partial \rho}{\partial \nu} &= 0 & \text{on} & \Sigma, \\ \rho(x,T) &= 0 & \text{in} & \Omega. \end{cases}$$

Then the following result holds.

Proposition 3.1. [5] There exist λ^*, σ^* and K only depending on Ω and ω such that, for any $\lambda \geq \lambda^*$ and any $s \geq s^*(\lambda) = \sigma^*(e^{2\lambda|\Psi|_\infty} T + T^2)$, the solution to (15) satisfies

(16)

$$\int_0^T \int_\Omega e^{-2s\eta} \left((s\varphi)^{-1} \left(\left| \frac{\partial \rho}{\partial t} \right|^2 + |\Delta \rho|^2 \right) + s\lambda^2 \varphi \, |\nabla \rho|^2 + s^3 \lambda^4 \varphi^3 \, |\rho|^2 \right) dx \, dt$$
$$\leq K \left[\int_0^T \int_\Omega e^{-2s\eta} \, |L_0^* \rho|^2 \, dx \, dt + s^3 \lambda^4 \int_0^T \int_\omega e^{-2s\eta} \varphi^3 \, |\rho|^2 \, dx \, dt \right].$$

Now, if we write $L_0^* \rho = L^* \rho - \alpha \rho$, the inequality (16) holds for fixed for any $\lambda \geq \lambda^*$ and any $s \geq s^*(\lambda) = \sigma^*(e^{2\lambda|\Psi|_\infty} T + T^2)$. Therefore, observing that

$$\int_0^T \int_\Omega e^{-2s\eta} |L_0^* \rho|^2 \, dx \, dt \leq$$
$$2 \left[\int_0^T \int_\Omega e^{-2s\eta} |L^* \rho|^2 \, dx \, dt + \alpha^2 \int_0^T \int_\Omega e^{-2s\eta} |\rho|^2 \, dx \, dt \right],$$

and choosing s and λ sufficiently large depending on α, we absorb the term $2\alpha^2 \displaystyle\int_0^T \int_\Omega e^{-2s\eta} |\rho|^2 \, dx \, dt$ in the left hand side and we deduce from (16), the following result.

Lemma 3.2. *There exist λ^*, σ^* and K only depending on Ω, ω and α such that, for any $\lambda \geq \lambda^*$ and any $s \geq s^*(\lambda) = \sigma^*(e^{2\lambda|\Psi|_\infty} T + T^2)$, the solution to (15) satisfies*

$$(17)\qquad \begin{aligned} &\int_0^T \int_\Omega e^{-2s\eta} \varphi^3 \, |\rho|^2 \, dx \, dt \\ &\leq K \left[\int_0^T \int_\Omega s^{-3} \lambda^{-4} e^{-2s\eta} \, |L^* \rho|^2 \, dx \, dt + \int_0^T \int_\omega e^{-2s\eta} \varphi^3 \, |\rho|^2 \, dx \, dt \right]. \end{aligned}$$

Remark 3.3. From now on, we will denote by $K(X)$ a generic positive constant whose value varies from a line to another but depending on X.

In view of the definitions of η and φ given respectively by (12) and (13), we have that $e^{-2s\eta}\varphi^3$ and $e^{-2s\eta}$ belong to $L^\infty(Q)$. Therefore, we set

$$\frac{1}{\theta^2} = \varphi^3 e^{-2s\eta} \tag{18}$$

and we deduce from (17) that

$$\int_0^T \int_\Omega \frac{1}{\theta^2} |\rho|^2 \, dx\, dt \leq K(\Omega, \omega, \alpha, T) \left[\int_0^T \int_\Omega |L^*\rho|^2 \, dx\, dt + \int_0^T \int_\omega |\rho|^2 \, dx\, dt \right]. \tag{19}$$

Moreover, proceeding as in [20] we can prove using (17) that the following inequality of observability holds:

$$\int_\Omega |\rho(0)|^2 \, dx\, dt \leq K(\Omega, \omega, \alpha, T) \left[\int_0^T \int_\Omega |L^*\rho|^2 \, dx\, dt + \int_0^T \int_\omega |\rho|^2 \, dx\, dt \right]. \tag{20}$$

We can now construct a solution to (8) and (9). Thus, consider the following symmetric bilinear form

$$a(\rho, \hat{\rho}) = \int_0^T \int_\Omega L^*\rho \, L^*\hat{\rho} \, dx\, dt + \int_0^T \int_\omega \rho \, \hat{\rho} \, dx\, dt. \tag{21}$$

According to (19), this symmetric bilinear form is a scalar product on $\mathcal{V}$. Let V be the completion of $\mathcal{V}$ with respect to the norm:

$$\rho \mapsto \|\rho\|_V = \sqrt{a(\rho, \rho)}. \tag{22}$$

The closure of $\mathcal{V}$ is the Hilbert space V.

