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Abstract

Under Approval Voting (AV), each voter just distinguishes the candidates he approves of from those appearing as unacceptable. The *Preference Approval Voting* (PAV) is a hybrid version of the approval voting first introduced by Brams and Sanver (2009). Under PAV, each voter ranks all the candidates and then indicates the ones he approves. In this paper, we provide analytical representations for the probability that PAV elects the Condorcet winner when she exists in three-candidate elections with large electorates. We also provide analytical representations for the probability that PAV elects the Condorcet loser. We perform our analysis by assuming the assumption of the Extended Impartial Culture. Under this assumption, it comes that AV seems to perform better than PAV on electing the Condorcet winner and that in most of the cases, PAV seems to be less likely to elect the Condorcet loser than AV.

Keywords: Approval Voting, Ranking, Condorcet, Extended Impartial Culture, Probability.

1. Introduction

Popularized by Brams and Fishburn (1978), the *Approval Voting* (AV) rule is a voting system under which each voter approves (any number of) candidates that he considers as acceptable and the winner is the most-approved candidate. This rule has made (and continues to be) the subject of numerous research works in political science, economics and computer science. To have a quick overview of these works, the reader may refer to the books of Brams and Fishburn (2007), Brams (2008) and to the *Handbook of Approval Voting* edited by Laslier and Sanver (2010). Under AV, there is no need to rank the candidates as under the scoring rules[1]. This absence of rankings gave rise to a controversy between Saari and van Newenhizen (1988a,b) and Brams et al. (1988a,b). Saari and van Newenhizen (1988b) blame AV of hiding the real preferences of the voters which can be strict between the candidates approved by a voter. Brams and Sanver (2009) may have brought what appears as a possible response to this criticism by introducing the *Preference Approval Voting* (PAV). Under PAV, each voter ranks all the candidates then indicates the ones he approves.[2] According to Brams and Sanver (2009), the winner under PAV is determined by two rules:

Rule 1: The PAV winner is the AV winner if[3]

i. no candidate receives a majority of approval votes (*i.e* approved by more than half of the electorate)
ii. exactly one candidate receives a majority of approval votes.

Rule 2: In the case that two or more candidates receive a majority of approval votes,

i. The PAV winner is the ones among these candidates who is preferred by a majority to every other majority-approved candidate.

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[1] A scoring rule is a voting rule under which voters give points to candidates according to the ranks they have in voter's preferences. The winner is the candidate with the highest total number of points.

[2] Brams and Sanver (2009) also introduced the *Fallback Voting* under which voters only rank the candidates they approve. In this paper, we are not concerned with this rule.

[3] Here, we have chose to split *Rule 1* into two. This will be helpful for our analysis.

ii. In the case of a cycle among the majority-approved candidates, then the AV winner among them is the PAV winner.

Brams and Sanver (2009) noticed that it is Rule 2 that clearly differentiate PAV from AV. They pointed out that for some situations where a *Condorcet winner exists*, this candidate may not be a PAV winner under each of the subcases of Rule 1 and Rule 2. When she exists, a *Condorcet winner* is a candidate who defeats each of the other candidates in pairwise comparisons. We know that AV always elects the Condorcet winner, when she exists given that voters' preferences are dichotomous (Ju, 2010, Xu, 2010). This is no more the case when the voters' true preferences are assumed to be strict orderings (Gehrlein and Lepelley, 1998) or when indifference are allowed in the voters' true preferences (Diss et al., 2010). For large electorates, Gehrlein and Lepelley (1998) found that AV has the same Condorcet efficiency (probability of electing the Condorcet winner when she exists) as both the Plurality rule and the Antiplurality rule.[4] Going from a more general framework, Diss et al. (2010) found that for large electorates and three candidates, AV performs better that both the Plurality rule and the Antiplurality rule on the Condorcet efficiency; they also found some scenarios under which AV performs better than the Borda rule. Their results were strongly reinforced by Gehrlein and Lepelley (2015).

As PAV appears as a possible response to the criticism of Saari and van Newenhizen (1988b) on AV, does it perform better than AV on the likelihood of electing the Condorcet winner when she exists? Up to our knowledge, no work has questioned the Condorcet efficiency PAV. This paper tries to provide an answer to the question for voting situations with three candidates by computing the Condorcet efficiency of PAV when indifference are allowed as in Diss et al. (2010). Computations are done under the extended impartial culture defined by Diss et al. (2010). This assumption will be defined later.

Some works have looked on the probability that AV elects the *Condorcet loser* when she exists. A *Condorcet loser* is a candidate, when she exists, who is defeated by each of the other candidates in pairwise comparisons. Gehrlein and Lepelley (1998) showed that with more than three candidates and under the impartial culture assumption, AV is more likely to elect the Condorcet loser than the Plurality rule. For three-candidate elections, they showed that AV has the same probability of electing the Condorcet loser as both the Plurality rule and the Antiplurality rule. This result is a bit challenged by a recent paper by Gehrlein et al. (2016). Using impartial anonymous culture-like assumptions[5] and considering a range of scenarios, Gehrlein et al. (2016) concluded that in three-candidate elections, AV is less likely to elect the Condorcet loser than both the Plurality rule and the Antiplurality rule. The second objective of this paper is to focus on the probability that PAV elects the Condorcet loser when she exists. We provide for AV and for PAV, analytical representations of the probability of electing the Condorcet loser when she exists under the extended impartial culture assumption.

The rest of the paper is structured as follows: Section 2 is devoted to basic notations and definitions. Section 3 presents our results on the Condorcet efficiency. Section 4 deals with our results on the probability of electing the Condorcet loser. Section 5 concludes.

2. Preliminaries

2.1. Preferences in three-candidate elections

Let N be a set of n voters ($n \geq 2$) and $A = \{a, b, c\}$ a set of three candidates. We assume that voters rank all the candidates, indifference is allowed and they indicate which candidates they approve by underlining the names of the candidates[6]. So, there are 19 possible types of preferences. Following Diss et al. (2010), these 19 types of preferences can be partitioned into five classes of preferences:

- **Class I:** this class is made of voters with strict rankings and who only approve their top ranked candidates. These voters are labelled 1 to 6 in Table 1.

[4] The Plurality rule is a scoring rule under which each voter votes only for (gives one point to) his top ranked candidate and the winner is the ones with the highest total number first places; under the Antiplurality rule, the winner is the candidate with the fewest total number of last places.

[5] Under impartial anonymous culture-like assumptions, voting situations are assumed to be equally likely.

[6] It is assumed that voters vote sincerely. So, we are not concerned with strategic behaviors.

- **Class II:** voters in this class also have strict ranking and they approve their top two ranked candidates. These types of voters are labelled 7 to 12 in Table 1.
- **Class III:** in this class, voters are indifferent between their two preferred candidates or do not consider the difference significant enough to reveal their true strict preference. These types of voters are labelled 13 to 15 in Table 1.
- **ClassIV:** Voters rank one candidate strictly above the two other between whom they are indifferent. These types of voters are labelled 16 to 17 in Table 1.
- **Class V:** the voters of this class are indifferent between the three candidates, thus they approve all the three candidates(type 19).

