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Abstract

Most multicriteria aggregation functions are designed in a mono-decision-maker context. Using them for multi-decision-maker problems requires a prior transformation of the individual data of each decision-maker into a collective datum. Recently a method for the aggregation of data in the context of social choices has been introduced by Ngoie et al. (2015a): The Mean-Median Compromise Method (MMCM). In this paper, we suggest an adaptation of the MMCM to multi-criteria multi-decision-maker problems: the Mean and Median for Multi-Criteria Decision (3MCD). We also examine some properties of this rule.

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1 Introduction

Decision-making aid occupies a place of choice in our contemporary societies. More often than not, making a decision can be a difficult task that is accompanied in most times by a lot of hesitation and reflection. Sometimes, the help of a third person or even several individuals consulted individually is sought in order to make a "good decision". As noticed by Adla (2010), for a long time, the decision-making aid was handled in a mono-decision-making framework. Nowadays, it is interested in decisions taken by several decision-makers and which depend on several criteria that are often contradictory. In the literature, there are many methods of multi-criteria help to group decision. These methods are determined by mathematical formulas called aggregation functions. The most complex task in this case is to determine a consensus-based method (aggregation function) that is easy to use. These types of methods are popular in companies that want to improve their services. The increasing demand for these methods has stimulated the search for those which, if not all, fulfill the most desirable properties.

In his seminal works on the social choice theory, Arrow (1950, 1951) proposes to evaluate the mechanisms of aggregation on the basis of normative criteria deemed desirable. The purpose of aggregating preferences is to determine a collective ranking that is transitive, Arrow (1950, 1951) bases his analysis on four criteria or conditions. The first condition called Universality (**U**) presupposes a certain freedom of the individuals taking part in the decision-making process; they are free to have any ranking on the options subject to their appreciation. The condition of Independence with irrelevant options (**I**) requires that the collective decision between two options must depend only on individual preferences on these two options only; even if individuals modify their preferences among other options while leaving their preferences between the two options unchanged. The third condition is that of Pareto optimum (**P**) according to which if all the individuals involved in the collective decision-making process prefer a a option to a b option, this same decision must be transcribed at the preference level collective. The last criterion is that of the absence of Dictatorship (**D**) which excludes any situation in which an individual would be able to impose his choice to all the individuals involved in the decision process. Arrow (1950, 1951) came to conclude that there is no aggregation method that simultaneously satisfies the four conditions **I**, **U**, **P** and **D** when at least two individuals have to decide on at least three options.

Unfortunately, this result is also verified in multi-criteria analysis where no multi-criteria aggregation function is free from reproach. The best that one can get from an aggregation function is a "good" solution, the one that best meets the requirements of the decision maker. In the multi-criteria context, such a solution is not necessarily the best one on all the criteria, but the one that achieves the best compromise on these criteria. Thus, multi-criteria aggregation functions are naturally compromise functions. They arbitrate between several alternatives evaluated on several criteria. In this sense, they are intended to be fair and impartial. This is why the criteria of justice, equity and democracy are often the most verified for these functions. In this paper, we propose to adapt a voting function to obtain a multi-criteria aggregation method. Indeed, given that the voting function considered has passed several tests on the demands of justice, equity and democracy, we believe that most of its qualities would be preserved even after adaptation to the multi-criteria paradigm.

The Mean-Median Compromise Method (MMCM) is a social choice function recently introduced by Ngoie et al. (2015a,b). This function is obtained by combining the mean and the median. As a social choice function in itself, the MMCM jointly retains the qualities of the mean and the median. Given the qualities of the MMCM as highlighted by its authors and in particular its resistance to some strategic behavior, we propose an extension of the MMCM for the resolution of multi-decision multi-criteria problems. The aggregation function we suggest is named the *Mean and Median for Multi-Criteria Decision* method (3MCD). As we are going to see, it simply consists in apply the MMCM twice.

The rest of the paper is organized as follows: Section 2 is devoted to some basic definitions prior to the presentation of the MMCM which is done in Section 3. In Section 4, we introduce the 3MCD our extension of the MMCM. Section 5 concludes.

2 Basic terminology

Let $A = a_1, a_2, \ldots, a_i, \ldots$ a finite set of m competitors or candidates (with $m \geq 2$) and $J = 1, 2, \ldots, j, \ldots, n$ a finite set of n judges (voters).

