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Riv. Mat. Univ. Parma, Vol. **x** (2014), 000-000

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Abstract. In this paper we study some structure properties of primitive weird numbers in terms of their factorization. We give sufficient conditions to ensure that a positive integer is weird. Two algorithms for generating weird numbers having a given number of distinct prime factors are presented. These algorithms yield primitive weird numbers of the form $mp_1 \dots p_k$ for a suitable deficient positive integer m and primes $p_1, \dots, p_k$ and generalize a recent technique developed for generating primitive weird numbers of the form $2^n p_1 p_2$. The same techniques can be used to search for odd weird numbers, whose existence is still an open question.

Keywords. Abundant numbers, semiperfect numbers, almost perfect numbers, sum-of-divisor function, Erdős problems.

Mathematics Subject Classification (2010): 11A25, 11B83.

1 - Introduction

Let $n \in \mathbb{N}$ be a natural number, and let $\sigma(n) = \sum_{d|n} d$ be the sum of its divisors. If $\sigma(n) > 2n$, then n is called *abundant*, whereas if $\sigma(n) < 2n$, then n is called *deficient*. *Perfect numbers* are those for which $\sigma(n) = 2n$. According to [5], we will refer to $\Delta(n) = \sigma(n) - 2n$ as the *abundance* of n, and to $d(n) = 2n - \sigma(n) = -\Delta(n)$ as the *deficience* of n. If n can be expressed as a sum of distinct proper divisors, then n is called *semiperfect*, or sometimes

also *pseudoperfect*. Slightly abundant numbers with $\Delta(n) = 1$ are called *quasi-perfect*, and slightly deficient numbers with $d(n) = 1$ are called *almost perfect*.

A *weird number* is a number which is abundant but not semiperfect. In other words, $n \in \mathbb{N}$ is weird if it is abundant and it cannot be written as the sum of some of its proper divisors.

Weird numbers have been defined in 1972 by Benkoski [1], and appear to be rare: for instance, up to 10^4 we have only 7 of them [10]. Despite this apparent rarity, which is the reason of the name, weird numbers are easily proven to be infinite: if n is weird and p is a prime larger than $\sigma(n)$, then np is weird (see for example [4, page 332]). But a much stronger property is true: Benkoski and Erdős, in their joint 1974 paper [2], proved that the set of weird numbers has positive asymptotic density.

Several questions on weird numbers have not been settled yet. For instance, if we look for *primitive weird numbers*, that is, that are not multiple of other weird numbers, we don't know whether they are infinite or not:

Conjecture 1.1. *[2, end of page 621] There exist infinitely many primitive weird numbers.*

In this respect, the third author recently proved in [8] that the infiniteness of primitive weird numbers follows by assuming the classic Cramér conjecture on gaps between consecutive primes [3].

Another open question is the existence of odd weird numbers. Erdős offered \$ 10 for an example of odd weird number, and \$ 25 for a proof that none can exist [1]. Recently Wenjie Fang [10, Sequence A006037] claimed that there are no odd weird numbers up to 10^{21}, and no odd weird numbers up to 10^{28} with abundance not exceeding 10^{14}.

Moreover, very little is known about a pattern in the prime factorization of primitive weird numbers. As of now, most of the known primitive weird numbers are of the form $2^n pq$ with p and q primes, and all the papers on primitive weird numbers deal with numbers of this form [6, 7, 8, 9]. Relatively few examples of primitive weird numbers with more than three distinct prime factors are known up to now: for instance among the 657 primitive weird numbers not exceeding $1.8 \cdot 10^{11}$ there are 531 primitive weird numbers having three distinct prime factors; 69 having four distinct prime factors; 54 having five distinct prime factors and only 3 having six distinct prime factors.

This paper considers primitive weird numbers that have several distinct prime factors. In particular, we give sufficient conditions in order to ensure that a positive integer of the form $mp_1 \dots p_k$ is weird, where m is a deficient number and $p_1, \dots, p_k$ are primes (see Theorem 3.1).

We then apply Theorem 3.1 to search for new primitive weird numbers, looking in particular at four or more prime factors. We find hundreds of primitive weird numbers with four distinct prime factors of the form $2^m p_1 p_2 p_3$, 75 primitive weird numbers with five distinct prime factors of the form $2^m p_1 p_2 p_3 p_4$, and 9 primitive weird numbers with six distinct prime factors (see Section 4).

