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On the Likelihood of the Borda Effect: The Overall Probabilities for General Weighted Scoring Rules and Scoring Runoff Rules

Eric Kamwa

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Abstract The *Borda Effect*, first introduced by Colman and Poutney (1978), occurs in a preference aggregation process using the Plurality rule if given the (unique) winner there is at least one loser that is preferred to the winner by a majority of the electorate. Colman and Poutney (1978) distinguished two forms of the Borda Effect: -the *Weak Borda Effect* describing a situation under which the unique winner of the Plurality rule is majority dominated by only one loser; and -the *Strong Borda Effect* under which the Plurality winner is majority dominated by each of the losers. The *Strong Borda Effect* is well documented in the literature as the Strong Borda Paradox. Colman and Poutney (1978) showed that the probability of the *Weak Borda Effect* is not negligible; they only focused on the Plurality rule. In this note, we extend the work of Colman and Poutney (1978) by providing in three-candidate elections, the representations for the limiting probabilities of the *(Weak) Borda Effect* for the whole family of the scoring rules and scoring runoff rules. We highlight that there is a relation between the *(Weak) Borda Effect* and the Condorcet efficiency. We perform our analysis under the Impartial Culture and the Impartial Anonymous Culture which are two well-known assumptions often used for such a study.

Keywords Borda Effect · Rankings · Scoring rules · Probability · Impartial Culture · Impartial and Anonymous Culture

E Kamwa
LC2S UMR CNRS 8053 and Université des Antilles
Faculté de Droit et d'Economie de la Martinique
Campus de Schoelcher, F-97275 Schoelcher Cedex
E-mail: eric.kamwa@univ-antilles.fr

1 Introduction

Borda (1781) and Condorcet (1785) who were members of the Paris Royal Academy of Sciences, proposed alternative voting rules to the one that was in use in the academy. With $m \geq 3$ candidates, the *Borda rule* gives $m-k$ points to a candidate each time she is ranked k^{th} by a voter and the winner is the candidate with highest total number of points. This rule belongs to the family of scoring rules containing all the voting systems under which candidates receive points according to the position they have in voters' rankings and the total number of points received by a candidate defines her score; the winner is the candidate with the highest score. The well-known scoring rules are the Plurality rule, the Borda rule and the Antiplurality rule. Under the Antiplurality rule, the winner is the candidate with the fewest number of last places in the voters' rankings. Condorcet (1785) criticized the *Borda* rule in that it can exist a candidate that is preferred by more than half of the electorate to the Borda winner. Condorcet (1785) proposed the *Pairwise Majority Rule* based on pairwise comparisons[1]. According to this rule, a candidate should be declared the winner if she beats all the other candidates in pairwise comparisons; such a candidate is called the *Condorcet winner*. Nonetheless, the *Condorcet principle* has a main drawback: the Condorcet winner does not always exists and one can end with *majority cycles*.

Borda (1784) showed that for a given voting rule, the Plurality rule can elect the *Condorcet loser*, a candidate who loses all his pairwise comparisons. The Borda-Condorcet debate just emphasized the fact that the Pairwise Majority Rule may not agree with the *scoring rules*. The possible disagreements gave rise to the definition of the following phenomena or voting paradoxes: *i)* the *Strong Borda Paradox* which occurs when a scoring rule elects the Condorcet loser when she exists; *ii)* the *Strict Borda Paradox* occurs when the collective rankings of a scoring is completely the reversal of that of the PMR; and *iii)* the *Weak Borda Paradox* in which a scoring rule reverses the ranking of the PMR on some pairs of candidates without necessarily electing the Condorcet loser; in other words, this paradox occurs if given that there is a Condorcet loser, she is not ranked last by the scoring rule. The study of the likelihood of each of these three paradoxes is well addressed in the social choice literature. Without been exhaustive, the reader may refer to the theoretical works of Diss and Gehrlein (2012), Diss and Tlidi (2018), Gehrlein and Fishburn (1976, 1978a), Gehrlein and Lepelley (2017, 2011, 2010b, 1998), Lepelley (1996, 1993), Lepelley et al. (2000a,b), Saari (1994), Saari and Valognes (1999), Tataru and Merlin (1997). We can also mention that there are some empirical works that looked after these paradoxes in real-world data; we refer to the works of Bezembinder (1996), Colman and Poutney (1978), Riker (1982), Taylor (1997), Van Newenhizen (1992), Weber (1978). A summary on the results of these empirical researches can be found in Gehrlein and Lepelley (2011, p.15).