Table 1: The 19 possible preferences types with three candidates

Class I:	$\underline{a} \succ b \succ c$	p_1	**Class II:**	$\underline{a} \succ \underline{b} \succ c$	p_7
	$\underline{a} \succ c \succ b$	p_2		$\underline{a} \succ \underline{c} \succ b$	p_8
	$\underline{b} \succ a \succ c$	p_3		$\underline{b} \succ \underline{a} \succ c$	p_9
	$\underline{b} \succ c \succ a$	p_4		$\underline{b} \succ \underline{c} \succ a$	p_{10}
	$\underline{c} \succ a \succ b$	p_5		$\underline{c} \succ \underline{a} \succ b$	p_{11}
	$\underline{c} \succ b \succ a$	p_6		$\underline{c} \succ \underline{b} \succ a$	p_{12}
Class III:	$\underline{a} \sim \underline{b} \succ c$	p_{13}	**Class IV:**	$\underline{a} \succ b \sim c$	p_{16}
	$\underline{a} \sim \underline{c} \succ b$	p_{14}		$\underline{b} \succ a \sim c$	p_{17}
	$\underline{b} \sim \underline{c} \succ a$	p_{15}		$\underline{c} \succ a \sim b$	p_{18}
Class V:	$\underline{a} \sim \underline{b} \sim \underline{c}$	p_{19}			

If we denote by n_t the number of voter of type t, a *voting situation* is an 19-tuple $\tilde{n} = (n_1, n_2, \ldots, n_t, \ldots, n_{19})$ that indicates the total number n_t of voters casting each type of preferences such that $\sum_{t=1}^{19} n_t = n$. We denote by p_t the probability that a voter chooses the preference type t such that $\sum_{t=1}^{19} p_t = 1$. In Table 2, $S(a)$ denotes the AV score candidate a given the labels of Table 1.

Table 2: The AV score of the candidates

$S(a)$	$=$	$n_1 + n_2 + n_7 + n_8 + n_9 + n_{11} + n_{13} + n_{14} + n_{16} + n_{19}$
$S(b)$	$=$	$n_3 + n_4 + n_7 + n_9 + n_{10} + n_{12} + n_{13} + n_{15} + n_{17} + n_{19}$
$S(c)$	$=$	$n_5 + n_6 + n_8 + n_{10} + n_{11} + n_{12} + n_{14} + n_{15} + n_{18} + n_{19}$

Given $a, b \in A$, we denote by n_{ab} the total number of voters who strictly prefer a to b. If $n_{ab} > n_{ba}$, we say that a majority dominates candidate b; or equivalently, a beats b in a pairwise majority voting. In such a case, we will simply write $a\mathbf{M}b$. Candidate a is said to be the *Condorcet winner* (resp. the *Condorcet loser*) if for all $b \in A \setminus \{a\}$, $a\mathbf{M}b$ (resp. $b\mathbf{M}a$). If for a given voting situation we get $a\mathbf{M}b$, $b\mathbf{M}c$ and $c\mathbf{M}a$, this describes a majority cycle.

2.2. *PAV, the Condorcet winner and the Condorcet loser*

By the definition of PAV, it is obvious that with three candidates, if there is a Condorcet winner who belongs to the subset of majority-approved candidates, she is always elected if Rule 2i applies while rule 2ii will never apply. So, Rule 1i and 1ii can fail to elect the Condorcet winner. It is also obvious that if there is a Condorcet loser in a three-candidate election, she cannot be elected under Rule 2i and 2ii; so, PAV may elect the Condorcet loser only when Rule 1i or 1ii apply.

In order to motivate the paper, let us take the following two voting profiles[7] each with 9 voters V_i ($i = 1..9$) in order to illustrate that in three-candidate elections, PAV can fail to select the Condorcet winner when she exists (under Rules 1i, 1ii and 2i) and that it can select the Condorcet loser (under Rules 1i and 1ii).

Profile 1

V_1:$\underline{a} \succ c \succ b$	V_2:$\underline{a} \succ c \succ b$	V_3:$\underline{b} \succ c \succ a$
V_4:$\underline{b} \succ c \succ a$	V_5:$\underline{c} \succ a \succ b$	V_6:$\underline{c} \succ a \succ b$
V_7:$\underline{b} \succ a \succ c$	V_8:$\underline{c} \succ b \succ a$	V_9:$\underline{a} \succ \underline{b} \succ c$

Profile 2

V_1:$\underline{a} \succ c \succ b$	V_2:$\underline{b} \succ a \succ c$	V_3:$\underline{b} \succ c \succ a$
V_4:$\underline{b} \succ c \succ a$	V_5:$\underline{c} \succ a \succ b$	V_6:$\underline{c} \succ a \succ b$
V_7:$\underline{c} \succ a \succ b$	V_8: $\underline{a} \succ \underline{b} \succ c$	V_9:$\underline{c} \succ \underline{b} \succ a$

Under both profiles, the reader can check that c is the Condorcet winner and b is the Condorcet loser. Under the first profile, we get $S(a) = S(c) = 3$ and $S(b) = 4$; no candidate gets the majority of the approvals (5 votes), according to Rule 1i, b is the winner since she is the AV winner. Thus, PAV under Rule 1i fails to select the Condorcet winner but selects the Condorcet loser. Under the second profile, b is the unique majority-approved candidate with 5 votes; Rule 1ii applies and b is the PAV winner: PAV under Rule 1i fails to elect the Condorcet winner but can select the Condorcet loser.

To get a profile under which Rule 2i applies and that PAV fails to select the Condorcet winner, the reader only need to add the following groups of voters to Profile 1: 3 voters with $\underline{a} \succ \underline{b} \succ c$, 2 voters with $\underline{a} \succ c \succ b$, 3 voters with $\underline{c} \succ \underline{a} \succ b$ and 4 voters with $\underline{c} \succ \underline{b} \succ a$.

The profiles we just used illustrate that under some voting situations, PAV can fail to elect the Condorcet winner when she exists and that it can elect the Condorcet loser when she exists. These two behaviors of PAV are just rare oddities or are a common occurrence? The aim of this paper is then to provide an answer to this question. So, we compute the Condorcet efficiency of PAV and its probability of electing the Condorcet loser for voting situations with three candidates. Before starting this task, we need to define a probability model for this.

2.3. *The probability model: the Extended Impartial Culture assumption*

The Impartial Culture (IC) assumption is one of the assumptions used in the social choice literature when computing the likelihood of voting events. Under IC, it is assumed that each voter chooses her preference according to a uniform probability distribution. When only strict ranking are allowed with m candidates, IC gives a probability $\frac{1}{m!}$ for each of the $m!$ rankings to be chosen independently. The likelihood of a given voting situation $\tilde{n} = (n_1, n_2, ..., n_t, ..., n_{m!})$ is

$$Prob(\tilde{n} = (n_1, n_2, ..., n_t, ..., n_{m!})) = \frac{n!}{\prod_{t=1}^{m!} n_t!} \times (m!)^{-n}$$

When indifference is allowed, the Impartial Weak Ordering Culture (IWOC) was introduced by Gehrlein and Fishburn (1980) as an extension of IC. Diss et al. (2010) provide an extension of IC that allows the possibility that voters could have dichotomous preferences with complete indifference between two of the candidates and also the possibility of a complete indifference between all three candidates. This extension is called the Extended Impartial Culture (EIC) assumption. Let us describes how it works. Consider the 5 classes of preferences described in Table 1. We denote by k_1 the probability that a voter's preference belongs to Class **I**; k_2 is the probability that a voter's preference belongs to Class **II**; k_3 is the probability that a voter's preference belongs to Class **III**; k_4 is the probability that a voter's preference belongs to Class **IV** and k_5 is the probability that a voter's preference belongs to Class **V** such that $k_1 + k_2 + k_3 + k_4 + k_5 = 1$. Under EIC, it is assumed that the rankings within a class are equally likely: $p_t = \frac{k_1}{6}$ for $t = 1, 2, ..., 6$, $p_t = \frac{k_2}{6}$ for $t = 7, 8, ..., 12$, $p_t = \frac{k_3}{3}$ for $t = 13, 14, 15$, $p_t = \frac{k_4}{3}$ for $t = 16, 17, 18$ and $p_{19} = k_5$.