Prior to introduce the *Mean and Median for Multi-Criteria Decision* method (3MCD) our extension of the Mean-Median Compromise Method (MMCM), we need first to present the two aggregation rules : The Majority Judgment of Balinski and Laraki (2007, 2010) and the Borda Majority Count of Zahid and De Swart (2015).

2.1 The Majority Judgment

The Majority Judgment (MJ) was introduced by Balinski and Laraki (2007, 2010) as a voting rule under which voters have to grade the candidates using a *common language* or a well-defined grading system[1]. The grading system can be made by a range of positive integers, a set of letters, words or phrases denoting the opinion or how the voter (judge) finds the candidates. Following Balinski and Laraki (2007, 2010), a *common language* $\Lambda = \{g_1, g_2, \ldots, g_p\}$ is a set of strictly ordered grades. Given and, a profile $\Phi(A, J)$ is an $m \times n$ matrix of the grades $\Phi(a_i, j) \in \Lambda$ assigned by each $j \in J$ to each of the competitors $a_i \in A$.

Definition 1 (*The method of grading*)**.** A method of grading is a function F defined as follows:

$$F: \quad \Lambda^{m\times n} \rightarrow \Lambda^m$$

$$\Phi(A,J) = \begin{pmatrix} g_{11} & \cdots & g_{1j} & \cdots & g_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{i1} & \cdots & g_{ij} & \cdots & g_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{m1} & \cdots & g_{mi} & \cdots & g_{mn} \end{pmatrix} \mapsto \big(f(g_{11}, g_{12}, \ldots, g_{1n}), \ldots, f(g_{m1}, g_{m2}, \ldots, g_{mj}, \ldots, g_{mn})\big)$$

with g_{ij} the grade given by judge j to candidate a_i, f the aggregation function and $f(g_{i1}, \ldots, g_{ij}, \ldots, g_{in})$ the final grade of a_i.

[1] A similar or related rule has been suggested by Smith (2009), named the “Range Voting”. For more on this rule, please refer to Smith (2009)

Definition 2 (*The majority-grade*)**.** The majority-grade of $a_i \in A$ is given by

$$f^{\text{maj}}(a_i) = \begin{cases} f^{\frac{n+1}{2}}(g_{i1}, \ldots, g_{ij}, \ldots, g_{in}) & \text{if } n \text{ is odd} \\ \\ f^{\frac{n+2}{2}}(g_{i1}, \ldots, g_{ij}, \ldots, g_{in}) & \text{if } n \text{ is even} \end{cases}$$

where $f^k(.)$ is a k^{th} *order function* that returns the k^{th} highest grade.

As we can notice, the MJ's winner is the competitor with the highest median grade. Given two competitors a_i and a_s , if $f^{\text{maj}}(a_i) > f^{\text{maj}}(a_s)$ then $a_i \succ_{maj} a_s$. If $f^{\text{maj}}(a_i) = f^{\text{maj}}(a_s)$ $a_i \succ_{maj} a_s$, the majority-grade is dropped from the grades of each of the competitors and the procedure is repeated. Balinski and Laraki (2010) admitted that this tie-breaking mechanism can become very arduous and they suggested another tie-breaking mechanism based on the concept of "*majority gauge*". For the forma definition of this concept, we refer to Balinski and Laraki (2010).

Balinski and Laraki (2007, 2010, 2016) have shown that their rule meets some desirable properties; among others, we can list the following ones : *Voter-expressivity*[2], *Anonymity*[3], *Neutrality*[4], *Unanimity*[5], *Transitive ordering*[6], *Independence of irrelevant alternatives*[7], *Monotonicity*[8], *Immunity to candidate cloning*[9], *Resolvability*[10] and *Sincere voting*. Nonetheless, the MJ has been the subject of many criticisms. Most of these criticisms are contained in Felsenthal and Machover (2008), Felsenthel (2012), and Zahid (2009).

2.2 The Borda Majority

The Borda Majority Count (BMC) introduced by Zahid and De Swart (2015) is based on the well-known Borda rule. The Borda rule is a voting rule under which, given the preferences (rankings) of the voters, a candidate receives $m - p$ points each time he is ranked p^{th}; the Borda score of a candidate is the total number of points received and the winner is the one with th highest Borda score. Under the BMC, the system of grades are converted into scores as under the Borda rule.