In addition to the just listed declensions of the Borda Paradox, we find the less-known *Borda Effect* first introduced by Colman and Poutney (1978). Colman and Poutney (1978) distinguished the *Strong Borda Effect* and the *Weak Borda Effect*: the *Strong Borda Effect* describes a situation in which the Plurality rule elect the Condorcet loser while the *Weak Borda Effect* is related to a situation under which the Plurality winner is majority dominated by only one of the Plurality losers. As one can notice, the *Strong Borda Effect* is equivalent to the *Strong Borda Paradox*. The *Weak Borda Effect* is a little bit special and subtle. If one is not careful one can misunderstand this phenomenon and therefore end with a bad evaluation of its occurrence. This is indeed what happened to Gillett (1986, 1984).

Gillett (1984) criticized Colman (1980) of misusing the *Weak Borda Effect* as an indicator of the likelihood of the Plurality rule to produce an outcome inconsistent with the wishes of the majority. Then he showed that the likelihood of the Weak Borda Effect provides an inadequate, poor and misleading index of the propensity of the Plurality/Majority disagreement. Colman (1984) replied that this criticism is based on a misunderstanding as he "...had proposed it (the Weak Borda Effect) not as an index of Plurality-majority disagreement, but rather as an index of the propensity of the Plurality voting procedure to select a unique winner when a majority of a committee or an electorate... prefer one of the defeated alternatives to the plurality winner". This misunderstanding

[1] See Young (1988) for a modern interpretation of Condorcet's rule.

appears clearly in the introduction of Gillett (1986) where one can read what follows: "The *Weak Borda Effect* refers to a situation which can occur under the plurality voting system whereby *at least one* of the losing candidates is preferred to the winning candidate by a simple majority of the voters...". As one can notice, this definition refers to the overall *Borda Effect*. This misunderstanding obviously led Gillet to question the probability of the *Weak Borda Effect* calculated by Colman (1980). Colman (1986) cleverly fixed all the misunderstandings and criticisms of Gillett (1986, 1984).

Colman and Poutney (1978, p.17) reported for three-candidate elections, the exact probabilities of the *Strong Borda Effect* and the *Weak Borda Effect* for groups of voters ranging in size from 7 to 301. According to their results "... the smallest committee size in which the Strong or Weak Borda Effect can occur is seven, probabilities 0.018 and 0.126, respectively. In a committee of eight members it is useful to know that the effect cannot occur, but in groups of nine or more there is a significant probability of its occurrence. The likelihood of the strong and weak effects tends to rise as the number of voters increases until with 301 voters the probabilities are 0.029 and 0.276, respectively,with no obvious asymptote in sight...". With the use of survey data regarding voters' preference rankings, Colman and Poutney (1978) found the occurrence of the *Borda Effect* in fifteen instances out of 261 three-cornered contests in the results of the 1966 British General Election. A similar experiment was conducted by Nurmi and Suojanen (2004).

As their analysis was only focused on the Plurality rule, the results of Colman and Poutney (1978, p.17) are quite limited in the scope as the Borda Effect can also be observed with all the scoring rules and scoring runoff rules. Up to our knowledge, apart from Colman and Poutney (1978) no other paper has investigated the *Weak Borda Effect* under other scoring rules nor under scoring runoff rules. The main objective of this paper is to fill this gap in the literature by providing, for three-candidate elections, the representation overall limiting probabilities for general weighted scoring rules and scoring runoff rules. We show that these representation can be deduced from the well-known results on the likelihood of the Strong Borda Paradox and on the Condorcet efficiency. The Condorcet efficiency of a voting procedure is the conditional probability that it will elect the Condorcet winner, given that a Condorcet winner exists. We perform our analysis under the Impartial Culture (IC) and the Impartial Anonymous Culture (IAC) which are two well-known assumptions under which such a study is often driven in the social choice literature. These assumptions are defined in Section 2.4.

The rest of the paper is structured as follows: Section 2 is devoted to basic notations and definitions. Section 3 presents our results. Section 4 concludes.

2 Preliminaries

2.1 Preferences

Let N be a set of n voters ($n \geq 2$) and A a set of m candidates ($m \geq 3$). Individual preferences are linear orders, these are complete, asymmetric and transitive binary relations on A. With m candidates, there are exactly $m!$ linear orders $P_1,\ P_2,\ ,\ldots, P_{m!}$ on A. A *voting situation* is an $m!$-tuple $\pi = (n_1, n_2, ..., n_t, ..., n_{m!})$ that indicates the total number n_t of voters casting each complete linear order $P_t, t = 1, 2, \ldots, m!$ in such a way that $\sum_{t=1}^{m!} n_t = n$. In the sequel, we consider three candidates a, b and c. In this case, we will simply write abc to denote the linear order on A according to which a is strictly preferred to b, b is strictly preferred to c; and by transitivity a is strictly preferred to c. Table 1 describes a voting situation with three candidates: there are six preference types and for $t = 1, 2, \ldots, 6$, n_t is the total number of voters having type t.