This paper generalizes to several factors a technique developed in [8]. This approach, as far as we know, is the first that can be used to generate primitive

weird numbers with several distinct prime factors. Moreover, since there are many odd deficient numbers, Theorem 3.1 can be used to hunt for the first example of an odd weird number (see Section 5).

2 - Basic ideas

We recall a fundamental lemma that will be extensively used, and that corresponds to an equivalent definition of weird number.

Lemma 2.1. *An abundant number w is weird if and only if $\Delta(w)$ cannot be expressed as a sum of distinct divisors of w.*

Proof. For a proof one can see [8, Lemma 2]. □

We will need another technical lemma, which will be used in the proof of the main theorems.

Lemma 2.2. *If $w = mq$ is an abundant number, m is deficient, q is prime and $q \geq \sigma(p^\alpha) - 1$ for each $p^\alpha || m$, then w is primitive abundant.*

Proof. Since a multiple of an abundant number is abundant, in order to prove that m is a primitive abundant number, (i.e., an abundant number whose proper divisors are all deficient), it suffices to prove that w/p is deficient for each $p|w$. If $p = q$, then $w/q = m$, and we assumed that m is deficient. Otherwise, if $p^\alpha || m$, then

$$\begin{aligned} \frac{\sigma(w/p)}{w/p} &= \frac{\sigma(w)(p^\alpha - 1)p}{w(p^{\alpha+1} - 1)} = \frac{\sigma(w)}{w} \cdot \left(1 - \frac{p-1}{p^{\alpha+1} - 1}\right) \\ &\leq \frac{\sigma(w)}{w} \cdot \left(1 - \frac{1}{q+1}\right) = \frac{\sigma(w)}{w} \cdot \frac{q}{q+1} = \frac{\sigma(w/q)}{w/q} < 2. \end{aligned}$$

□

3 - Main result

In this section we provide two ways for generating primitive weird numbers.

Theorem 3.1. *Let $m > 1$ be a deficient number and $k > 1$. Let $p_1, \ldots, p_k$ be primes with $\sigma(m) + 1 < p_1 < \cdots < p_k$. Let*

$$h^* = \left\lfloor \frac{p_1 - \sigma(m)}{p_k - p_1} \right\rfloor.$$

Let $\tilde{w} = mp_1 \ldots p_k$, and

$$U_{m,p_1,p_k} := \bigcup_{j=0}^{h^*} \{n \in \mathbb{N} \mid jp_k + \sigma(m) < n < (j+1)p_1\}. \tag{1}$$

3.1.1 If $\tilde{w}$ is abundant and $\Delta(\tilde{w}) \in U_{m,p_1,p_k}$, then $w = \tilde{w}$ is a primitive weird number.

3.1.2 If $\tilde{w}$ is deficient, let

$$p < \frac{2\tilde{w}}{d(\tilde{w})} - 1$$

a prime with $p > p_k$. Then $w = \tilde{w}p$ is abundant. Furthermore, if $\Delta(w) \in U_{m,p_1,p_k}$, then

$$p > \frac{2\tilde{w}}{d(\tilde{w})} - 1 - \frac{(1+h^*)p_1}{d(\tilde{w})}$$

and moreover, if $p > \Delta(w)$ then w is a primitive weird number.

P r o o f o f 3.1.1 Since $\sigma(m) + 2 \leq p_1$, the set union in the right side of (1) is not empty. The sets of consecutive integers involved in the union in the right side of (1) are pairwise disjoint. If

$$h < \frac{p_1 - \sigma(m)}{p_k - p_1},$$

then

$$\sigma(m) + hp_k < (h+1)p_1.$$

Now, let $j \leq h^*$ and let n be an integer with $jp_k + \sigma(m) < n < (j+1)p_1$. We will prove that n cannot be expressible as a sum of distinct divisors of w.