Let θ be defined as in (18) and $\beta \in L^\infty(0, T)$ be such that $\theta\beta \in L^\infty(Q)$. Then, thanks to Cauchy-Schwartz's inequality, (19) and (20), the following linear form defined on V by:

$$\rho \mapsto \int_0^T \int_\Omega \beta(t) S_0(x) b(x,t) \, \rho \, dx\, dt + \int_\Omega C_0 \rho(0) \, dx$$

is continuous on V. Therefore, Lax-Milgram's theorem allows us to say that, for every $C_0 \in L^2(\Omega)$, $S_0 \in L^\infty(\Omega)$, $b \in L^2(Q)$ and for any $\beta \in L^\infty(0, T)$ such that $\theta\beta \in L^\infty(Q)$, there exists one and only one solution ρ_θ in V of the variational equation :

$$a(\rho_\theta, \rho) = \int_Q \beta(t) S_0(x) b(x,t) \, \rho \, dx\, dt + \int_\Omega C_0 \rho(0) \, dx, \ \ \forall \rho \in V. \tag{23}$$

In others words

$$\int_Q L^*\rho_\theta \, L^*\rho \, dx\, dt + \int_{\omega_T} \rho \, \rho_\theta \, dxdt = \int_Q \beta(t) S_0(x) b(x,t) \, \rho \, dx\, dt + \int_\Omega C_0 \rho(0) \, dx, \ \ \forall \rho \in V.$$

Proposition 3.4. For any $C_0 \in L^2(\Omega)$, $S_0 \in L^\infty(\Omega)$ $b \in L^2(Q)$ and for any $\beta \in L^\infty(0,T)$ such that $\theta\beta \in L^\infty(Q)$, let ρ_θ be the unique solution of (23). Set

$$v_\theta = -\rho_\theta \chi_\omega \tag{24}$$

and

$$C_\theta = L^* \rho_\theta. \tag{25}$$

Then the pair (v_θ, C_θ) is such that (8) and (9) hold. Moreover, there exists $K = K(\Omega, \omega, T, \alpha) > 0$ such that

$$\|\rho_\theta\|_V \leq K\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}|b|_{L^2(Q)}\right], \tag{26a}$$

$$\|v_\theta\|_{L^2(\omega_T)} \leq K\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}|b|_{L^2(Q)}\right], \tag{26b}$$

$$\|C_\theta\|_{L^2(0,T;H^1(\Omega))} \leq K\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}|b|_{L^2(Q)}\right]. \tag{26c}$$

Proof. One proceeds as in [14, 15], using the variational equation (23) and inequalities (19) and (20).

Remark 3.5. Since v_θ and y_θ depends on θ, (v_θ, y_θ) is a family of solution of the null controllability problem (8) and (9).

Proposition 3.6. Under the assumption of Proposition 3.4, there exists a unique control $\hat{u}$ such that

$$\|\hat{u}\|_{L^2(\omega_T)} = \min_{\bar{v}\in\mathcal{E}} |\bar{v}|_{L^2(\omega_T)} \tag{27}$$

where

$$\mathcal{E} = \left\{\bar{v} \in L^2(\omega_T) | \, (\bar{v}, \bar{C} = C(\bar{v})) \text{ verifies } (8), (9)\right\}.$$

Moreover, there exists $K = K(\Omega, \omega, \sigma, T, \alpha) > 0$ such that

$$\|\hat{u}\|_{L^2(\omega_T)} \leq K\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}|b|_{L^2(Q)}\right]. \tag{28}$$

Proof. According to Proposition 3.4, the pair (v_θ, C_θ) satisfies (8) and (5). Consequently, the set $\mathcal{E}$ is non empty. Since $\mathcal{E}$ is also a closed convex subset of $L^2(\omega_T)$, we deduce that there exists a unique *control variable* $\hat{u}$ of minimal norm in $L^2(\omega_T)$ such that $\left(\hat{u}, \hat{C} = C(\hat{u})\right)$ solves (8) and (9). This means that

$$\|\hat{u}\|_{L^2(\omega_T)} \leq \|v_\theta\|_{L^2(\omega_T)}.$$

Hence, using (26b), we obtain (28). ■

Proposition 3.7. Let $\hat{u}$ be the unique control verifying (27). Then

$$\hat{u} = -\hat{\rho}\chi_\omega \tag{29}$$

where $\hat{\rho} \in V$ is solution of

$$L^*\hat{\rho} = 0, \text{ in } Q, \tag{30a}$$

$$\frac{\partial \hat{\rho}}{\nu} = 0, \text{ on } \Sigma. \tag{30b}$$

Moreover, there exists $K = K(\Omega, \omega, \alpha, T, \alpha) > 0$ such that

$$\|\hat{u}\|_{L^2(\omega_T)} \leq K \left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)} |S_0|_{L^\infty(\Omega)} |b|_{L^2(Q)} \right]. \tag{31}$$

To prove Proposition 3.7, we use a penalization argument and we proceed in three steps.

Step 1. For every $\varepsilon > 0$, we consider the functional

$$J_\varepsilon(v) = \frac{1}{2}|v|^2_{L^2(\omega_T)} + \frac{1}{2\varepsilon}|C(T)|^2_{L^2(\Omega)}. \tag{32}$$

The functional J_ε is well defined since $C \in \mathcal{C}([0,T], L^2(\Omega))$.
The optimal control problem is then to find $v_\varepsilon \in L^2(\omega_T)$ such that

$$J_\varepsilon(v_\varepsilon) = \min_{v \in L^2(\omega_T)} J_\varepsilon(v). \tag{33}$$

It is classical to show that there exists a unique solution v_ε to (33) (see for example [12]). If we write C_ε the solution to (8) corresponding to v_ε, using an adjoint state ρ_ε, the triplet $(C_\varepsilon, \rho_\varepsilon, v_\varepsilon)$ is solution of the following first order optimality system

$$\left\{ \begin{array}{lcll} LC_\varepsilon & = & \beta(t)S_0(x)b(x,t) + v_\varepsilon \chi_\omega & \text{in} \quad Q, \\ \dfrac{\partial C_\varepsilon}{\partial \nu} & = & 0 & \text{on} \quad \Sigma, \\ C_\varepsilon(0) & = & C_0 & \text{in} \quad \Omega, \end{array} \right. \tag{34}$$

$$\left\{ \begin{array}{lcll} L^*\rho_\varepsilon & = & 0 & \text{in} \quad Q, \\ \dfrac{\partial \rho_\varepsilon}{\partial \nu} & = & 0 & \text{on} \quad \Sigma, \\ \rho_\varepsilon(T) & = & \dfrac{1}{\varepsilon} C_\varepsilon(T) & \text{in} \quad \Omega. \end{array} \right. \tag{35}$$

$$v_\varepsilon \quad = \quad -\rho_\varepsilon \chi_\omega. \tag{36}$$

Step 2. We give estimates on the control v_ε and on the state and adjoint state C_ε and ρ_ε.