[7]Other examples are provided in Brams (2008), Brams and Sanver (2009).

With the 19 preference types of Table 1, the likelihood of a given voting situation $\tilde{n} = (n_1, n_2, ..., n_t, ..., n_{19})$ is given by

$$Prob(\tilde{n} = (n_1, n_2, ..., n_t, ..., n_{19})) = \frac{n!}{\prod_{t=1}^{19} n_t!} \times \prod_{t=1}^{19} p_t$$

Diss et al. (2010) used EIC to analyze the Condorcet efficiency of AV and all the scoring rules. They also provided the limiting probability that a Condorcet winner exists as follows[8]:

$$P^{\infty}_{Con} = \frac{3}{4} + \frac{3}{2\pi} \arcsin\left(\frac{k_1 + k_2 + k_3 + k_4}{3k_1 + 3k_2 + 2k_3 + 2k_4}\right)$$

Given that $k_1 + k_2 + k_3 + k_4 + k_5 = 1$, we can rewrite P^{∞}_{Con}:

$$\begin{aligned} P^{\infty}_{Con} &= \frac{3}{4} + \frac{3}{2\pi} \arcsin\left(\frac{1 - k_5}{3 - k_3 - k_4 - 3k_5}\right) \\ &= \frac{3}{4} + \frac{3}{2\pi} \arcsin\left(\frac{1 - k_5}{3 - k_{34} - 3k_5}\right) \text{ with } k_{34} = k_3 + k_4 \end{aligned}$$

3. Probability that PAV elects the Condorcet winner

By computing the Condorcet efficiency of PAV, it will we interesting to compare it to that of AV. Diss et al. (2010) compute the Condorcet efficiency of AV under EIC assumption by assuming that $p_{19} = 0$. They made this assumption because the preference type of Class **V** has no impact on the outcome under AV; this is not the case under PAV where type 19 can really matter. In order to consider a comparison between AV and PAV, we propose to recalculate the Condorcet efficiency of AV assuming that $p_{19} \geq 0$.[9]

Given the voting situation $\tilde{n}$ on $A = \{a, b, c\}$, assume that candidate a is the Condorcet winner; this means that $a\mathbf{M}b$ and $a\mathbf{M}b$. Using the labels of Table 1 these are respectively equivalent to Equations 1 and 2.

$$n_1 + n_2 - n_3 - n_4 + n_5 - n_6 + n_7 + n_8 - n_9 - n_{10} + n_{11} - n_{12} + n_{14} - n_{15} + n_{16} - n_{17} > 0 \tag{1}$$

$$n_1 + n_2 + n_3 - n_4 - n_5 - n_6 + n_7 + n_8 + n_9 - n_{10} - n_{11} - n_{12} + n_{13} - n_{15} + n_{16} - n_{18} > 0 \tag{2}$$

Candidate a is also the AV winner means that $S(a) > S(b)$ and $S(a) > S(c)$ which are respectively equivalent to Equations 3 and 4.

$$n_1 + n_2 - n_3 - n_4 + n_8 - n_{10} + n_{11} - n_{12} + n_{14} - n_{15} + n_{16} - n_{17} > 0 \tag{3}$$

$$n_1 + n_2 - n_5 - n_6 + n_7 + n_9 - n_{10} - n_{12} + n_{13} - n_{15} + n_{16} - n_{18} > 0 \tag{4}$$

So, a voting situation under which AV elects the Condorcet winner is fully described by Equations 1 to 4. We derive Theorem 1.

Theorem 1. *With three candidates and an infinite number of voters, the Condorcet efficiency of AV is given by:*

$$CE^{\infty}_{AV}(k_{34}, k_5) = 3\left(\frac{\Phi(R_4)}{P^{\infty}_{Con}}\right)$$

where $R_4 = \begin{pmatrix} 1 & \frac{x}{y} & \sqrt{\frac{2x}{y}} & \sqrt{\frac{x}{2y}} \\ & 1 & \sqrt{\frac{x}{2y}} & \sqrt{\frac{2x}{y}} \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{pmatrix}$ *with* $x = 1 - k_5$, $y = 3 - 3k_5 - k_{34}$ *and* $\Phi(R_4)$ *the positive-orthant probability associated with matrix* R_4.

[8]Notice that P^{∞}_{Con} is also the probability that a Condorcet loser exists.
[9]This is also done by Gehrlein and Lepelley (2015).

. See Appendix A. $\square$

Table 3 reports some computed values of $CE^{\infty}_{\text{AV}}(k_{34}, k_5)$. We recover the same figures as Diss et al. (2010) for $k_5 = 0$. In this table, one can notice that for a given value of one parameter, the probability tends to increase with the other parameter.

Table 3: Some values of the probabilities $CE^{\infty}_{\text{AV}}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.75720	0.75720	0.75720	0.75720	0.75720	0.75723	0.75723	0.75720	0.75720	0.75720	–
0.1	0.76626	0.76725	0.76860	0.77031	0.77271	0.77604	0.78132	0.79074	0.81249	–	–
0.2	0.77604	0.77835	0.78135	0.78525	0.79074	0.79890	0.81249	0.84033	–	–	–
0.3	0.78684	0.79074	0.79578	0.80265	0.81252	0.82827	0.85815	–	–	–	–
0.4	0.79890	0.80469	0.81249	0.82353	0.84036	0.87078	–	–	–	–	–
0.5	0.81252	0.82095	0.83262	0.85005	0.88044	–	–	–	–	–	–
0.6	0.82824	0.84036	0.85812	0.88818	–	–	–	–	–	–	–
0.7	0.84705	0.86490	0.89448	–	–	–	–	–	–	–	–
0.8	0.87081	0.89979	–	–	–	–	–	–	–	–	–
0.9	0.90435	–	–	–	–	–	–	–	–	–	–
1	1	–	–	–	–	–	–	–	–	–	–

Let us turn to the Condorcet efficiency of PAV. Assume that a is the PAV winner. This requires to analyze what may happen under Rule 1i, under Rule 1ii and under Rule 2i.

- **Case 1:** *Rule 1i applies*. In this case, no candidate receives a majority of the approvals and a, the Condorcet winner, gets the highest AV score. This is fully described by the following inequalities system:

$$\begin{cases} a\mathbf{M}b \\ a\mathbf{M}b \\ S(b) < S(a) \\ S(c) < S(a) \\ S(a) < \frac{n}{2} \\ S(b) < \frac{n}{2} \\ S(c) < \frac{n}{2} \end{cases} \Rightarrow \begin{cases} a\mathbf{M}b \\ a\mathbf{M}b \\ S(b) < S(a) \\ S(c) < S(a) \\ S(a) < \frac{n}{2} \end{cases} \Rightarrow \begin{cases} n_{ab} - n_{ba} > 0 \\ n_{ac} - n_{ca} > 0 \\ S(a) - S(b) > 0 \\ S(a) - S(c) > 0 \\ 2S(a) - n > 0 \end{cases}$$

The first four inequalities of the final system respectively correspond to Equations 1 to 4. We derive the last inequality as follows:

$$-n_1 - n_2 + n_3 + n_4 + n_5 + n_6 - n_7 - n_8 - n_9 + n_{10} - n_{11} + n_{12} - n_{13} - n_{14} + n_{15} - n_{16} + n_{17} + n_{18} - n_{19} > 0 \quad (5)$$

So, a voting situation under which Rule 1i applies and that PAV elects the Condorcet winner is fully described by Equations 1 to 5. This event is described by five constraints. In most of the social choice literature, when computing under IC the probability of voting events described by more than four constraints most of the authors rely on Monte-Carlo simulations. Thanks to Gehrlein (2017), the computations are made possible for events described by five constraints through a formula based on normal positive orthants. As all the events in this paper are described by five constraints, we then rely on this formula. In Appendix, while providing the proof of Theorem 2, we will say a few words on how this formula is obtained.