Note that $n < p_1^2$. This is because $n < (j+1)p_1 \leq (h^*+1)p_1 < (p_1/2+1)p_1$ and $p_1/2+1 < p_1$. This means that if n is expressible as a sum of distinct divisors of w, these divisors must be of the form dp with $d \mid m$ and $p \in \{p_1, \ldots, p_k\}$, or simply of the form d, with $d \mid m$. Let's say $n = d_1p_1 + \cdots + d_Np_N + d'_1 + d'_2 + \cdots + d'_M$, where $d'_1, \ldots, d'_M$ are distinct divisors of m. Then, necessarily $d_1 + d_2 + \cdots + d_N \leq j$, since $n < (j+1)p_1$. As a consequence, we have:

$$d'_1 + \cdots + d'_M = n - (d_1p_1 + \ldots d_Np_N) \geq n - jp_k > \sigma(m)$$

and this is in contradiction with the assumption on $d'_1, \ldots, d'_M$. So n cannot be expressible as a sum of distinct divisors of w.

No elements in U_{m,p_1,p_k} can be expressed as a sum of distinct divisors of w. Since $\Delta(w) \in U_{m,p_1,p_k}$, by Lemma 2.1 this implies that w is weird.

In order to prove that w is a primitive weird number, by Lemma 2.2 it suffices to prove that w/p_k is deficient. If $k = 2$, then $\Delta(w/p_k) = \Delta(mp_1) = \Delta(m)p_1 + \sigma(m)$. Since $\Delta(m) \leq -1$ and $p_1 > \sigma(m)$, then $\Delta(w/p_2) < 0$. We may assume that $k \geq 3$. Then we have:

$$\begin{aligned} \Delta\left(\frac{w}{p_k}\right) &= \sigma\left(\frac{w}{p_k}\right) - 2\frac{w}{p_k} = \frac{\sigma(w)}{p_k+1} - \frac{2w}{p_k} \\ &= \frac{p_k(\sigma(w) - 2w) - 2w}{p_k(p_k+1)} = \frac{\Delta(w) - \frac{2w}{p_k}}{p_k+1} \end{aligned}$$

Now, $\Delta(w) \in U_{m,p_1,p_k}$. Since $k \geq 3$, between p_1 and p_k there is at least an odd integer that is not prime, and therefore $h^* < p_1/2k$. This means that

$$\Delta(w) < (h^* + 1)p_1 < \left(1 + \frac{p_1}{2k}\right) p_1.$$

On the other hand $2w/p_k \geq 2mp_1^2$, and since $2mp_1 > p_1 + p_1 > 1 + p_1/2k$, this means that $\Delta(w) < 2w/p_k$ and therefore $\Delta(w/p_k) < 0$. In particular, since w/p_k is deficient, by Lemma 2.2, w is a primitive weird number. □

Proof of 3.1.2 Note that p is the largest prime that divides w, and that $w/p = \tilde{w}$ is deficient. So by Lemma 2.2, in order to prove that w is a primitive weird number, it suffices to prove that w is indeed abundant and weird.

Since $(\tilde{w}, p) = 1$ and $2\tilde{w} - d(\tilde{w})p - d(\tilde{w}) > 0$ by hypothesis, we have

$$\begin{aligned} \Delta(w) &= \sigma(w) - 2w \\ &= \sigma(\tilde{w})(p+1) - 2p\tilde{w} \\ &= (\sigma(\tilde{w}) - 2\tilde{w})p + \sigma(\tilde{w}) \\ &= -d(\tilde{w})p + 2\tilde{w} - d(\tilde{w}) \ > \ 0 \end{aligned}$$

This proves that w is abundant.

Now assume that $\Delta(w) \in U_{m,p_1,p_k}$. Since $\max U_{m,p_1,p_k} < (1 + h^*)p_1$, then $\Delta(w) = -d(\tilde{w})p + 2\tilde{w} - d(\tilde{w}) < (1 + h^*)p_1$ and therefore

$$p > \frac{2\tilde{w}}{d(\tilde{w})} - 1 - \frac{(1 + h^*)p_1}{d(\tilde{w})}$$

Let $p > \Delta(w)$. In order to prove that w is weird, by Lemma 2.1, we have to prove that $\Delta(w)$ is not a sum of proper divisors of w.

Since $\Delta(w) < p$, if $\Delta(w)$ is a sum of proper divisors of w, then all divisors involved must be divisors of $\tilde{w}$. On the other hand $\Delta(w) \in U_{m,p_1,p_k}$ and as seen above, no element in U_{m,p_1,p_k} can be expressed as a sum of distinct divisors of $\tilde{w}$. This completes the proof. □

Remark 3.1. Very often, when conditions of Theorem 3.1.2 hold, it is $(1 + h^*)p_1 < d(\tilde{w})$. This means that in these cases, $p = [2\tilde{w}/d(\tilde{w}) - 1]$.