Multiplying the state equation (34) by ρ_ε and integrating by parts over Q, we obtain that

$$\begin{aligned} \int_\Omega C_\varepsilon(T)\, \rho_\varepsilon(T)\, dx \quad &= \quad \int_\Omega C_\varepsilon(0)\, \rho_\varepsilon(0)\, dx + \int_0^T \int_\Omega \beta(t) S_0(x) b(x,t)\, \rho_\varepsilon(x,t) dx\, dt \\ &+ \quad \int_0^T \int_\omega v_\varepsilon\, \rho_\varepsilon dx\, dt \end{aligned}$$

and in view of $(35)_3$ and (36) we get,

$$\begin{aligned}\frac{1}{\epsilon}|C_\varepsilon(T)|^2_{L^2(\Omega)} + |v_\varepsilon|^2_{L^2(\omega_T)} &= \int_\Omega C_0\,\rho_\varepsilon(0)\,dx \\ &+ \int_0^T\int_\Omega \beta(t)S_0(x)b(x,t)\,\rho_\varepsilon(x,t)dx\,dt.\end{aligned}$$

Therefore, using (19), (20) and $(35)_1$, we have

$$\begin{aligned}2J_\varepsilon(v_\varepsilon) &\leq K(\Omega,\omega,T,\alpha)|C_0|_{L^2(\Omega)}|\rho_\varepsilon|_{L^2(\omega_T)} \\ &+ |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}|b|_{L^2(Q)}\left|\frac{1}{\theta}\rho_\varepsilon\right|_{L^2(Q)} \\ &\leq K(\Omega,\omega,T,\alpha)\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}|b|_{L^2(Q)}\right]|\rho_\varepsilon|_{L^2(\omega_T)}\end{aligned}$$

which in view of the definition of J_ε and (36) implies that

$$\text{(37a)}\quad |C_\varepsilon(T)|_{L^2(\Omega)} \leq \sqrt{\epsilon}K(\Omega,\omega,T,\alpha)\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}|b|_{L^2(Q)}\right],$$

$$\text{(37b)}\quad |v_\varepsilon|_{L^2(\omega_T)} \leq K(\Omega,\omega,T,\alpha)\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}|b|_{L^2(Q)}\right].$$

Hence, in view of (34), we deduce that there exists

$$K = K(\Omega,\omega,T,\alpha,|\theta\beta|_{L^\infty(Q)},|\beta|_{L^\infty(0,T)},|S_0|_{L^\infty(\Omega)}) > 0$$

such that

$$\text{(38)}\quad |C_\varepsilon|_{L^2(0,T;H^1(\Omega))} \leq K\left[|C_0|_{L^2(\Omega)} + |b|_{L^2(Q)}\right],$$

and by standard arguments, we can prove that,

$$\text{(39)}\quad \left|\frac{\partial C_\varepsilon}{\partial t}\right|_{L^2(0,T;H^{-1}(\Omega))} \leq K\left[|C_0|_{L^2(\Omega)} + |b|_{L^2(Q)}\right].$$

From (38) and (39), we obtain that

$$\text{(40)}\quad |C_\varepsilon|_{W(0,T))} \leq K\left[|C_0|_{L^2(\Omega)} + |b|_{L^2(Q)}\right]$$

where

$$\text{(41)}\quad W(0,T) = \left\{\rho \in L^2((0,T),H^1(\Omega)),\ \frac{\partial\rho}{\partial t} \in L^2(0,T;H^{-1}(\Omega))\right\}$$

On the other hand, as ρ_ε is solution of (35), using (19), (36), (37b) and the definition of the norm on V given by (22), we deduce that

$$\text{(42)}\quad \left|\frac{1}{\theta}\rho_\epsilon\right|_{L^2Q} \leq K(\Omega,\omega,T,\alpha)\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}|b|_{L^2(Q)}\right],$$

$$\text{(43)}\quad |\rho_\epsilon|_V \leq K(\Omega,\omega,T,\alpha)\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}|b|_{L^2(Q)}\right].$$

Step 3. In view of (37), (40) and (42), we can extract subsequences of (v_ε), (C_ε) and ρ_ε (still called (v_ε), $(C_\varepsilon)$and ρ_ε) such that

$$C_\varepsilon(T) \to \delta = 0 \qquad \text{strongly in} \quad L^2(\Omega), \tag{44}$$

$$v_\varepsilon \rightharpoonup \tilde{v} \qquad \text{weakly in} \quad L^2(\omega_T), \tag{45}$$

$$C_\varepsilon \rightharpoonup \tilde{C} \qquad \text{weakly in} \quad W(0,T), \tag{46}$$

$$\rho_\varepsilon \rightharpoonup \tilde{\rho} \qquad \text{weakly in} \quad L^2(\frac{1}{\theta}, Q). \tag{47}$$