Recall that as $\Lambda = \{g_1, g_2, \ldots, g_p\}$ is an strictly ordered set, we get $g_1 > g_2 > \ldots > g_p$. We denote by $\widetilde{\Phi}(a_i, j)$ the ordered vector of $\Phi(a_i, j) \in \Lambda$. The BMC defines $\Lambda^\star = \{g_1^\star, g_2^\star, \ldots, g_p^\star\}$ with $(g_1^\star > g_2^\star > \ldots > g_p^\star)$ the natural set associated to Λ such that $g_1^\star = p - 1$, $g_2^\star = p - 2$, $\ldots$, $g_p^\star = 0$. Similarly $g_{ij}^\star$ is defined as the natural number associated to g_{ij} the grade given by judge j to candidate a_i. The Borda Majority Count of a_i is equal to the mean of the sum of the natural numbers associated to g_{ij} given all the n judges.

[2] It allows voters to award (ordinal) grades to all candidates.

[3] It treats voters equally.

[4] It treats competitors equally.

[5] If all voters award candidate a_i a higher grade than to every other candidate, then a_i is elected.

[6] Candidates are ranked in a transitive ordering; one candidate is necessarily ranked ahead or behind another, unless they have identical sets of grades.

[7] If candidate a_i wins, then he would still win if another candidate is removed, ceteris paribus.

[8] If a candidate wins, he would still win if at least one of his grades is increased, ceteris paribus.

[9] If candidate a_i wins, he would still win if another candidate is added with grade distribution identical to that of a_i or of another candidate, ceteris paribus.

[10] The probability of ties quickly tends to zero as the number of voters increases.

Definition 3 (*The Borda Majority Count*)**.** For $a_i \in A$ and $(g_{i1}, \ldots, g_{ij}, \ldots, g_{in})$, the Borda Majority Count of $a_i \in A$ denoted by $f^{\text{mean}}(a_i)$ is given by :

$$f^{\text{mean}}(a_i) = \frac{1}{n}\sum_{j=1}^{n} g_{ij}^{\star}$$

For two competitors a_i and a_s, if $f^{\text{mean}}(a_i) > f^{\text{mean}}(a_s)$ then $a_i \succ_{mean} a_s$. In the case of $f^{\text{mean}}(a_i) = f^{\text{mean}}(a_s)$, drop all the reject grades and recalculate the Borda Majority Count. The procedure is repeated step by step by dropping grades from lower to higher until a winner among a_i and a_s is found.

Zahid and De Swart (2015) established a list of properties satisfied or failed by the BMC. They also provided a characterization of this rule. For more details, the reader may refer to their paper.

The MJ and the BMC rules has served as multi-criteria aggregation methods. A multi-criteria aggregation method is an application of a multi-criteria decision support problem to a set of actions. The objectives that the decision-maker seeks to achieve define the problematic of decision-support. The objectives are multiple. Roy (1985) categorizes them into four benchmark problems: the choice (α-Problem), the sorting (β-Problem), the ranking (γ-Problem) and the description (δ-Problem). Thus, all methods of multi-criteria analysis are constructed to solve a specific type of problem and do not all meet the same objectives. In this paper, we pay a particular attention to the ranking problem. The objective of the ranking problem is to classify potential actions by equivalence classes in order to define the most appropriate strategies or treatments. This is the case under the function introduced by Ngoie et al. (2015a): the Mean-Median Compromise Method (MMCM).

3 The Mean-Median Compromise Method

Recently introduced by Ngoie et al. (2015a), the Mean-Median Compromise Method (MMCM) combines the Majority Judgement of Balinski and Laraki (2007, 2010) and the Borda Majority Count of Zahid and De Swart (2015) in the sense that it is based simultaneously on the median and the average of the grades. To set how the MMCM works, we need some additional notations.

Given $\widetilde{\Phi}(a_i, j)$ the ordered vector of the grades assigned to a_i, this distribution is divided into 2^k intervals of the same amplitude; k, called the *degree of division*, is an integer set in advance ($k \geq 2$). With n the number of judges, we define the *amplitude of a division* as the real number $\varrho = \frac{n+1}{2^k}$.

Definition 4 (Inter-median grade)**.** Given a candidate a_i and $\widetilde{\Phi}(a_i, j) \in \Lambda$, g_{ij} is the inter-median grade if $\exists t \in \mathbb{N}$ such that $1 \leq t \leq 2^k - 1$, $Rd(t \times \varrho) = j$; with $Rd(.)$ the value rounded to the nearest integer.