Theorem 2. *With three candidates and an infinite number of voters, the Condorcet efficiency of PAV under EIC when* Rule 1i *applies is given by*

$$CE^{\infty}_{PAV_{1i}}(k_{34}, k_5) = 3\Big(\frac{\Phi(R_5^{1i})}{P^{\infty}_{Con}}\Big)$$

where $R_5^{1i} = \begin{pmatrix} 1 & \frac{x}{y} & -\varphi & \sqrt{\frac{2x}{y}} & \sqrt{\frac{x}{2y}} \\ & 1 & -\varphi & \sqrt{\frac{x}{2y}} & \sqrt{\frac{2x}{y}} \\ & & 1 & -\sqrt{\frac{2x}{3}} & -\sqrt{\frac{2x}{3}} \\ & & & 1 & \frac{1}{2} \\ & & & & 1 \end{pmatrix}$ *and* $\varphi = \frac{2\sqrt{3}}{3}(\frac{x\sqrt{y}}{y})$.

. See Appendix B. □

The proofs of all the subsequent Theorems are omitted since they follow the same scheme as that of Theorem 2. Some computed values of $CE^{\infty}_{\mathrm{PAV}_{1i}}(k_{34}, k_5)$ are provided in Table 4.

Table 4: Some values of the probabilities $CE^{\infty}_{\mathrm{PAV}_{1i}}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.02036	0.03998	0.06107	0.08340	0.10709	0.13240	0.15988	0.19044	0.22599	0.27138	–
0.1	0.02064	0.04067	0.06230	0.08533	0.10997	0.13666	0.16627	0.20074	0.24557	–	–
0.2	0.02098	0.04151	0.06378	0.08770	0.11358	0.14219	0.17519	0.21733	–	–	–
0.3	0.02139	0.04250	0.06560	0.09065	0.11828	0.14988	0.18946	–	–	–	–
0.4	0.02190	0.04375	0.06789	0.09455	0.12486	0.16222	–	–	–	–	–
0.5	0.02255	0.04535	0.07094	0.10003	0.13542	–	–	–	–	–	–
0.6	0.02339	0.04751	0.07530	0.10890	–	–	–	–	–	–	–
0.7	0.02456	0.05064	0.08244	–	–	–	–	–	–	–	–
0.8	0.02634	0.05595	–	–	–	–	–	–	–	–	–
0.9	0.02951	–	–	–	–	–	–	–	–	–	–
1	–	–	–	–	–	–	–	–	–	–	–

- **Case 2:** *Rule 1ii applies*. Here, it is assumed that a the Condorcet winner is the only candidate with a majority of approval votes. This is described by the following inequalities system:

$$\begin{cases} a\mathbf{M}b \\ a\mathbf{M}c \\ S(a) > \frac{n}{2} \\ S(b) < \frac{n}{2} \\ S(c) < \frac{n}{2} \end{cases} \Rightarrow \begin{cases} n_{ab} - n_{ba} > 0 \\ n_{ac} - n_{ca} > 0 \\ 2S(a) - n > 0 \\ n - 2S(b) > 0 \\ n - 2S(c) > 0 \end{cases}$$

From these inequalities, we were able to derive the five constraints that fully characterize a voting situation under which Rule 1ii applies and that PAV elects the Condorcet winner. Using the same reasoning described above, we get Theorem 3.

Theorem 3. *With three candidates and an infinite number of voters, the Condorcet efficiency of the PAV when* Rule 1ii *applies is given by*

$$CE^{\infty}_{PAV_{1ii}}(k_{34}, k_5) = 3\left(\frac{\Phi(R_5^{1ii})}{P^{\infty}_{Con}}\right)$$

where $R_5^{1ii} = \begin{pmatrix} 1 & \frac{x}{y} & \varphi & \varphi & 0 \\ & 1 & \varphi & 0 & \varphi \\ & & 1 & \frac{4x-3}{3} & \frac{4x-3}{3} \\ & & & 1 & \frac{3-4x}{3} \\ & & & & 1 \end{pmatrix}$

Some computed values of $CE^{\infty}_{\mathrm{PAV}_{1ii}}(k_{34}, k_5)$ are provided in Table 5.

- **Case 3:** *Rule 2i applies*. It implies the following two subcases:

Table 5: Some values of the probabilities $CE^{\infty}_{\text{PAV}_{1ii}}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.39270	0.36729	0.34193	0.31617	0.28958	0.26169	0.23183	0.19895	0.16104	0.11293	–
0.1	0.39712	0.37177	0.34650	0.32089	0.29449	0.26692	0.23751	0.20539	0.16889	0.12469	–
0.2	0.40184	0.37654	0.35140	0.32595	0.29984	0.27264	0.24384	0.21277	0.17847	–	–
0.3	0.40685	0.38164	0.35665	0.33142	0.30563	0.27893	0.25097	0.22137	–	–	–
0.4	0.41220	0.38713	0.36231	0.33736	0.31198	0.28594	0.25909	–	–	–	–
0.5	0.41796	0.39303	0.36949	0.34384	0.31900	0.29387	–	–	–	–	–
0.6	0.42417	0.39944	0.37516	0.35100	0.32693	–	–	–	–	–	–
0.7	0.43094	0.40646	0.38255	0.35907	–	–	–	–	–	–	–
0.8	0.43836	0.41423	0.39093	–	–	–	–	–	–	–	–
0.9	0.44664	0.42307	–	–	–	–	–	–	–		–
1	0.45613	–	–	–	–	–	–	–	–	–	–

- *two candidates receive a majority of approval votes*. Let assume that candidates a and b are the majority-approved; since $a\mathbf{M}b$, it follows that a is the PAV winner. Thus, such a voting situation is fully described by the following system:

$$\begin{cases} a\mathbf{M}b \\ a\mathbf{M}c \\ S(a) > \frac{n}{2} \\ S(b) > \frac{n}{2} \\ S(c) < \frac{n}{2} \end{cases} \Rightarrow \begin{cases} n_{ab} - n_{ba} > 0 \\ n_{ac} - n_{ca} > 0 \\ 2S(a) - n > 0 \\ 2S(b) - n > 0 \\ n - 2S(c) > 0 \end{cases}$$

- *all the three candidates receive a majority of approval votes*. Since $a\mathbf{M}b$ and $a\mathbf{M}b$ it follows that a is the PAV winner. This is described by the following system:

$$\begin{cases} a\mathbf{M}b \\ a\mathbf{M}c \\ S(a) > \frac{n}{2} \\ S(b) > \frac{n}{2} \\ S(c) > \frac{n}{2} \end{cases} \Rightarrow \begin{cases} n_{ab} - n_{ba} > 0 \\ n_{ac} - n_{ca} > 0 \\ 2S(a) - n > 0 \\ 2S(b) - n > 0 \\ 2S(c) - n > 0 \end{cases}$$

As the two subcases are disjoints, we then derive Theorem 4.