Therefore, we can prove by passing (34) to the limit when ε tends to 0 that $\tilde{C}$ is solution of

$$\left\{ \begin{array}{rcll} L\tilde{C} &=& \beta(t)S_0(x)b(x,t) + \tilde{v}\chi_\omega & \text{in} \quad Q, \\ \dfrac{\partial \tilde{C}}{\partial \nu} &=& 0 & \text{on} \quad \Sigma, \\ \tilde{C}(0) &=& C_0 & \text{in} \quad \Omega, \end{array} \right. \tag{48}$$

and verifies

$$\tilde{C}(T) = 0$$

since by (44) and (46),

$$C_\varepsilon(T) \to \delta = \tilde{C}(T) = 0 \;\; \text{strongly} \quad \text{in } Ł^2(\Omega).$$

Therefore it is clear that $(\tilde{v}, \tilde{C})$ verifies (8)-(9) and there exists a solution to the null controllability problem. Moreover, because of (37b), (43) and (36) we see that

$$\rho_\varepsilon \rightharpoonup \tilde{\rho} \qquad \text{weakly in} \quad L^2(\omega_T), \tag{49}$$

$$\rho_\varepsilon \rightharpoonup \tilde{\rho} \qquad \text{weakly in} \quad V. \tag{50}$$

Therefore, it is clear from (35) that $\tilde{\rho}$ satisfies

$$\left\{ \begin{array}{l} L^*\tilde{\rho} = 0 \quad \text{in} \quad Q, \\ \dfrac{\partial \tilde{\rho}}{\partial \nu} = 0 \quad \text{on} \quad \Sigma \end{array} \right. \tag{51}$$

on the one hand, and on the other hand,

$$\tilde{v} = -\tilde{\rho}\chi_\omega \tag{52}$$

since (36), (45) and (49) hold.

As it has been shown in Proposition 3.6 that we can find a unique $\hat{u} \in \mathcal{E}$ (admissible control) such that $\hat{u}$ is of minimal norm in $L^2(\omega_T)$. As $\tilde{v} \in \mathcal{E}$,

$$\frac{1}{2}|v_\varepsilon|^2_{L^2(\omega_T)} \leq J_\varepsilon(v_\varepsilon) \leq J_\varepsilon(\hat{u}) = \frac{1}{2}|\hat{u}|^2_{L^2(\omega_T)}$$

and

$$\frac{1}{2}|\hat{u}|^2_{L^2(\omega_T)} \leq \frac{1}{2}|\tilde{v}|^2_{L^2(\omega_T)}.$$

But because of (45),

$$\liminf_{\varepsilon\to 0}\frac{1}{2}|v_\varepsilon|^2_{L^2(\omega_T)} \geq \frac{1}{2}|\tilde{v}|^2_{L^2(\omega_T)}.$$

Therefore we have

$$\tilde{v} = \hat{u}$$

and

$$v_\varepsilon \to \hat{u} \qquad \text{strongly in} \quad L^2(\omega_T). \tag{53}$$

Writing $\hat{\rho} = \tilde{\rho}$ we have

$$\hat{u} = \hat{\rho}\chi_\omega. \tag{54}$$

Finally from (37b), (45) and (53) we deduce (31). This finishes the proof of Proposition 3.7. ■

4. Proof of the main result

For $z \in L^2(Q)$, let $b = F(z)$. Then according to Proposition 3.7 there exists a control $\hat{u}$ verifying (31) such that the pair $(\hat{u}, \hat{C} = C(\hat{u}))$ satisfies the null controllability (8) and (9). Then we defined for $z \in L^2(Q)$, $\Pi(z)$, the nonempty set of all $\int_0^t C(x,s)ds$ where $C = C(v)$ solves (8) and verifies (9), the control v verifies (31). It suffices now to prove that Π which is a multivalued mapping of $L^2(Q)$ has a fixed point to complete the proof of Theorem 2.1. To this end we use the generalisation of the Leray-Schauder fixed point theorem [4]. So, set

$$N = \{z \in L^2(Q),\, \exists\zeta \in (0,1),\, z \in \zeta\Pi(z)\}.$$

Then we have the following results

Proposition 4.1. Let F be defined as in Section 1. Then

(i) Π is a compact multivalued mapping of $L^2(Q)$.
(ii) For all $z \in L^2(Q)$, $\Pi(z)$ is a nonempty closed convex subset of $L^2(Q)$.
(iii) N is bounded in $L^2(Q)$.
(iv) Π is upper semicontinuous on $L^2(Q)$.

Proof. • **We prove the compactness of Π.**.

Let $z \in L^2(Q)$ such that $|z|_{L^2(Q)} \leq r$, $r > 0$. Consider $(\varphi_n)_n \subset \Pi(z)$. Then from the definition of Π, for all n, there exists a pair $(v_n, C_n) \in L^2(\omega_T) \times L^2(\mathcal{Q})$ such that $\varphi_n = \int_0^t C_n(x,s)\,ds$, v_n verifies (31) and the associate state C_n is solution of (8) with $b(x,t) = F\left(\int_0^t z(x,s)\,ds\right)$ and satisfies (9) . This means that the pair (v_n, C_n) satisfies

$$
\left\{
\begin{array}{rcll}
\dfrac{\partial C_n}{\partial t} - \Delta C_n + \alpha C_n + \beta(t) S_0(x) F\left(\displaystyle\int_0^t z(x,s)\,ds\right) &=& v_n \chi_\omega, & \text{in} \quad Q,\\
\dfrac{\partial C_n}{\partial \nu} &=& 0, & \text{on} \quad \Sigma,\\
C_n(0) &=& C_0, & \text{in} \quad \Omega,
\end{array}
\right. \tag{55}
$$

and

$$
C_n(T) = 0 \quad \text{in } \Omega. \tag{56}
$$

Moreover, in view of (31), there exists $K = K(\Omega, \omega, \alpha, T) > 0$ such that