We denote by $\mathcal{M}_k(a_i)$ the vector of non-redundant inter-median grades of a_i given k:

$$\mathcal{M}_k(a_i) = \{g'_{i1}, \ldots, g'_{il}, \ldots, g'_{it}\} = \{f^{Rd(1\times\varrho)}, f^{Rd(2\times\varrho)}, \ldots, f^{Rd((2^k-1)\times\varrho)}\}$$

According to Ngoie et al. (2015a,b), the smallest value of the integer k such that $\mathcal{M}_k = \widetilde{\Phi}(a_i, j)$, is given by $\nu = Rd(\frac{1}{2} + \log_2(n+1))$; ν is called the *maximal division index* or the *total division index.*

Definition 5 (Average Majority Compromise)**.** Given a candidate a_i and $\mathcal{M}_k(a_i)$, the Average Majority Compromise of a_i is given by

$$f^{\text{mm}}(a_i) = \frac{1}{t}\sum_{l=1}^{t} g'_{il}$$

For two competitors a_i and a_s, if $f^{\text{mm}}(a_i) > f^{\text{mm}}(a_s)$ then $a_i \succ_{mm} a_s$. In the case of $f^{\text{mean}}(a_i) = f^{\text{mean}}(a_s)$, repeat the process for $k+1$.

Let us now use an example to illustrate how the MMCM operates.

Example 1. *Assume that* 8 *judges respectively grade a competitor* a_i *with* $\Phi(a_i, j) = (9, 7, 3, 6, 5, 4, 5, 8)$ *and it follows that* $\widetilde{\Phi}(a_i, j) = (9, 8, 7, 6, 5, 5, 4, 3)$ *Let us fix* $k = 3$*; we get* $\varrho = \frac{8+1}{2^3} = 1.125$ *and*

$$\begin{aligned}
\mathcal{M}_3 &= \left(f^{Rd(1\times1.125)}, f^{Rd(2\times1.125)}, f^{Rd(3\times1.125)}, f^{Rd(4\times1.125)}, f^{Rd(5\times1.125)}, f^{Rd(6\times1.125)}, f^{Rd(7\times1.125)}\right)\\
&= \left(f^{Rd(1.125)}, f^{Rd(2.25)}, f^{Rd(3.375)}, f^{Rd(4.5)}, f^{Rd(5.625)}, f^{Rd(6.75)}, f^{Rd(7.875)}\right)\\
&= \left(f^1, f^2, f^3, f^5, f^6, f^7, f^8\right)\\
&= (9, 8, 7, 5, 5, 4, 3)
\end{aligned}$$

Then, $f^{mm}(a_i) = \frac{9+8+7+5+5+4+3}{7} = \frac{41}{7} = 5.8$

According to Ngoie et al. (2015a,b), when the degree of division is set at $k = 1$, the MMCM is always equivalent to the Majority Judgement *i.e* $f^{\text{mm}}(a_i) = f^{\text{maj}}(a_i)$ for all $a_i \in A$; when the degree of division is set at its maximum ($k = \varrho$), the MMCM is always equivalent to the Borda Majority Count *i.e* $f^{\text{mm}}(a_i) = f^{\text{mean}}(a_i)$. So, the MMCM appears as an intermediate method between the Majority Judgement (based on the highest median) and the Borda Majority Count (based on the highest mean). For values of k varying between 1 and ϱ, it allows to qualify and balance the advantages and disadvantages of one or the other method.

We can now introduce our adaptation of the MMCM to multi-decision multi-criterion problems.

4 Mean and Median for Multi-Criteria Decision

In the real world, most decision-making problems involve several criteria and several decision-makers. Such problems are characterized by:

- a finite set of s decider-makers $D = \{d_1, d_2, \ldots, d_s\}$,
- a finite set of actions $A = \{a_1, a_2, \ldots, a_n\}$,
- a finite set of criteria $C = \{g_1, g_2, \ldots, g_m\}$,

- a vector $P_j = \{p_j(g_1), p_j(g_2), \ldots, p_j(g_m)\}$ of the weights assigned to each of the criteria by the decision-maker j.
- $X = \{x_{ij} = g_j(a_i), i = 1, \ldots, n; j = 1, \ldots, m\}$ a set comprising the scores of the action i on the criterion j;
- $X^l = \{x^l_{ij} = g_j(a_i), i = 1, \ldots, n; j = 1, \ldots, m; l = 1, \ldots, s\}$ designating the performance of the alternative i on the criterion j for the l-th decision-maker.
- No decision maker is dictator and no action dominates all on each criterion.