Remark 3.2. If $m, p_1, \ldots, p_k$ is a sequence such that $w = mp_1 \ldots p_k$ is primitive weird according to Theorem 3.1.1, $k > 2$ and $\Delta(w) < p_k$, then $\tilde{w} = p_1 \ldots p_{k-1}$ and $p = p_k$ verify the conditions of Theorem 3.1.2

Indeed, if w satisfies the condition of Theorem 3.1.1, then w is abundant and $\tilde{w} = w/p_k$ is deficient. This implies that $p_k < 2\tilde{w}/d(\tilde{w}) - 1$. Moreover, since $\Delta(w) \in U_{m,p_1,p_k}$, there is a $j \leq h^* = \left[\frac{p_1 - \sigma(m)}{p_k - p_1}\right]$ such that $jp_k + \sigma(m) < \Delta(w) < (j+1)p_1$. Since $p_k > p_{k-1}$, then $jp_{k-1} + \sigma(m) < \Delta(w)$, with $j \leq \left[\frac{p_1 - \sigma(m)}{p_{k-1} - p_1}\right]$. This means $\Delta(w) \in U_{m,p_1,p_{k-1}}$ as for Theorem 3.1.2. Finally, if $\Delta(w) < p_k$ all requirements of Theorem 3.1.2 are satisfied.

Despite of the above remark, the conditions of Theorem 3.1.1. and 3.1.2 are not equivalent. For example, $m = 2^{11}$, $p_1 = 11321$, $p_2 = 12583$ and $p_3 = 13093$ verify conditions 3.1.1 (so $w = mp_1p_2p_3$ is a primitive weird number) but not 3.1.2, because $p_3 < \Delta(w) = 43936$. In the other sense, for $m = 2^{12}$, $p_1 = 23143$, $p_2 = 24043$, $p_3 = 27061$ and $p = 3077507$, conditions 3.1.2 are satisfied, but the weird number $\tilde{w} = mp_1p_2p_3p_4$ does not verify the conditions 3.1.1, because $\Delta(\tilde{w}) = 39680 \notin U_{m,p_1,p_4}$.

4 - The application of Theorem 3.1

Theorem 3.1 may be used to develop an algorithm which searches for primitive weird numbers with many different prime factors. The sufficient conditions are computationally much easier to check than the standard definition of weird number. The question, however, is in which range the prime numbers $p_1, \ldots, p_k$ should be chosen. The following theorem gives a partial answer.

T h e o r e m 4.1. *Let m be a deficient number and let d be its deficience. Let $p_1, \ldots, p_k$ be primes, with $m < p_1 < p_2 < \cdots < p_k$. Let $\tilde{w} = mp_1 \ldots p_k$.*

(i) If $p_k < 2km/d - (k+2)/2$ then $\tilde{w}$ is abundant.

(ii) If $p_1 > 2km/d - k/2$ then $\tilde{w}$ is deficient.

P r o o f. *(i).* We first prove that if $p_1, p_2, \ldots, p_k$ are distinct primes with $m < p_1 < \cdots < p_k < 2km/d - (k+2)/2$ then $w = mp_1 \ldots p_k$ is abundant.
Since $(m, p_i) = 1$, one has $\sigma(w) = \sigma(m)\sigma(p_1) \ldots \sigma(p_k)$. This implies that:

$$\frac{\sigma(w)}{w} = \left(2 - \frac{d}{m}\right) \prod_{h=1}^{k} \left(1 + \frac{1}{p_h}\right) > \left(2 - \frac{d}{m}\right) \cdot \left(1 + \frac{1}{p_k}\right)^k > 2.$$

The above inequalities hold because for positive x, the function $x \to 1 + 1/x$ is decreasing, and because the equation

$$\left(2 - \frac{d}{m}\right) \cdot \left(1 + \frac{1}{q}\right)^k = 2$$

holds for

$$q = \frac{1}{\sqrt[k]{\dfrac{2}{2 - \frac{d}{m}}} - 1} = \frac{2k}{d}m - \frac{k+1}{2} + O\left(\frac{d}{m}\right) > p_k.$$