(57)

$$
\|v_n\|_{L^2(\omega_T)} \quad \leq \quad K\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)} |S_0|_{L^\infty(\Omega)} \left|F\left(\textstyle\int_0^t z(x,s)\,ds\right)\right|_{L^2(Q)}\right].
$$

Since $|F(z)| \leq \sigma |z|$, this latter inequality yields

$$
\|v_n\|_{L^2(\omega_T)} \leq K\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)} |S_0|_{L^\infty(\Omega)} \sigma\, r\right]. \tag{58}
$$

Multiplying (55) by C_n and integrating by parts over Q, we obtain

$$
\begin{aligned}
\int_0^T \int_\Omega |\nabla C_n|^2 dx\,dt + \alpha \int_0^T \int_\Omega |C_n|^2 dx\,dt \quad &= \quad \frac{1}{2}\int_\Omega |C_0|^2\,dx + \int_0^T \int_\Omega v_n \chi_\omega\, C_n dx\,dt \\
&+ \quad \int_0^T \int_\Omega \beta(t) S_0(x) F\left(\int_0^t z(x,s)\,ds\right) C_n dx\,dt,
\end{aligned}
$$

which by using Young inequality gives

$$
\begin{aligned}
&\int_0^T \int_\Omega |\nabla C_n|^2 dx\,dt + \left(\alpha - \frac{|\beta|^2_{L^\infty(0,T)} |S_0|^2_{L^\infty(\Omega)}}{2\mu_0} - \frac{\alpha}{2}\right) \int_0^T \int_\Omega |C_n|^2 dx\,dt \leq \\
&\frac{1}{2}|C_0|^2_{L^2(\Omega)} + \frac{1}{2\alpha}\int_0^T \int_\omega |v_n|^2 dx\,dt + \frac{\mu_0}{2} |\beta|^2_{L^\infty(0,T)} |S_0|^2_{L^\infty(\Omega)} \sigma^2 r^2,
\end{aligned}
$$

for any $\mu_0 > 0$. Therefore choosing μ_0 such that

$$
\left(\frac{\alpha}{2} - \frac{|\beta|^2_{L^\infty(0,T)} |S_0|^2_{L^\infty(\Omega)}}{2\mu_0}\right) > \alpha/4
$$

. That is, $\mu_0 > \dfrac{2}{\alpha}|\beta|^2_{L^\infty(0,T)} |S_0|^2_{L^\infty(\Omega)}$ and using (58), we have that

$$
\begin{aligned}
&\min(1, \alpha/4)\left(\int_0^T \int_\Omega |\nabla C_n|^2 dx\,dt + \int_0^T \int_\Omega |C_n|^2 dx\,dt\right) \leq \\
&\frac{1}{2}|C_0|^2_{L^2(\Omega)} + \frac{1}{2\alpha} K^2 \left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)} |S_0|_{L^\infty(\Omega)} \sigma\, r\right]^2 + \\
&\frac{\mu_0}{2} |\beta|^2_{L^\infty(0,T)} |S_0|^2_{L^\infty(\Omega)} \sigma^2 r^2 \leq \\
&\frac{1}{2}|C_0|^2_{L^2(\Omega)} + \frac{1}{\alpha} K^2 \left[|C_0|^2_{L^2(\Omega)} + |\theta\beta|^2_{L^\infty(Q)} |S_0|^2_{L^\infty(\Omega)} \sigma^2\, r^2\right] + \\
&\frac{\mu_0}{2} |\beta|^2_{L^\infty(0,T)} |S_0|^2_{L^\infty(\Omega)} \sigma^2 r^2
\end{aligned}
$$

where $K = K(\Omega, \omega, \alpha, T) > 0$. Hence, there exists $K = K(\Omega, \omega, \alpha, T) > 0$ such that

(59)
$$
\begin{aligned}
|C_n|^2_{L^2(0,T;H^1(\Omega))} \quad &\leq \quad \frac{1}{\min(1,\alpha/4)} \left[\left(\frac{1}{2} + \frac{K^2}{\alpha} \right) |C_0|^2_{L^2(\Omega)} \right. \\
&+ \quad \left. \left(\frac{K^2}{\alpha} |\theta\beta|^2_{L^\infty(Q)} |S_0|^2_{L^\infty(\Omega)} + \frac{\mu_0}{2} |\beta|^2_{L^\infty(0,T)} |S_0|^2_{L^\infty(\Omega)} \right) \sigma^2 r^2 \right].
\end{aligned}
$$

Consequently, by standard argument we can prove that for any $\mu_0 > \dfrac{2}{\alpha} |\beta|^2_{L^\infty(0,T)} |S_0|^2_{L^\infty(\Omega)}$, there exists a constant $K = K\left(\Omega, \omega, \alpha, T, |\beta|_{L^\infty(0,T)}, |\theta\beta|_{L^\infty(Q)}, |S_0|_{L^\infty(\Omega)}, \mu_0\right) > 0$ such that such that

$$
\left| \frac{\partial C_n}{\partial t} \right|^2_{L^2(0,T;H^{-1}(\Omega))} \leq K \left(|C_0|^2_{L^2(\Omega)} + \sigma^2 r^2 \right).
$$

Thus, combining this latter inequality with (59) we obtain that

$$
|C_n|^2_{W(0,T)} \leq K \left(|C_0|^2_{L^2(\Omega)} + \sigma^2 r^2 \right) \tag{60}
$$

with $K = K\left(\Omega, \omega, \alpha, T, |\beta|_{L^\infty(0,T)}, |\theta\beta|_{L^\infty(Q)}, |S_0|_{L^\infty(\Omega)}, \mu_0\right) > 0$.