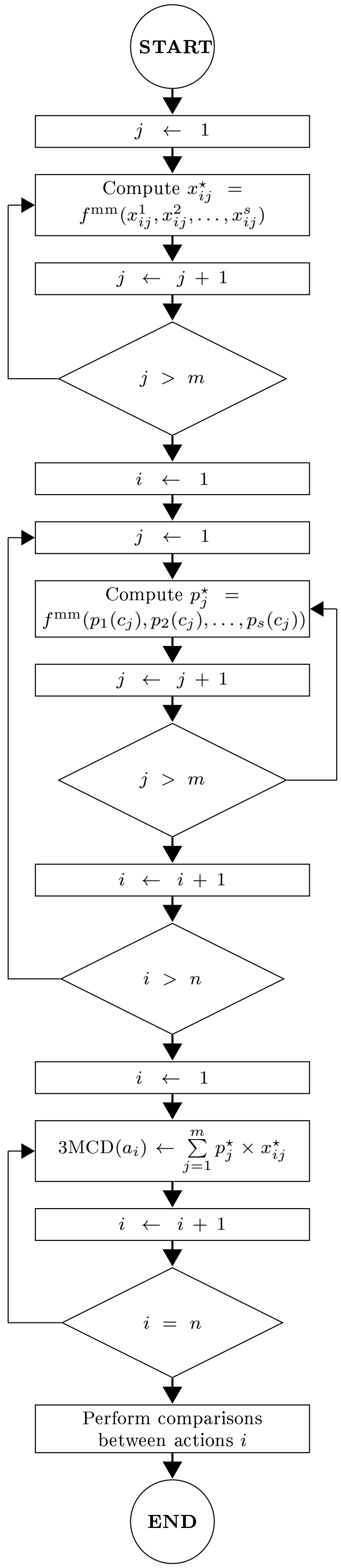
Faced with such a problem, we need multi-criteria decision support to establish a good compromise solution. The Mean and Median for Multi-Criteria Decision (3MCD) that we suggest seems well suited. The 3MCD method is related to a ranking problem. It uses MMCM twice: first to aggregate the weights of the criteria, then to aggregate the performance of the actions on the criteria. The result of these two aggregations is a single decision table which makes it possible to calculate the final “scores” of the actions. The method finally produces a complete ranking of the actions.

The 3MCD operates as follows:

1. apply the MMCM function on the weights of each criterion to find the overall weight;
2. apply the MMCM function on the scores of each action on each criterion to find the global scores of the actions on each of the selected criteria;
3. apply the weighted sum to determine the overall performance of the actions;
4. stop the process when comparisons are made.

At the end of the process, a ranking is obtained on A the set of actions. As one can notice, the 3MCD simply consists in applying the MMCM twice: once on the weights of the criteria and then on the judgments. A Flow Chart of the 3MCD is given in Figure 1.

Figure 1: Flow Chart of the 3MCD



Let us provide an example illustrating the 3MCD method.

Example 2. *Assume that five members (judges) of the executive committee of a company audition four candidates (named H1, H2, H3 and H4) in order to hire a new marketing director. This choice depends on three criteria: academic background (C1), professional experience (C2) and skills (C3). Each member of the committee builds his judgment matrix and the score scale is* $0-10$. *The matrices are listed as follows:*

Judge 1

Weights	8	4	6
Criteria	*C1*	*C2*	*C3*
H1	6	5	7
H2	5	7	5
H3	6	8	4
H4	7	6	7

Judge 2

Weights	6	6	7
Criteria	*C1*	*C2*	*C3*
H1	7	6	5
H2	6	5	5
H3	7	7	6
H4	6	6	5

Judge 3

Weights	7	5	6
Criteria	*C1*	*C2*	*C3*
H1	8	10	6
H2	7	7	6
H3	6	6	7
H4	7	5	6

Judge 4

Weights	6	4	5
Criteria	*C1*	*C2*	*C3*
H1	7	10	8
H2	7	6	7
H3	8	7	7
H4	7	6	7

Judge 5

Weights	6	7	5
Criteria	*C1*	*C2*	*C3*
H1	7	7	6
H2	6	9	7
H3	8	8	7
H4	7	6	5

Assume $k=2$. *First of all, we sort the weights of each criterion in descending order; so, for C1 we get* $(8,7,6,6,6)$, *for C2 we get* $(7,6,5,4,4)$ *and for C3 we get* $(7,6,6,5,5)$. *Then, we apply the MMCM function on the weights of each criterion to find the overall weight:*

$$\begin{aligned}
\textit{overall weight of C1} &= \frac{7+6+6}{3} = 6.33\\
\textit{overall weight of C2} &= \frac{6+5+4}{3} = 5.00\\
\textit{overall weight of C3} &= \frac{6+6+5}{3} = 5.66
\end{aligned}$$