Therefore w is abundant.
(ii). The proof is analogous. □

If one wants to generate weird numbers by means of Theorem 3.1 (3.1.1 or 3.1.2), i.e., abundant numbers $\tilde{w}$ with $\Delta(\tilde{w}) \in U_{m,p_1,p_k}$ on a hand U_{m,p_1,p_k} have to be as large as possible. Good choices are with $p_1 - \sigma(m)$ as large as

possible. On the other hand, in order to get higher values of h^*, $p_k - p_1$ have to be as small as possible. This leads to consider k-tuples of primes $p_1, \dots p_k$ in an interval that, by Theorem 4.1, includes $2km/d - (k+1)/2$. However, if $k \leq d$ then $2km/d$ might be smaller than $\sigma(m)$, and if $p_1 < \sigma(m)$ then $h^* < 0$ and U_{m,p_1,p_k} is empty. Therefore, $k > d$ is generally preferable, and all new weird numbers we have found enjoy this property.

The primitive weird numbers generated with Theorem 1 in [8] are particular cases of Theorem 3.1.1, with $m = 2^h$, and $k = 2$. When p_1 and p_2 are chosen according to that theorem, then conditions of Theorem 3.1.1 are fulfilled and $w = 2^h p_1 p_2$ is a primitive weird number.

However, Theorems 3.1.1 and 3.1.2 become more interesting when applied to search weird numbers with several prime factors.

Theorem 3.1.1 yields primitive weird numbers of the form $mp_1 \dots p_k$ where m is a deficient number, k is an integer larger than the deficience of m and the p_i's are suitably chosen. It is relatively easy to generate primitive weird numbers up to four distinct prime factors. The table below shows some of the primitive weird numbers having at least five distinct prime factors we were able to generate with Theorem 3.1.1

w	prime factorization	m	$\Delta(w)$
9210347984	$2^4 \cdot 83 \cdot 89 \cdot 149 \cdot 523$	2^4	32
9772585048	$2^3 \cdot 17 \cdot 317 \cdot 419 \cdot 541$	$2^3 \cdot 17$	304
23941578736	$2^4 \cdot 73 \cdot 103 \cdot 127 \cdot 1567$	2^4	32
109170719992	$2^3 \cdot 19 \cdot 79 \cdot 2731 \cdot 3329$	2^3	16
359214428128	$2^5 \cdot 127 \cdot 211 \cdot 509 \cdot 823$	2^5	64
446615164768	$2^5 \cdot 163 \cdot 167 \cdot 331 \cdot 1549$	2^5	64
83701780710848	$2^6 \cdot 181 \cdot 563 \cdot 3407 \cdot 3767$	2^6	128
823548808494656	$2^6 \cdot 139 \cdot 3631 \cdot 4441 \cdot 5741$	2^6	128
31871420410521385088	$2^7 \cdot 257 \cdot 90803 \cdot 98221 \cdot 108631$	$2^7 \cdot 257$	78464
32852586770937891968	$2^7 \cdot 257 \cdot 87443 \cdot 90803 \cdot 125777$	$2^7 \cdot 257$	85184
32892333375893455232	$2^7 \cdot 257 \cdot 79841 \cdot 109943 \cdot 113909$	$2^7 \cdot 257$	76736
33622208489084493184	$2^7 \cdot 257 \cdot 76757 \cdot 107873 \cdot 123439$	$2^7 \cdot 257$	72832

Table 1: The above primitive weird numbers have five distinct prime factors and have been found by means of Theorem 3.1.1 with $k = 4$ for $m = 2^h$ and with $k = 3$ for $m = 136 = 2^3 \cdot 17$ or $m = 32896 = 2^7 \cdot 257$.

An implementation of Theorem 3.1.2 yields in minutes hundreds of primitive weird numbers of the form $2^h p_1 p_2 p$. By going deeper in the implementation of Theorem 3.1.2, we have been able to find 65 primitive weird numbers w having five distinct prime factors of the form $w = 2^h p_1 p_2 p_3 p$; and nine primitive weird numbers having six distinct prime factors, that are shown in the table below. As a comparison, before our computations only three primitive weird numbers having six distinct prime factors were known at the OEIS database [10].