On the other, we have that $\varphi_n = \int_0^t C_n(x,s)\,ds$ verifies

$$
\left\{
\begin{array}{rcll}
\dfrac{\partial \varphi_n}{\partial t} - \Delta \varphi_n + \alpha \varphi_n & = & \Theta_n & \text{in} \quad Q, \\
\dfrac{\partial \varphi_n}{\partial \nu} & = & 0 & \text{on} \quad \Sigma, \\
\varphi_n(0) & = & 0 & \text{in} \quad \Omega,
\end{array}
\right. \tag{61}
$$

where

$$
\Theta_n = C_0 + \int_0^t v_n(x,s) \chi_\omega ds + S_0(x) \int_0^t \beta(s) F \left(\int_0^s z(x,\tau)\, d\tau \right) ds.
$$

Observing in the one hand that,

$$
\begin{aligned}
\left| \int_0^t v_n(x,s) \chi_\omega ds \right|^2_{L^2(Q)} \quad &\leq \quad T^3 |v_n|^2_{L^2(\omega_T)} \\
&\leq \quad 2T^3 K(\Omega, \omega, \alpha, T)^2 \left(|C_0|^2_{L^2(\Omega)} + |\theta\beta|^2_{L^\infty(Q)} |S_0|^2_{L^\infty(\Omega)} \sigma^2 r^2 \right)
\end{aligned}
$$

since (58) holds, and on the other hand that,

$$
\left| S_0(x) \int_0^t \beta(s) F \left(\int_0^s z(x,\tau)\, d\tau \right) ds \right|^2_{L^2(Q)} \quad \leq \quad |\beta|^2_{L^\infty(Q)} |S_0|^2_{L^\infty(\Omega)} \sigma^2 r^2 T^2,
$$

we deduce that

$$
\begin{aligned}
|\Theta_n|^2_{L^2(Q)} \quad &\leq \quad 3(T + 2T^3 K(\Omega, \omega, \alpha, T)^2) |C_0|^2_{L^2(\Omega)} \\
&+ \quad 3 \left(2T^3 K(\Omega, \omega, \alpha, T)^2 |\theta\beta|^2_{L^\infty(Q)} + |\beta|^2_{L^\infty(Q)} T^2 \right) |S_0|^2_{L^\infty(\Omega)} \sigma^2 r^2.
\end{aligned}
$$

Thus, there exists $K = K\left(\Omega, \omega, \alpha, T, |\beta|_{L^\infty(0,T)}, |\theta\beta|_{L^\infty(Q)}, |S_0|_{L^\infty(\Omega)}\right) > 0$ such that

$$|\Theta_n|^2_{L^2(Q)} \leq K\left(|C_0|^2_{L^2(\Omega)} + \sigma^2\, r^2\right). \tag{62}$$

Multiplying (61) by φ_n and integrating by parts over Q, we deduce that

$$\int_0^T\int_\Omega |\nabla\varphi_n|^2 dx\,dt + \frac{1}{2}(2\alpha-\alpha)\int_0^T\int_\Omega |\varphi_n|^2 dx\,dt \quad\leq\quad \frac{1}{2\alpha}\int_0^T\int_\omega |\Theta_n|^2 dx\,dt.$$

This implies that

$$\min(1,\alpha/2)\left(\int_0^T\int_\Omega |\nabla\varphi_n|^2 dx\,dt + \int_0^T\int_\Omega |\varphi_n|^2 dx\,dt\right) \quad\leq\quad \frac{1}{2\alpha}\int_0^T\int_\omega |\Theta_n|^2 dx\,dt.$$

Hence, in view of (62), there exists $K = K\left(\Omega,\omega,\alpha,T,|\beta|_{L^\infty(0,T)},|\theta\beta|_{L^\infty(Q)},|S_0|_{L^\infty(\Omega)}\right) > 0$ such that

$$|\varphi_n|^2_{L^2(0,T;H^1(\Omega))} dx\,dt \leq K\left(|C_0|^2_{L^2(\Omega)} + \sigma^2\, r^2\right). \tag{63}$$

By standard arguments, we deduce that $\dfrac{\partial\varphi_n}{\partial t}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. This means that φ_n is bounded in $W(0,T)$. As by the Aubin-Lions Lemma, the imbedding of $W(0,T)$ in $L^2(\mathcal{Q})$ is compact, we conclude that Π is compact.

• **We prove that $\Pi(z)$ is a nonempty closed convex subset of $L^2(Q)$.**

It is clear that for all $z \in L^2(\mathcal{Q})$, we have that $\Pi(z)$ is a nonempty convex set. Let $(\varphi_n) \subset \Pi(z)$ such that $\varphi_n \to \varphi$ in $L^2(\mathcal{Q})$. It suffices to prove that $\varphi \in \Pi(z)$ to obtain that $\Pi(z)$ is a closed subset of $L^2(Q)$. Since $\varphi_n = \int_0^t C_n(x,s)\,ds$ where C_n is solution of (55) and satisfies (56) with v_n verifying (57), we can say that v_n and C_n verifies (58) and (60) respectively. Consequently, there exists subsequences of (v_n) and (C_n) still denoted by (v_n) and (C_n) such that

$$v_n \rightharpoonup v \text{ weakly in } L^2(\omega_T), \tag{64}$$

$$C_n \rightharpoonup C \text{ weakly in } W(0,T). \tag{65}$$

Since the imbedding of $W(0,T)$ in $L^2(\omega_T)$ is compact, we have

$$C_n \to C \text{ strongly in } L^2(\omega_T). \tag{66}$$

Therefore, we have

$$\varphi_n = \int_0^t C_n(x,s)\,ds \to \varphi = \int_0^t C(x,s)\,ds$$

and since F is continuous on $\mathbb{R}$,

$$F(\varphi_n) = F\left(\int_0^t C_n(x,s)\,ds\right) \to F(\varphi) = F\left(\int_0^t C(x,s)\,ds\right).. \tag{67}$$