Following the other steps of the 3MCD, we end with the following outcome:

Criteria	*C1*	*C2*	*C3*	***Score 3MCD***
Weights	6.33	5.00	5.66	
H1	7.00	7.66	6.66	120.3056
H2	6.00	6.33	6.00	103.5900
H3	7.00	7.33	6.66	118.6556
H4	6.66	5.66	6.00	104.4178

Let us explain how we get the figures in the outcome matrix. For example, to obtain the score of candidate H3 on the criterion C2, we proceed as follows: On criterion C2, the 5 decision-makers assign H3, respectively with grades 8, 7, 6, 7, 7*. In descending order, we have:* 8, 7, 7, 7, 6*. If we apply MMCM to these sorted data, we see that the inter-median are the second, third and fifth data; namely* 7, 7 *and* 6*. Thus, the overall score of H3 on C2 is* $\frac{7+7+6}{3} = 6.66$*. The last column gives the 3MCD's scores of the candidates; it is obtained by calculating the weighted sum of the overall score of each candidate. For instance, the 3MCD's scores of candidate H4 is obtained by the calculation:* $6.66 \times 6.33 + 5.66 \times 5.00 + 6.00 \times 5.66 = 104.4178$*. Given the scores of the Candidates, we can conclude that, on the position, the committee will rank candidate H1 first, candidate H3 second, candidate H4 third and candidate H2 last.*

Ngoie (2016), Ngoie et al. (2015b) showed that the MMCM meets some appealing requirements of choice functions namely the neutrality condition, the anonymity condition, the monotonicity condition, the Pareto condition, the Independence of Irrelevant Alternatives and the clone-resistance condition. Nonetheless, the MMCM fails among others the reinforcement criterion and the participation criterion. As the 3MCD consists in applying the MMCM twice, it is obvious a priori that the 3MCD has the same properties as the MMCM. Nonetheless, we want to pay a particular attention to the following properties as they need to be redefined to fit clearly into the 3MCD context: the majority condition, the Condorcet winner criterion, the Condorcet loser criterion, the homogeneity criterion, the reinforcement criterion and the participation criterion.

Definition 6. Given the set of the judges N and a criteria g, an action a_i majority dominates an action a_k on criterion g_j if $\sharp\{j \in N \backslash g_j(a_i) \geq g_j(a_k)\} > \frac{n}{2}$. We denote it by $a_i M_{g_j} a_k$. If for all the criteria, $a_i M_g a_k \ \ \forall\ a_k \in A, \forall g_j \in C$, a_i is the *Condorcet winner*; if for all the criteria, $a_k M_g a_i \ \ \forall\ a_k \in A, \forall g_j \in C$, a_i is the *Condorcet loser*.

We will say that the 3MCD satisfy the *majority condition* if whenever $\forall g_j \in C, \ \ a_i M_{g_j} a_k$ implies that a_i scores better than a_k. The 3MCD will meet the *Condorcet winner criterion* if it always select the Condorcet winner when it exists; it meets the *Condorcet loser criterion* if it never selects the Condorcet loser when it exists.

Proposition 1. *The 3MCD fails the majority requirement, the Condorcet winner criterion and the Condorcet loser criterion.*

Proof. Let us consider the following judgement matrices of 3 judges on three candidates (named H1, H2 and H3) based on two criteria (C1) and (C2).

Judge 1

Weights		
Criteria	C1	C2
H1	2	2
H2	3	3
H3	4	4

Judge 2

Weights		
Criteria	C1	C2
H1	6	6
H2	0	1
H3	1	2

Judge 3

Weights		
Criteria	C1	C2
H1	2	1
H2	3	2
H3	4	3

The reader can easily check that H1 is the Condorcet loser while H3 is the Condorcet winner. By computing the 3MCD scores with $k = 2$, we end with following table:

Criteria	C1	C2
Weights		
H1	3.33	3
H2	2	2
H3	3	3

So, no matter what the weights, H1 is ranked first followed by H3 and then H2. It follows that the 3MCD fails the majority requirement, the Condorcet winner criterion and the Condorcet loser criterion[11]. □

An aggregation function is homogeneous if given a preference profile and the corresponding outcome, replicating this profile λ times ($\lambda > 1$, $\lambda \in \mathbb{N}$) does not change the outcome. In our framework, we will say that the 3MCD is homogeneous if whenever we replicate λ times the original profile, the outcome remains unchanged. Notice that nothing is known about the MMCM concerning the homogeneity condition.