Theorem 4. *With three candidates and an infinite number of voters, the Condorcet efficiency PAV when* Rule 2 *applies is given by*

$$CE^{\infty}_{PAV_2}(k_{34}, k_5) = 3\left(\frac{\Phi(R_5^{2i}) + \Phi(R_5^{2i'})}{P^{\infty}_{Con}}\right)$$

where

$$R_5^{2i} = \begin{pmatrix} 1 & \frac{x}{y} & \varphi & -\varphi & 0 \\ & 1 & \varphi & 0 & \varphi \\ & & 1 & \frac{-4x+3}{3} & \frac{4x-3}{3} \\ & & & 1 & \frac{4x-3}{3} \\ & & & & 1 \end{pmatrix} \quad \textit{and} \quad R_5^{2i'} = \begin{pmatrix} 1 & \frac{x}{y} & \varphi & -\varphi & 0 \\ & 1 & \varphi & 0 & -\varphi \\ & & 1 & \frac{-4x+3}{3} & \frac{4x-3}{3} \\ & & & 1 & \frac{4x-3}{3} \\ & & & & 1 \end{pmatrix}$$

Some computed values of $CE^{\infty}_{\text{PAV}_2}(k_{34}, k_5)$ are provided in Table 6.

From Tables 4, 5 and 6, it comes that PAV is more likely to select the Condorcet winner when Rule 2 applies. As all the events described above are all mutually exclusive, we then derive in Corollary 1, the overall Condorcet efficiency of PAV.

Corrolary 1. *With three candidates and an infinite number of voters, the Condorcet efficiency of PAV is given by*

$$CE^{\infty}_{PAV}(k_{34}, k_5) = CE^{\infty}_{PAV_{1i}}(k_{34}, k_5) + CE^{\infty}_{PAV_{1ii}}(k_{34}, k_5) + CE^{\infty}_{PAV_2}(k_{34}, k_5)$$

Table 6: Some values of the probabilities $CE^{\infty}_{\text{PAV}_2}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.26388	0.28030	0.29649	0.31267	0.32914	0.34620	0.36421	0.38385	0.40625	0.43445	–
0.1	0.26444	0.28086	0.29707	0.31331	0.32982	0.34695	0.36509	0.38489	0.40770	0.43771	–
0.2	0.26500	0.28144	0.29769	0.31398	0.33059	0.34786	0.36620	0.38645	0.40940	–	–
0.3	0.26560	0.28206	0.29836	0.31475	0.33149	0.34897	0.36776	0.38834	–	–	–
0.4	0.26620	0.28272	0.29912	0.31565	0.33263	0.35052	0.37045	–	–	–	–
0.5	0.26685	0.28344	0.29998	0.31674	0.33414	0.35306	–	–	–	–	–
0.6	0.26753	0.28426	0.30103	0.31818	0.33653	–	–	–	–	–	–
0.7	0.26826	0.28523	0.30239	0.32046	–	–	–	–	–	–	–
0.8	0.26912	0.28647	0.30453	–	–	–	–	–	–	–	–
0.9	0.27018	0.28847	–	–	–	–	–	–	–	–	–
1	0.27194	–	–	–	–	–	–	–	–	–	–

Table 7 provides the Condorcet efficiency of PAV. From the comparison of figures in Tables 3 and 7, we notice that for $k_5 < 0.7$, AV performs better than PAV and we get the reverse for $k_5 > 0.7$. It comes that, voters with total indifference really matter as their proportion determine the dominance between AV and PAV in terms of the Condorcet efficiency.

Table 7: Some values of the probabilities $CE^{\infty}_{\text{PAV}}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.67693	0.68757	0.69949	0.71224	0.72580	0.74028	0.75591	0.77323	0.79328	0.81876	–
0.1	0.68220	0.69331	0.70587	0.71953	0.73429	0.75052	0.76888	0.79102	0.82216	0.56241	–
0.2	0.68782	0.69949	0.71287	0.72762	0.74401	0.76268	0.78523	0.81655	0.58788	–	–
0.3	0.69384	0.70620	0.72061	0.73683	0.75540	0.77779	0.80819	0.60972	–	–	–
0.4	0.70031	0.71360	0.72932	0.74756	0.76947	0.79868	0.62955	–	–	–	–
0.5	0.70736	0.72182	0.74041	0.76061	0.78856	0.64694	–	–	–	–	–
0.6	0.71509	0.73121	0.75149	0.77809	0.66347	–	–	–	–	–	–
0.7	0.72376	0.74234	0.76738	0.67954	–	–	–	–	–	–	–
0.8	0.73382	0.75665	0.69547	–	–	–	–	–	–	–	–
0.9	0.74633	0.71155	–	–	–	–	–	–	–	–	–
1	0.72807	–	–	–	–	–	–	–	–	–	–

4. Probability that PAV elects the Condorcet loser

Lepelley (1993) showed under an extension of IC assumption that if preferences are single-peaked, the election of the Condorcet loser is much less frequent with AV than with the Plurality rule. More recently, Gehrlein et al. (2016) built a framework to compare AV and the Plurality rule and they found under impartial anonymous culture-like assumptions that AV is much less susceptible to elect the Condorcet loser than the Plurality rule. Notice that Gehrlein et al. (2016) investigated different scenarios on voters' preferences included the one assumed in this paper. In this section, we first reconsider the likelihood of AV to elect the Condorcet loser when she exists under the EIC assumption in three-candidate election.

Theorem 5. *With three candidates and an infinite number of voters, the probability that AV elects the Condorcet loser under the EIC assumption is given by*

$$CL^{\infty}_{AV}(k_{34}, k_5) = 3\Big(\frac{\Phi(\overline{R}_4)}{P^{\infty}_{Con}}\Big)$$

$$where\ \ \overline{R}_4 = \begin{pmatrix} 1 & \frac{x}{y} & -\sqrt{\frac{2x}{y}} & -\sqrt{\frac{x}{2y}} \\ & 1 & -\sqrt{\frac{x}{2y}} & -\sqrt{\frac{2x}{y}} \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{pmatrix}$$

Table 8 report the probability $CL^{\infty}_{\text{AV}}(k_{34}, k_5)$. In this table, one can notice that for a given value of one parameter, the probability tends to decrease with the other parameter.

Table 8: Some values of the probabilities $CL^{\infty}_{\text{AV}}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.03709	0.03709	0.03709	0.03709	0.03709	0.03709	0.03709	0.03709	0.03709	0.03709	–
0.1	0.03393	0.03355	0.03311	0.03253	0.03174	0.03063	0.02896	0.02609	0.02013	–	–
0.2	0.03063	0.02989	0.02896	0.02775	0.02609	0.02374	0.02013	0.01379	–	–	–
0.3	0.02724	0.02609	0.02464	0.02272	0.02013	0.01637	0.01051	–	–	–	–
0.4	0.02374	0.02216	0.02013	0.01746	0.01379	0.00847	–	–	–	–	–
0.5	0.02013	0.01805	0.01541	0.01193	0.0071	–	–	–	–	–	–
0.6	0.01637	0.01379	0.01051	0.00612	–	–	–	–	–	–	–
0.7	0.01249	0.00938	0.00538	–	–	–	–	–	–	–	–
0.8	0.00847	0.00475	–	–	–	–	–	–	–	–	–
0.9	0.00436	–	–	–	–	–	–	–	–	–	–
1	–	–	–	–	–	–	–	–	–	–	–

Let us now turn to the probability that PAV elects the Condorcet loser in order to envisage the comparison with AV. We will proceed as we did for the Condorcet efficiency by computing the likelihood when the event is susceptible to occur under the variety rules of PAV. We already know that the Condorcet loser can only be elected by PAV if Rule 1i or Rule 1ii apply. For each of these rules, Theorems 6 and 7 gives the likelihood of the election of the Condorcet loser.