Then, using (64), (65), (66) and (67) while passing (55) and (56) to the limit when n tends to ∞, one obtains that the pair $(v, C = C(v))$ satisfies

$$(68)\quad \begin{cases} \dfrac{\partial C}{\partial t} - \Delta C + \alpha C + \beta(t) S_0(x) F\left(\displaystyle\int_0^t z(x,s)\, ds\right) = v\chi_\omega, & \text{in} \quad Q, \\ \dfrac{\partial C}{\partial \nu} = 0, & \text{on} \quad \Sigma, \\ C(0) = C_0, & \text{in} \quad \Omega \end{cases}$$

and (9). Moreover, in view of (57) and (64), we deduce that

$$(69)\quad \|v\|_{L^2(\omega_T)} \leq K\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}\left|F\left(\int_0^t z(x,s)\, ds\right)\right|_{L^2(Q)}\right].$$

where $K = K(\Omega, \omega, \alpha, T)$. Hence, $\varphi \in \Pi(z)$.

• **We prove that N is bounded in $L^2(Q)$.**

Let $z \in N$. Then there exists $\zeta \in (0,1)$ such that $\frac{1}{\zeta} z \in \Pi(z)$. Consequently there exists a pair (v,C=C(v))$\in L^2(\omega_T) \times L^2(\mathcal{Q})$ such that $z = \zeta \int_0^t C(x,s)\, ds$ where C satisfies (68) and (9) with $b(x,t) = F\left(\zeta \displaystyle\int_0^t C(x,s)\, ds\right)$, the control v verifies (31). This means that

$$\begin{aligned} \|v\|_{L^2(\omega_T)} &\leq K(\Omega, \omega, \alpha, T)\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}\,|F(z)|_{L^2(Q)}\right] \\ &\leq K(\Omega, \omega, \alpha, T)\left[|C_0|_{L^2(\Omega)} + \sigma|\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}\,|z|_{L^2(Q)}\right]. \end{aligned}$$

Therefore, proceeding as for (59) we obtain that for any $\mu_0 > \frac{2}{\alpha}|\beta|^2_{L^\infty(0,T)}|S_0|^2_{L^\infty(\Omega)}$ there exists there exists $K = K(\Omega, \omega, \alpha, T) > 0$ such that

$$\begin{aligned} |C|^2_{L^2(0,T;H^1(\Omega))} &\leq \frac{1}{\min(1, \alpha/4)}\left[\left(\frac{1}{2} + \frac{K^2}{\alpha}\right)|C_0|^2_{L^2(\Omega)}\right. \\ &+ \left.\left(\frac{K^2}{\alpha}|\theta\beta|^2_{L^\infty(Q)}|S_0|^2_{L^\infty(\Omega)} + \frac{\mu_0}{2}|\beta|^2_{L^\infty(0,T)}|S_0|^2_{L^\infty(\Omega)}\right)\sigma^2|z|^2_{L^2(Q)}\right]. \end{aligned}$$

This implies that

(70)

$$\begin{aligned} |C|^2_{L^2(Q)} &\leq \frac{1}{\min(1, \alpha/4)}\left[\left(\frac{1}{2} + \frac{K^2}{\alpha}\right)|C_0|^2_{L^2(\Omega)}\right. \\ &+ \left.\left(\frac{K^2}{\alpha}|\theta\beta|^2_{L^\infty(Q)}|S_0|^2_{L^\infty(\Omega)} + \frac{\mu_0}{2}|\beta|^2_{L^\infty(0,T)}|S_0|^2_{L^\infty(\Omega)}\right)\sigma^2|z|^2_{L^2(Q)}\right]. \end{aligned}$$

with $K = K(\Omega, \omega, \alpha, T) > 0$. As

$$(71)\quad |z|^2_{L^2(Q)} = \left|\zeta\int_0^t c(x,s)\, ds\right|^2_{L^2(Q)} \leq T^3|C|^2_{L^2(Q)},$$

using (70), we deduce that

$$\left[1-\frac{T^3\sigma^2|S_0|^2_{L^\infty(\Omega)}}{\min(1,\alpha/4)}\left(\frac{K^2}{\alpha}|\theta\beta|^2_{L^\infty(Q)}+\frac{\mu_0}{2}|\beta|^2_{L^\infty(0,T)}\right)\right]|z|^2_{L^2(Q)}\leq$$
$$\frac{T^3}{\min(1,\alpha/4)}\left(\frac{1}{2}+\frac{K^2}{\alpha}\right)|C_0|^2_{L^2(\Omega)}$$

choosing μ_0 such that

$$\left[1-\frac{T^3\sigma^2|S_0|^2_{L^\infty(\Omega)}}{\min(1,\alpha/4)}\left(\frac{K^2}{\alpha}|\theta\beta|^2_{L^\infty(Q)}+\frac{\mu_0}{2}|\beta|^2_{L^\infty(0,T)}\right)\right]>\frac{1}{2}.$$