Proposition 2. *The 3MCD fails the homogeneity condition, the reinforcement criterion and the participation condition.*

Proof. Let us consider the following profile with five judges, two actions. We assume that $k = 2$; no matter what are the weights on the criteria.

	Judges									
	J1		J2		J3		J4		J5	
Criteria	C1	C2	C1	C2	C1	C2	C1	C2	C1	C2
H1	2	8	2	7	7	9	4	6	3	5
H2	3	7	2	8	2	6	6	7	6	7

Assume that we replicate each of the judge two (or three) times. The reader can check that the output matrices for the original and the replicated profiles are as follows:

original profile

Criteria	C1	C2
H1	4	7
H2	3.66	7

replicated profile

Criteria	C1	C2
H1	3	7
H2	4	7

Under the original profile, H1 wins but under the replicated profile H2 wins. So, by replicating the set of the judges, the outcome changes. Thus, the 3MCD fails the homogeneity condition. To show that this is also the case for the MMCM, just erase C2 in the above profile and we get the same conclusion.

Assume that we add two new judges who are in favor of H1; they both assign 7 and 8 to H1 on C1 and C2 respectively while they assign 6 and 7 to H2 on C1 and C2 respectively. So, if we consider the profile only without these two judges, H1 wins. The reader can check that the even though these two judges are in favor of H1, their arrival favors H2 who is the new winner. Thus, the 3MCD rule also fails the Participation condition and the reinforcement criterion. □

[11] Example 2 can also be used to show the failure of the majority requirement and of the Condorcet winner criterion. In this example, the reader can check that on each of the criteria, H3 is the Condorcet winner and he is not chosen.

Although the 3MCD does not meet the homogeneity criterion, we can describe the conditions under which this condition is satisfied.

Proposition 3. *Assume a population made of n judges such that* $n = 2^x$ $(x \geq 2, x \in \mathbb{N})$. *If we duplicate this population* λ *times with* $\lambda = 2^y$ $(y \in \mathbb{N})$*, the 3MCD is always homogeneous in this case.*

Proof. Assume n judges ($n = 2^x$ and $x \geq 2, x \in \mathbb{N}$) and their grades (listed decreasingly). For $k = 2$, $\varrho = \frac{2^x+1}{4}$ and

$$\begin{aligned}\mathcal{M}_2 &= \left(f^{Rd(1\times\frac{2^x+1}{4})}, f^{Rd(2\times\frac{2^x+1}{4})}, f^{Rd(3\times\frac{2^x+1}{4})}\right)\\ &= \left(f^{(2^{x-2})}, f^{(2^{x-1}+1)}, f^{(3\times 2^{x-2}+1)}\right)\end{aligned}$$

Assume that each grade of the initial population is replicated λ times ($\lambda = 2^y$ and $y \in \mathbb{N}$). Given the new population and the grades always being ordered in a decreasing way, the ith grade of the initial population will now appear at the positions $(i-1)\lambda + 1$ to $i \times \lambda$.

As we now have $\lambda n = 2^{x+y}$ judges in the duplicated population, we get $\varrho = \frac{2^{x+y}+1}{4}$ for $k = 2$ and

$$\begin{aligned}\mathcal{M}'_2 &= \left(f^{Rd(1\times\frac{2^{x+y}+1}{4})}, f^{Rd(2\times\frac{2^{x+y}+1}{4})}, f^{Rd(3\times\frac{2^{x+y}+1}{4})}\right)\\ &= \left(f^{(2^{x+y-2})}, f^{(2^{x+y-1}+1)}, f^{(3\times 2^{x+y-2}+1)}\right)\end{aligned}$$

As the ith grade of the initial population will now appear at the positions $(i-1)\lambda + 1$ to $i \times \lambda$, it follows that

- $f^{(2^{x-2})} = f^{(2^{x+y-2}-2^y+1)} = \ldots = f^{(2^{x+y-2})}$,
- $f^{(2^{x-1}+1)} = f^{(2^{x+y-1}+1)} = \ldots = f^{(2^{x+y-1}+2^y)}$,
- $f^{(3\times 2^{x-2}+1)} = f^{(3\times 2^{x+y-2}+1)} = \ldots = f^{(3\times 2^{x+y-2}+2^y)}$