Theorem 6. *With three candidates and an infinite number of voters, the probability that PAV elects the Condorcet loser when Rule 1i applies is given by*

$$CL^{\infty}_{PAV_{1i}}(k_{34}, k_5) = 3\left(\frac{\Phi(\hat{R}_5)}{P^{\infty}_{Con}}\right)$$

$$where\ \ \hat{R}_5 = \begin{pmatrix} 1 & \frac{x}{y} & -\varphi & -\sqrt{\frac{2x}{y}} & -\sqrt{\frac{x}{2y}} \\ & 1 & -\varphi & -\sqrt{\frac{x}{2y}} & -\sqrt{\frac{2x}{y}} \\ & & 1 & \frac{\sqrt{6x}}{3} & -\frac{\sqrt{6x}}{3} \\ & & & 1 & \frac{1}{2} \\ & & & & 1 \end{pmatrix}$$

In Table 9, we report the probabilities $CL^{\infty}_{\text{PAV}_{1i}}(k_{34}, k_5)$.

Theorem 7. *With three candidates and an infinite number of voters, the probability that PAV elects the Condorcet loser when Rule 1ii applies is given by*

$$CL^{\infty}_{PAV_{1ii}}(k_{34}, k_5) = 3\left(\frac{\Phi(\tilde{R}_5)}{P^{\infty}_{Con}}\right)$$

$$where\ \ \tilde{R}_5 = \begin{pmatrix} 1 & \frac{x}{y} & -\varphi & -\varphi & 0 \\ & 1 & -\varphi & 0 & -\varphi \\ & & 1 & \frac{4x-3}{3} & \frac{4x-3}{3} \\ & & & 1 & \frac{-4x+3}{3} \\ & & & & 1 \end{pmatrix}$$

In Table 10, we report the probabilities $CL^{\infty}_{\text{PAV}_{1ii}}(k_{34}, k_5)$.

Table 9: Some values of the probabilities $CL^{\infty}_{\mathrm{PAV}_{1i}}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.03134	0.02900	0.02735	0.02606	0.02499	0.02406	0.02320	0.02236	0.02152	0.02056	–
0.1	0.02839	0.02597	0.02413	0.02255	0.02106	0.01952	0.01776	0.01536	0.01129		–
0.2	0.02542	0.02283	0.02078	0.01893	0.01698	0.01482	0.01199	0.00781		–	–
0.3	0.02235	0.01964	0.01737	0.01518	0.01278	0.00989	0.06600		–	–	–
0.4	0.01920	0.01639	0.01388	0.01134	0.00848	0.00489		–	–	–	–
0.5	0.01598	0.01303	0.01038	0.00751	0.00416		–	–	–	–	–
0.6	0.01273	0.00964	0.00676	0.00361		–	–	–	–	–	–
0.7	0.00944	0.00631	0.00325		–	–	–	–	–	–	–
0.8	0.00609	0.00301		–	–	–	–	–	–	–	–
0.9	0.00285		–	–	–	–	–	–	–	–	–
1		–	–	–	–	–	–	–	–	–	–

Table 10: Some values of the probabilities $CL^{\infty}_{\mathrm{PAV}_{1ii}}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.01076	0.00891	0.00755	0.00647	0.00557	0.00477	0.00403	0.00332	0.00258	0.00175	–
0.1	0.00962	0.00779	0.00646	0.00539	0.00446	0.00362	0.00281	0.00198	0.00105	–	–
0.2	0.00846	0.00668	0.00537	0.00431	0.00337	0.00250	0.00165	0.00077	–	–	–
0.3	0.00729	0.00557	0.00429	0.00324	0.00232	0.00145	0.00063	–	–	–	–
0.4	0.00611	0.00446	0.00323	0.00223	0.00134	0.00055	–	–	–	–	–
0.5	0.00493	0.00337	0.00222	0.00128	0.00051	–	–	–	–	–	–
0.6	0.00376	0.00232	0.00128	0.00048	–	–	–	–	–	–	–
0.7	0.00262	0.00135	0.00048	–	–	–	–	–	–	–	–
0.8	0.00155	0.00051	–	–	–	–	–	–	–	–	–
0.9	0.00059	–	–	–	–	–	–	–	–	–	–
1	–	–	–	–	–	–	–	–	–	–	–

From Tables 9, 10, it comes that PAV is more likely to select the Condorcet loser when Rule 1i applies. As all the events described in Theorems 6 and 7 are mutually exclusive, we then derive in Corollary 2, the overall probability that the Condorcet loser is elected by PAV.

Corrolary 2. *With three candidates and an infinite number of voters, the probability that PAV elects the Condorcet loser under EIC assumption is given by*

$$CL^{\infty}_{PAV}(k_{34}, k_5) = CL^{\infty}_{PAV_{1i}}(k_{34}, k_5) + CL^{\infty}_{PAV_{1ii}}(k_{34}, k_5)$$

Comparing the figures of Table 8 and 11, it comes that PAV seems to be less likely to elect the Condorcet loser than AV for $k_5 \geq 0.1$. When there is no voter with total indifference ($k_5 = 0$), AV is less likely to elect the Condorcet loser than PAV.

5. Conclusion

In this paper, we focused on the *Preference Approval Voting* which is a rule that combines approval and preferences. Under the extended impartial culture (EIC) assumption, we provided representations of the probability that PAV elects the Condorcet winner when she exists in three-candidate elections under all the subcases that PAV can induce then we derive the overall probability. It comes that PAV is more likely to select the Condorcet winner when more candidates receive a majority of approval votes (Rule 2 applies). In each of the subcases, we noticed that for any fixed number of voters with dichotomous preferences the probabilities tend to increase as the number of voters with

Table 11: Some values of the probabilities $CL^{\infty}_{\text{PAV}}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.04210	0.03791	0.03490	0.03253	0.03056	0.02883	0.02723	0.02568	0.02410	0.02231	–
0.1	0.03801	0.03376	0.03059	0.02794	0.02552	0.02314	0.02057	0.01734	0.01234	–	–
0.2	0.03388	0.02951	0.02615	0.02324	0.02035	0.01732	0.01364	0.00858	–	–	–
0.3	0.02964	0.02521	0.02166	0.01842	0.01510	0.01134	0.00663	–	–	–	–
0.4	0.02531	0.02085	0.01711	0.01357	0.00982	0.00544	–	–	–	–	–
0.5	0.02091	0.01640	0.01260	0.00879	0.00467	–	–	–	–	–	–
0.6	0.01649	0.01196	0.00804	0.00409	–	–	–	–	–	–	–
0.7	0.01206	0.00766	0.00373	–	–	–	–	–	–	–	–
0.8	0.00764	0.00352	–	–	–	–	–	–	–	–	–
0.9	0.00344	–	–	–	–	–	–	–	–	–	–
1	–	–	–	–	–	–	–	–	–	–	–

total indifference increases (the number of voters with strict ranking decreases). Comparing the Condorcet efficiency of PAV to that of AV, we noticed that when the proportion of voters with total indifference is at least equal to 70% of the electorate, AV is more likely to elect the Condorcet winner than PAV.

We also provided under EIC, a representation of the probability that AV elect the Condorcet loser when she exists in three-candidate elections. For any fixed number of voters with dichotomous preferences this probability tend to decrease as the number of voters with total indifference increases (the number of voters with strict ranking decreases). We get the same picture with PAV. It comes that PAV is more likely to select the Condorcet loser when no candidate receives a majority of approval votes (Rule 1i applies). We noticed that PAV seems to be less likely to elect the Condorcet loser than AV when the proportion of voter with total indifference is greater or equal to 1% of the electorate; when there is no voter with total indifference, AV is less likely to elect the Condorcet loser than PAV.