Table 1 possible strict rankings on $A = \{a, b, c\}$

$n_1 : abc$	$n_2 : acb$	$n_3 : bac$
$n_4 : bca$	$n_5 : cab$	$n_6 : cba$

Table 2 Scores of candidates

$S(\pi, \lambda, a) = n_1 + n_2 + \lambda(n_3 + n_5)$
$S(\pi, \lambda, b) = n_3 + n_4 + \lambda(n_1 + n_6)$
$S(\pi, \lambda, c) = n_5 + n_6 + \lambda(n_2 + n_4)$

Given $a, b \in A$ and a voting situation π , we denote by $n_{ab}(\pi)$ (or simply n_{ab}) the total number of voters who strictly prefer a to b. If $n_{ab} > n_{ba}$, we say that a majority dominates candidate b; or equivalently, a beats b in a pairwise majority voting. In such a case, we will simply write $a\mathbf{M}b$. Candidate a is said to be the *Condorcet winner* (resp. the *Condorcet loser*) if for all $b \in A \setminus \{a\}$, $a\mathbf{M}b$ (resp. $b\mathbf{M}a$). If for a given voting situation we get $a\mathbf{M}b$, $b\mathbf{M}c$ and $c\mathbf{M}a$, this describes a majority cycle.

2.2 Voting rules

Scoring rules are voting systems that give points to candidates according to the position they have in voters' ranking. For a given scoring rule, the total number of points received by a candidate defines her score for this rule. The winner is the candidate with the highest score. In general, with $m \geq 3$ and complete strict rankings, a scoring vector is an m-tuple $w = (w_1, w_2, ..., w_k, ..., w_m)$ of real numbers such that $w_1 \geq w_2 \geq ... \geq w_k \geq ... \geq w_m$ and $w_1 > w_m$. Given a voting situation π, each candidate receives w_k each time she is ranked k^{th} by a voter. The score of a candidate $a \in A$ is the sum $S(\pi, w, a) = \sum_{t=1}^{m!} n_t w_{r(t,a)}$ where $r(t, a)$ is the rank of candidate a according to voters of type t.

For uniqueness, we use the normalized form $(1, \frac{w_2 - w_m}{w_1 - w_m}, ..., \frac{w_k - w_m}{w_1 - w_m}, ..., 0)$ of each scoring vector w. With three candidates, a normalized scoring vector has the shape $w_\lambda = (1, \lambda, 0)$ with $0 \leq \lambda \leq 1$. For $\lambda = 0$, we obtain the *Plurality rule*. For $\lambda = 1$, we have the *Antiplurality rule* and for $\lambda = \frac{1}{2}$, we have the *Borda rule*. From now on, we will denote by $S(\pi, \lambda, a)$, the score of candidate a when the scoring vector is $w_\lambda = (1, \lambda, 0)$ and the voting situation is π; without loss of generality, w_λ will be used to refer to the voting rule. Table 2 gives the score of each candidate in $A = \{a, b, c\}$ given the voting situation of Table 1.

If for a given λ, candidate a scores better than candidate b, we denotes it by $a\mathbf{S}_\lambda b$. In one-shot voting, the winner is the candidate with the largest score. Runoff systems involve two rounds of voting: at the *first round*, the candidate with the smallest score is eliminated. At the *second round* , a majority contest determines who is the winner. Without loss of generality, we will denote by w_{λ^r} the runoff rule under which w_λ is used at the first stage. Runoff systems are widely used in the real world: in France, it is used for presidential, legislative and departmental elections; it is used for presidential elections in many other countries (Finland, Argentina, Austria, Egypt, etc) and organization such as the International Olympic Committee to designate the host city of the Olympic Games.

2.3 The Borda-likewise effects

Consider Tables 1 and 2 and let us assume that candidate a is the winner for the one-shot scoring rule w_λ. This means that $a\mathbf{S}_\lambda b$ and $a\mathbf{S}_\lambda c$. In such a case, we get the *Strong Borda Paradox* or the *Strong Borda Effect* if $b\mathbf{M}a$ and $c\mathbf{M}a$: candidate a is the Condorcet loser and she is elected by w_λ. If $b\mathbf{M}a$, $c\mathbf{M}a$, $b\mathbf{M}c$ and $c\mathbf{S}_\lambda b$ the collective ranking of w_λ is acb while that of the Pairwise Majority rule is bca; this defines the *Strict Borda Paradox*. If there is a Condorcet loser and she is not ranked last by w_λ, we get the *Weak Borda Paradox*. The *Weak Borda effect* happens if there is a unique candidate $x \in A \setminus a$ ($x = b$ or $x = c$) such that $x\mathbf{M}a$.

With runoff systems, it is obvious that the *Strong Borda Paradox* and the *Strict Borda Paradox* never occur for all λ; but this can be the case for the *Weak Borda Paradox* and the *Weak Borda Effect*. Under these rules, the *Borda Effect* is just equivalent to the *Weak Borda effect*.