Thus $\mathcal{M}'_2 = \mathcal{M}_2$. It follows that duplicating the population does not affect the outcome. □

5 Concluding remarks and discussion

The choosing of an aggregating function is dictated by the search for some requirements that the desired solution must satisfy. These requirements, presented as democratic criteria, are so numerous and often opposite such that no aggregating function, as complex as it can be, can satisfy them at all. The impossibility results show that sometimes a short list of fair criteria is enough to establish the incapacity for an aggregation function to satisfy all the desirable criteria. If all of the functions are struck by the impossibility results, all are not therefore bad. Indeed, considering the relevance of some criteria, we can then think to draw aside the functions which do not fulfill these criteria and choose among those that remain. The 3MCD Method introduced herein satisfies a long list of desirable properties. It fulfills inter alia the neutrality condition, the anonymity condition, the monotonicity condition, the Pareto condition, the Independence of irrelevant Alternatives and clone-resistance condition.

It is shown even more robust than the majority of grading methods. Nonetheless, it fails the conditions of homogeneity, reinforcement, participation and does not fulfill either the majority requirement, the Condorcet winner criterion and the Condorcet loser criterion. Nonetheless, we showed that, under certain conditions, 3MCD is (partially) homogeneous. Based on his numerous properties, 3MCD is an aggregating method dedicated to ranking problems that deserves a place among the functions which return acceptable compromise results. Thus, a thorough study must be devoted for analyzing and characterizing 3MCD.

References

Adla, A. (2010) Aide à la facilitation pour une prise de décision collective : Proposition d'un modèle et d'un outil. PhD thesis, Université de Toulouse.

Arrow, K. J. (1950) A difficulty in the concept of social welfare. Journal of Political Economy, 58(4):328-346.

Arrow, K. J. (1951) Social Choice and individual values. Yale University Press.

Balinski, M. and Laraki, R. (2007) A theory of measuring, electing and ranking. Proceedings of the National Academy of Sciences of the United States of America, 8720-8725.

Balinski, M. and Laraki, R. (2010) Majority Judgment: measuring, ranking and electing. MIT Press.

Balinski, M. and Laraki, R. (2016) Majority Judgment vs Majority Rule. Working paper Ecole Polytechnique no 2016-04.

Bouyssou, D., Marchant, T., Pirlot, M., Tsoukiàs, A. and Vincke, P. Decision models with multiple criteria. Stepping stones for the analyst. Springer, 2006.

Felsenthal, D.S. and Machover, M. (2008) The Majority Judgement Voting Procedure: A Critical Evaluation. Homo Oeconomicus 25 (3/4): 319-334.

Felsenthal, D. S. Review of paradoxes afflicting procedures for electing a single candidate. in Dan S. Felsenthal & Moshé Machover (eds.), Electoral Systems. Paradoxes, Assumptions, and Procedures, pages 19-91. Springer-Verlag, Berlin, 2012.

Ngoie, R.-B. M., Savadogo, Z. and Ulungu, B. E.-L. (2014) Median and average as tools for measuring, electing and ranking: new prospects. Fundamental Journal of Mathematics and Mathematical Sciences, 1(1):9-30.

Ngoie, R.-B. M., Savadogo, Z. and Ulungu, B. E.-L. (2015a) New prospects in Social Choice Theory: Median and average as tools for measuring, electing and ranking. Advanced Studies in Contemporary Mathematics, 25(1):19-38.

Ngoie, R.-B. M. and Ulungu, B. E.-L. (2015b) On Analysis and Characterization of the Mean-Median Compromise Method. International Journal of Scientific and Innovative Mathematical Research, 3(3):56-64.

Ngoie, R.-B. M. (2016) Nouvelle Approche basée sur la moyenne et la médiane en Théorie du Choix Social et Analyse Multicritère. PhD Thesis, Université Pédagogique Nationale.

Smith W.D. (2009) Tilburg Lecture on Range Voting. http://www.rangevoting.org/TBlecture.html

Roy, B. (1985) Méthodologie multicritère d'aide à la décision. Economica, Paris.

Zahid, M. A. Majority judgment and pardoxical results. International Journal of Arts and Sciences, 4(20) :121-131, 2009.

Zahid, M. A. and De Swart H. (2015) The Borda majority count. Information Sciences 295: 429-440.