Appendix

Here, we will only focus on the proof of Theorems 1 and 2 since all the other Theorems follow the same schemes.

Appendix A: Proof of Theorem 1

A situation under which AV elects the Condorcet winner is fully described by Equations 1 to 4. In order to get a representation of the Condorcet efficiency of AV, we follow the same technique as Gehrlein and Fishburn (1978a). So, considering each of Equations 1 to 4, we define the following four discrete variables:

$$X_1 = \begin{array}{rl} 1 & : p_1 + p_2 + p_5 + p_7 + p_8 + p_{11} + p_{14} + p_{16} \\ -1 & : p_3 - p_4 + p_6 + p_9 + p_{10} + p_{12} p_{15} + p_{17} \\ 0 & : p_{13} + p_{18} + p_{19} \end{array}$$

$$X_2 = \begin{array}{rl} 1 & : p_1 + p_2 + p_3 + p_7 + p_8 + p_9 + p_{13} + p_{16} \\ -1 & : p_4 + p_5 + p_6 + p_{10} + p_{11} + p_{12} + p_{15} + p_{18} \\ 0 & : p_{14} + p_{17} + p_{19} \end{array}$$

$$X_3 = \begin{array}{rl} 1 & : p_1 + p_2 + p_8 + p_{11} + p_{14} + p_{16} \\ -1 & : p_3 + p_4 + p_{10} + p_{12} + p_{15} + p_{17} \\ 0 & : p_5 + p_6 + p_7 + p_9 + p_{13} + p_{18} + p_{19} \end{array}$$

$$X_4 = \begin{array}{rl} 1 & : p_1 + p_2 + p_7 + p_9 + p_{13} + p_{16} \\ -1 & : p_5 + p_6 + p_{10} + p_{12} + p_{15} + p_{18} \\ 0 & : p_3 + p_4 + p_8 + p_{11} + p_{14} + p_{17} + p_{19} \end{array}$$

where p_i is the probability that a voter who is randomly selected from the electorate is associated with the i^{th} ranking of Table 1; $X_1 > 0$ indicates that a is preferred to b and $X_1 < 0$ indicates the reverse. Similarly, $X_2 > 0$ indicates that a is preferred to c. X_3 and X_4 respectively represent $S(a) - S(b)$ and $S(a) - S(c)$. Equations 1 to 4 fully describe a situation under which AV elects the Condorcet winner when the average value $\overline{X}_j$ of each of the X_j (for $j = 1, 2, 3, 4$) are positive. According the Gehrlein and Fishburn (1978a,b), the probability of such a situation is equal to the joint probability $\overline{X}_1 > 0$, $\overline{X}_2 > 0$, $\overline{X}_3 > 0$ and $\overline{X}_4 > 0$; when $n \to \infty$, it is equivalent to the quadrivariate normal positive orthant probability $\Phi(R_4)$ such that $\overline{X}_j\sqrt{n} \geq E(\overline{X}_j\sqrt{n})$ and R_4 is a correlation matrix between the variables X_j. The expectation value of X_j is $E(X_j) = 0$, the variances ($V(X_j) = E(X_j^2)$) and covariances ($Cov(X_j, X_k) = E(X_j X_k)$) are:

$$\begin{aligned}
V(X_1) &= V(X_2) = \frac{3k_1 + 3k_2 + 2k_3 + 2k_4}{3} = \frac{3 - 3k_5 - k_{34}}{3} \\
V(X_3) &= V(X_4) = \frac{2(k_1 + k_2 + k_3 + k_4)}{3} = \frac{2(1 - k_5)}{3} \\
Cov(X_1, X_2) &= Cov(X_1, X_4) = Cov(X_2, X_3) = Cov(X_3, X_4) = \frac{1 - k_5}{3} \\
Cov(X_1, X_3) &= Cov(X_2, X_4) = 2Cov(X_1, X_2)
\end{aligned}$$

We derive the correlation matrix R_4 where the components r_{jk} are $r_{jk} = r_{kj} = \frac{Cov(X_j,X_k)}{\sqrt{V(X_j)V(X_k)}}$:

$$R_4 = \begin{pmatrix} 1 & \frac{1-k_5}{3-3k_5-k_{34}} & \sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} & \sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} \\ & 1 & \sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} & \sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{pmatrix}$$

Gehrlein (1979) has developed a general representation of the orthant probabilities for obtaining numerical values of $\Phi(R_4)$ as a function of a series of bounded integrals over a single variable[10]. Given r_{jk} the correlation terms in the matrix R_4, this general representation is defined as follows:

$$\begin{aligned}
f(r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}) &= \frac{1}{16} + \frac{\arcsin(r_{12}) + \arcsin(r_{13}) + \arcsin(r_{23})}{8\pi} \\
&+ \frac{r_{14}}{4\pi^2}\left[\int_0^1 \frac{\arccos\left(\frac{r_{24}r_{34}z^2 - r_{13}r_{14}r_{24}z^2 + r_{12}r_{13} + r_{14}^2 r_{23}z^2 - r_{12}r_{14}r_{34}z^2 - r_{23}}{\sqrt{(1-r_{14}^2z^2 - r_{13}^2 - r_{34}^2z^2 + 2r_{13}r_{14}r_{34}z^2)(1 - r_{24}^2z^2 - r_{12}^2 - r_{14}^2z^2 + 2r_{12}r_{14}r_{24}z^2)}}\right)}{\sqrt{1 - r_{14}^2 z^2}} dz\right] \\
&+ \frac{r_{24}}{4\pi^2}\left[\int_0^1 \frac{\arccos\left(\frac{r_{14}r_{34}z^2 - r_{14}r_{23}r_{24}z^2 + r_{12}r_{23} + r_{24}^2 r_{13}z^2 - r_{12}r_{24}r_{34}z^2 - r_{13}}{\sqrt{(1-r_{24}^2z^2 - r_{23}^2 - r_{34}^2z^2 + 2r_{23}r_{24}r_{34}z^2)(1 - r_{24}^2z^2 - r_{12}^2 - r_{14}^2z^2 + 2r_{12}r_{14}r_{24}z^2)}}\right)}{\sqrt{1 - r_{24}^2 z^2}} dz\right] \\
&+ \frac{r_{34}}{4\pi^2}\left[\int_0^1 \frac{\arccos\left(\frac{r_{14}r_{24}z^2 - r_{14}r_{23}r_{34}z^2 + r_{13}r_{23} + r_{34}^2 r_{12}z^2 - r_{13}r_{24}r_{34}z^2 - r_{12}}{\sqrt{(1-r_{24}^2z^2 - r_{23}^2 - r_{34}^2z^2 + 2r_{23}r_{24}r_{34}z^2)(1 - r_{14}^2z^2 - r_{13}^2 - r_{34}^2z^2 + 2r_{13}r_{14}r_{34}z^2)}}\right)}{\sqrt{1 - r_{34}^2 z^2}} dz\right]
\end{aligned}$$

So, $\Phi(R_4) = f(r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34})$. We then gets $CE^{\infty}_{\text{AV}}(k_{34}, k_5) = 3\left(\frac{\Phi(R_4)}{P^{\infty}_{Con}}\right)$.