w	prime factorization	$\Delta(w)$
44257207676	$2^2 \cdot 11 \cdot 37 \cdot 59 \cdot 523 \cdot 881$	8
125258675788784	$2^4 \cdot 47 \cdot 149 \cdot 353 \cdot 1307 \cdot 2423$	32
147578947676144	$2^4 \cdot 43 \cdot 211 \cdot 367 \cdot 1091 \cdot 2539$	32
4289395775422432	$2^5 \cdot 127 \cdot 211 \cdot 401 \cdot 1861 \cdot 6703$	64
5976833582079328	$2^5 \cdot 181 \cdot 197 \cdot 353 \cdot 431 \cdot 34429$	64
1663944565537013728	$2^5 \cdot 131 \cdot 223 \cdot 311 \cdot 2179 \cdot 2626607$	64
2061779596379476178947690 24	$2^7 \cdot 257 \cdot 84691 \cdot 101891 \cdot 116041 \cdot 6259139$	74752
2996153601600440129026407808	$2^7 \cdot 257 \cdot 89021 \cdot 93239 \cdot 118621 \cdot 92505877$	83584
4808301947392627231482506 5088	$2^7 \cdot 257 \cdot 97213 \cdot 97973 \cdot 100957 \cdot 1520132521$	287264

Table 2: The above primitive weird numbers have six distinct prime factors and have been found by means of Theorem 3.1.2 with $k = 4$ for $m = 2^h$ and $k = 3$ for $m = 32896 = 2^7 \cdot 257$

5 - Tracking weird numbers with several distinct prime factors and eventual odd weird numbers

As we have seen, Theorems 3.1.1 and 3.1.2 provide two distinct strategies to track primitive weird numbers with several distinct prime factors. The same approaches could be applied to track odd weird numbers. If $m > 1$ is odd, the integers w that one creates by means of Theorem 3.1.1 or 3.1.2 are odd, and if the conditions are fulfilled, w would be a primitive odd weird number.

For tracking a weird number with several distinct prime factors (even or odd) both strategies (3.1.1 and 3.1.2) start with choosing a deficient number m with deficience d. As a general rule, since we want $k > d$, small values of d have to be preferred in order to keep the computational complexity low. Indeed, the case $d = 1$ corresponds to $m = 2^h$ assuming that no further almost perfect numbers exist. This case has been largely discussed in [6, 8]. The only known composite numbers with $d = 2$ are even. For example, $m = 136$, $m = 32896$ have deficience 2 [10, Sequence A191363] and, as shown in Table 1 and Table 2, Theorem 3.1 allows to find several primitive weird numbers starting with such values of m. We expect that primitive weird numbers could be generated also from $m = 2147516416$, whose deficience is 2, both with Theorem 3.1.1 and 3.1.2. Unfortunately the computation becomes dramatically longer.

As far as we know there are no known integers with deficience $d = 3$. The only known integers with deficience $d = 4$ are even, and the only known integer with $d = 5$ is 9 for which the above approaches are hard to apply.

An interesting case is $d = 6$. There are several integers whose deficience is 6, some of which are odd. The list starts with 7, 15, 52, 315, 592, 1155, 2102272, 815634435, and no other terms are known [10, Sequence A141548].

So, if one wants to track odd weird numbers, the starting m in both approaches 3.1.1 and 3.1.2, could be $m = 315$, $m = 1155$ or $m = 815634435$. Unfortunately our attempts to track an odd weird numbers from such a choice of m have been unfruitful.

6 - Conclusion

As one can argue from a table of primitive weird numbers, most of the primitive weird numbers are of the form $2^k pq$ for $k \in \mathbb{N}$, and primes p and q. This was already pointed out in [8] where the third author, among other things, conjectured that there are infinitely many primitive weird numbers of the form $2^k pq$.

It seems that primitive weird numbers that are not of this form become rarer. For example, between the 301st and the 400th, there are only 7 primitive weird numbers that are not of the form $2^k pq$. Five of them have four distinct prime factors and two of them have five distinct prime factors. The existence of several weird numbers having six distinct prime factors leads to the following conjecture.

Conjecture 6.1. *Given an integer $k \geq 3$, there exists a primitive weird number having at least k distinct prime factors.*

Of course, a positive answer would settle the question of the infiniteness of primitive weird numbers.

However, a proof that primitive weird numbers have a bounded number of distinct prime factors would not settle neither the question of the infiniteness of primitive weird numbers nor the question of the existence of odd weird numbers.

Acknowledgments. Part of this research was done while the first author was visiting the Department of Mathematics and Computer Science at Wesleyan University.

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