This means choosing

$$\frac{2}{\alpha}|\beta|^2_{L^\infty(0,T)}|S_0|^2_{L^\infty(\Omega)}<\mu_0<\frac{1}{T^3\sigma^2|S_0|^2_{L^\infty(\Omega)}|\beta|^2_{L^\infty(0,T)}},$$

we deduce that there exists $K=K\left(\Omega,\omega,\alpha,T\right)>0$ such that

$$|z|^2_{L^2(Q)}\leq\frac{2T^3}{\min(1,\alpha/4)}\left(\frac{1}{2}+\frac{K^2}{\alpha}\right)|C_0|^2_{L^2(\Omega)}.$$

• **We prove that Π is upper semicontinuous on $L^2(Q)$**

To this end we show that for any closed subset G of $L^2(\mathcal{Q})$, $\Pi^{-1}(G)$ is closed in $L^2(\mathcal{Q})$. Let $(z_n)\subset\Pi^{-1}(G)$ such that $z_n\to z$ in $L^2(\mathcal{Q})$. Then z_n is bounded in $L^2(\mathcal{Q})$ and for all n there exists $\varphi_n\in G$ such that $\varphi_n\in\Pi(z_n)$. Hence from the definition of Π, there exists a pair $(v_n,C_n=C(v_n))\in L^2(\omega_T)\times L^2(\mathcal{Q})$ such that $\varphi_n=\int_0^t C_n(x,s)\,ds$ where (v_n,C_n) satisfies

$$\tag{72}\left\{\begin{array}{rcll}\dfrac{\partial C_n}{\partial t}-\Delta C_n+\alpha C_n+\beta(t)S_0(x)F\left(\displaystyle\int_0^t z_n(x,s)\,ds\right)&=&v_n\chi_\omega,&\text{in}\quad Q,\\ \dfrac{\partial C_n}{\partial\nu}&=&0,&\text{on}\quad\Sigma,\\ C_n(0)&=&C_0,&\text{in}\quad\Omega\end{array}\right.$$

and

$$C_n(T)=0\text{ in }\Omega.$$

The control v_n verifies

$$\tag{73}\|v_n\|_{L^2(\omega_T)}\quad\leq\quad K\left[|C_0|_{L^2(\Omega)}+|\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}\left|F\left(\int_0^t z_n(x,s)\,ds\right)\right|_{L^2(Q)}\right].$$

Let μ_0 be chosen as above. That is

$$\frac{2}{\alpha}|\beta|^2_{L^\infty(0,T)}|S_0|^2_{L^\infty(\Omega)}<\mu_0<\frac{1}{T^3\sigma^2|S_0|^2_{L^\infty(\Omega)}|\beta|^2_{L^\infty(0,T)}}.$$

Since z_n is bounded, we know that there exists $r>$ such that $\|z_n\|_{L^2(\mathcal{Q})}\leq r$. Consequently, we can also prove as above that C_n and v_n satisfy respectively (60) and (58). This means that

$$|C_n|^2_{W(0,T)} \leq K\left(|C_0|^2_{L^2(\Omega)} + \sigma^2 r^2\right)$$

with $K = K\left(\Omega, \omega, \alpha, T, |\beta|_{L^\infty(0,T)}, |\theta\beta|_{L^\infty(Q)}, |S_0|_{L^\infty(\Omega)}, \mu_0\right) > 0$ and

$$\|v_n\|_{L^2(\omega_T)} \leq K\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}\sigma\, r\right].$$

with $K = K(\Omega, \omega, \alpha, T) > 0$. Thus, there exists a subsequence of (v_n, C_n) still called (v_n, C_n) such that

$$v_n \rightharpoonup v \text{ weakly in } L^2(\omega_T), \tag{74}$$

$$C_n \rightharpoonup C \text{ weakly in } W(0,T). \tag{75}$$

Since the imbedding of $W(0,T)$ in $L^2(\mathcal{Q})$ is compact, we have

$$C_n \rightarrow C \text{ strongly in } L^2(\mathcal{Q})$$

and

$$\varphi_n = \int_0^t C_n(x,s)\,ds \rightarrow \varphi = \int_0^t C(x,s).\,ds$$

The function F being a continuous, we have

$$F(z_n) \rightarrow F(z)$$

and one can prove using (74), (75) and (73) that $(v, C = C(v))$ satisfies

$$\left\{\begin{array}{rcll} \dfrac{\partial C}{\partial t} - \Delta C + \alpha C + \beta(t)S_0(x)F\left(\displaystyle\int_0^t z(x,s)\,ds\right) &=& v\chi_\omega, & \text{in} \quad Q, \\ \dfrac{\partial C}{\partial \nu} &=& 0, & \text{on} \quad \Sigma, \\ C(0) &=& C_0, & \text{in} \quad \Omega \end{array}\right. \tag{76}$$

and

$$C(T) = 0 \text{ in } \Omega,$$

the control v verifies

$$\|v\|_{L^2(\omega_T)} \leq K\left[|C_0|_{L^2(\Omega)} + |\theta\beta|_{L^\infty(Q)}|S_0|_{L^\infty(\Omega)}\sigma r\right] \tag{77}$$

with $K = K(\Omega, \omega, \alpha, T) > 0$. This implies that

$$\varphi \in \Pi(z). \tag{78}$$

On the other hand, z_n being bounded, one can prove as for (63) in Page 16 that φ_n is bounded in $W(0,T)$. Therefore, the Aubin-Lions Lemma allows us to conclude that

$$\varphi_n \rightarrow \varphi \text{ strongly in } L^2(\mathcal{Q}).$$

Since G is closed, we deduce that $\varphi \in G$. Finally, in view of (78), we deduce that $z \in \Pi^{-1}(G)$. The proof of Proposition 4.1 is then complete.

$\square$

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