Appendix B: Proof of Theorem 2

A voting situation under which Rule 1i applies and that PAV elects the Condorcet winner is fully described by Equations 1 to 5. We proceed as in Appendix A by defining for each equation, a discrete variable. As we have already defined the discrete variables X_1, X_2, X_3 and X_4 for Equations 1 to 4, it remains for us to define the discrete variable X_5 associated with Equation 5.

$$X_5 = \begin{array}{rl} 1 & : p_3 + p_4 + p_5 + p_6 + p_{10} + p_{12} + p_{15} + p_{17} + p_{18} \\ -1 & : p_1 p_2 + p_7 + p_8 + p_9 + p_{11} + p_{13} + p_{14} + p_{16} + p_{19} \end{array}$$

[10]Notice that $\Phi(R_4)$ can also be obtained following the works of David and Mallows (1961).

Equations 1 to 5 fully describe a situation under which AV elects the Condorcet winner when the average value $\overline{X}_j$ of each of the X_j (for $j = 1, 2, 3, 4, 5$) are positive. According the Gehrlein and Fishburn (1978a,b), the probability of such a situation is equal to the joint probability $\overline{X}_1 > 0$, $\overline{X}_2 > 0$, $\overline{X}_3 > 0$, $\overline{X}_4 > 0$ and $\overline{X}_5 > 0$; when $n \to \infty$, it is equivalent to the quadrivariate normal positive orthant probability $\Phi(R_5)$ such that $\overline{X}_j \sqrt{n} \geq E(\overline{X}_j \sqrt{n})$ and R_5 is a correlation matrix between the variables X_j. It remains for us to compute the following variances and covariances:

$$\begin{aligned} V(X_5) &= 1 \\ Cov(X_1, X_5) &= Cov(X_2, X_5) = Cov(X_3, X_5) = Cov(X_4, X_5) = -2Cov(X_1, X_2) \end{aligned}$$

We derive the correlation matrix R_5:

$$R_5^{1i} = \begin{pmatrix} 1 & \frac{1-k_5}{3-3k_5-k_{34}} & -\frac{2\sqrt{3}}{3}\left(\frac{(1-k_5)\sqrt{3-3k_5-k_{34}}}{3-3k_5-k_{34}}\right) & \sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} & \sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} \\ & 1 & -\frac{2\sqrt{3}}{3}\left(\frac{(1-k_5)\sqrt{3-3k_5-k_{34}}}{3-3k_5-k_{34}}\right) & \sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} & \sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} \\ & & 1 & -\sqrt{\frac{2(1-k_5)}{3}} & -\sqrt{\frac{2(1-k_5)}{3}} \\ & & & 1 & \frac{1}{2} \\ & & & & 1 \end{pmatrix}$$

where $\varphi = \frac{2\sqrt{3}}{3}\left(\frac{x\sqrt{y}}{y}\right)$.

Based on the Boole's Theorem, Gehrlein (2017, 2014) developed a general representation of the orthant probabilities for obtaining numerical values of $\Phi(R_5)$ as a linear combination of $\Phi(R_4)$ values, where matrices R_4 are obtained from correlation terms in R_5.

$$\begin{aligned} \Phi(R_5) = \frac{1}{2}\Big(& f(r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}) - f(r_{12}, r_{13}, -r_{15}, r_{23}, -r_{25}, -r_{35}) + f(r_{12}, -r_{14}, -r_{15}, -r_{24}, -r_{25}, r_{45}) \\ & - f(-r_{13}, -r_{14}, -r_{15}, r_{34}, r_{35}, r_{45}) + f(r_{23}, r_{24}, r_{25}, r_{34}, r_{35}, r_{45})\Big) \end{aligned}$$

Then we get $CE^{\infty}_{\mathrm{PAV}_{1i}}(k_{34}, k_5) = 3\left(\frac{\Phi(R_5^{1i})}{P^{\infty}_{Con}}\right)$.

References

Brams S. (2008) Mathematics and Democracy : Designing Better Voting and Fair-Division Procedures. Princeton University Press.

Brams S.J. and Fishburn P.C. (2007) Approval Voting. Springer-Verlag.

Brams S.J. and Fishburn P.C. (1978) Approval Voting. American Political Science Review 72 (3): 831-847

Brams S.J., Fishburn P.C. and Merril S. III (1988a) The responsiveness of approval voting: Comments on Saari and van Newenhizen. Public Choice 59(2): 121-131.

Brams S.J., Fishburn P.C. and Merril S. III (1988b) Rejoinder to Saari and van Newenhizen. Public Choice 59(2): 149.

Brams S.J. and Sanver R (2009) Voting systems that combine approval and preferences. In: Brams SJ, Gehrlein WV, Roberts FS (eds) The mathematics of preference, choice and order. Springer, Berlin, pp 215-237.

David F.N and Mallows C.L (1961) The variance of Spearman's rho in normal samples. Biometrika 48:19-28.

Diss M., Merlin, V., and Valognes F. (2010) On the condorcet efficiency of approval voting and extended scoring rules for three alternatives. In: Laslier J-F, Sanver RM (eds) Handbook on approval voting. Springer, Berlin, pp 255-283.

Gehrlein W.V. (2017) Computing Multivariate Normal Positive Orthant Probabilities with 4 and 5 Variables. Technical report. Available at `https://www.researchgate.net/publication/320467212_Computing_Multivariate_Normal_Positive_Orthant_Probabilities_with_4_and_5_Variables`.

Gehrlein W.V. (2004) The Effectiveness of Weighted Scoring Rules when Pairwise Majority Rule Cycles Exist. Mathematical Social Sciences 47: 69-85.

Gehrlein W.V. (1979) A representation for quadrivariate normal positive orthant probabilities. Communications in Statistics 8: 349-358.

Gehrlein, W.V. and Fishburn P.C. (1980) Robustness of positional scoring over subsets of alternatives. Applied Mathematics and Optimization 6: 241-255.

Gehrlein W.V. and Fishburn P.C. (1976) The probability of the paradox of voting: A computable solution. Journal of Economic Theory, 13: 14-25.

Gehrlein W.V. and Fishburn P.C. (1978a) Coincidence probabilities for simple majority and positional voting rules. Soc Sci Res 7:272-283.

Gehrlein W.V. and Fishburn P.C. (1978b) Probabilities of election outcomes for large electorates. Journal of Economic Theory 19(1): 38-49.

Gehrlein W.V. and Lepelley D. (2015) The Condorcet Efficiency Advantage that Voter Indifference Gives to Approval Voting Over Some Other Voting Rules. Group Decis Negot 24:243-269

Gehrlein W.V. and Lepelley D. (1998) The Condorcet efficiency of approval voting and the probability of electing the Condorcet loser. Journal of Math Econ 29:271-283.

Gehrlein W.V, Moyouwou I. and Lepelley D. (2016)A note on Approval Voting and electing the Condorcet loser. Mathematical Social Sciences 80:115-122.

Ju B-G (2010) Collective choices for simple preferences. In: Laslier J-F, Sanver MR (eds) Handbook on Approval Voting. Studies in Choice and Welfare, Springer-Verlag Berlin Heidelberg, pp 41-90

Laslier J-F. and Sanver R.M (2010) Handbook on Approval Voting. Studies in Choice and Welfare. Springer Berlin Heidelberg.

Lepelley D. (1993) On the probability of electing the Condorcet loser. Mathematical social Sciences 25: 105-116.

Saari D. G. and van Newenhizen J. (1988a) The problem of indeterminancy in approval, multiple and truncated voting systems. Public Choice 59(2):101-120

Saari D. G. and van Newenhizen J. (1988b) Is approval voting an unmitigated evil? A response to Brams, Fishburn and Merril. Public Choice 59(2): 133-147.

Xu Y. (2010) Axiomatizations of approval voting. In: Laslier J-F, Sanver MR (eds) Handbook on Approval Voting. Studies in Choice and Welfare, Springer-Verlag Berlin Heidelberg, pp 91-102