2.4 The probability models

As stated in Section 1, the likelihood of each of the declensions of the Borda paradox is well addressed in the social choice literature. Most of the time, the probability are obtained by assuming the Impartial Culture hypothesis (IC) or that of the Impartial and Anonymous Culture (IAC).

Under IC, it is assumed that each voter chooses her preference according to a uniform probability distribution and it gives a probability $\frac{1}{m!}$ for each ranking to be chosen independently. The likelihood of a given voting situation $\tilde{n} = (n_1, n_2, ..., n_t, ..., n_{m!})$ is given by $Prob(\tilde{n}) = \frac{n!}{\prod_{i=1}^{m!} n_i!} \times (m!)^{-n}$.

Under IAC, first introduced by Gehrlein and Fishburn (1976), the likelihood of a given event is calculated in respect with the ratio between the number of voting situations in which the event is likely over the total number of possible voting situations. It is known that the total number of possible voting situations in three-candidate elections is given by the following five-degree polynomial in n: $C^n_{n+3!-1} = \frac{(n+5)!}{n!5!}$. The number of voting situations associated with a given event can be reduced to the solutions of a finite system of linear constraints with rational coefficients. As recently pointed out in the social choice literature, the appropriate mathematical tools to find these solutions are the Ehrhart polynomials. The background of this notion and its connection with the polytope theory can be found in Gehrlein and Lepelley (2017, 2011), Lepelley et al. (2008), and Wilson and Pritchard (2007). This technique has been widely used in numerous studies analyzing the probability of electoral events in the case of three-candidate elections under the IAC assumption.

3 Likelihood of the weak Borda effect in three-candidate elections

Colman and Poutney (1978, p.17) reported for three-candidate elections, the exact probabilities of the Weak Borda Effect for groups of voters ranging in size from 7 to 301. Their calculations were performed under the IC hypothesis. For three-candidate elections, we provide representations for the limiting probabilities of the Weak Borda Effect for the whole family of the scoring rules and scoring runoff rules under IC and IAC.

3.1 Representations for the limiting probability for one-shot scoring rules

Given a voting situation on $A = \{a, b, c\}$ and $w_\lambda = (1, \lambda, 0)$, we denote by $P(a; b\mathbf{M}a)$ the probability of the situation described by the following inequalities:

$$\begin{cases} a\mathbf{S}_\lambda b \\ a\mathbf{S}_\lambda c \\ b\mathbf{M}a \end{cases} \Leftrightarrow \begin{cases} (1-\lambda)n_1 + n_2 + (\lambda-1)n_3 - n_4 + \lambda n_5 - \lambda n_6 > 0 \\ n_1 + (1-\lambda)n_2 + \lambda n_3 - \lambda n_4 + (\lambda-1)n_5 - n_6 > 0 \\ -n_1 - n_2 + n_3 + n_4 - n_5 + n_6 > 0 \end{cases} \tag{1}$$

Also, we denote by $P(a; b\mathbf{M}a; c\mathbf{M}a)$ the probability of the situation under which the winner is beaten in pairwise comparisons by the two other candidates.

It follows that with three candidates, $P^\lambda_{\text{WBE}}(3, \infty, \hbar)$ the limiting probability of the Weak Borda Effect under assumption $\hbar$ is[2] given by:

$$\begin{aligned} P^\lambda_{\text{WBE}}(3, \infty, \hbar) &= 3\Big(P(a; b\mathbf{M}a) + P(a; c\mathbf{M}a) - P(a; b\mathbf{M}a; c\mathbf{M}a)\Big) \\ &= 6P(a; b\mathbf{M}a) - 3P(a; b\mathbf{M}a; c\mathbf{M}a) \\ &= 6P(a; b\mathbf{M}a) - P_c(3, \infty, \hbar) \times P^\lambda_{\text{SgBP}}(3, \infty, \hbar) \end{aligned} \tag{2}$$

with $P^\lambda_{\text{SgBP}}(3, \infty, \hbar)$ the conditional probability of the Strong Borda Paradox and P_c the probability that a Condorcet winner (or Condorcet loser) exists; $P^\lambda_{\text{BE}}(3, \infty, \hbar)$ the limiting probability of the Borda Effect under assumption $\hbar$ is given by:

$$\begin{aligned} P^\lambda_{\text{BE}}(3, \infty, \hbar) &= P^\lambda_{\text{WBE}}(3, \infty, \hbar) + P(a; b\mathbf{M}a; c\mathbf{M}a)\Big) \\ &= 6P(a; b\mathbf{M}a) \end{aligned} \tag{3}$$

[2] $\hbar$ stands here for IC or IAC.

Representations for P_c are known in the literature both under IC and IAC.

$$P_c(3,\infty,IC)=\frac{3}{4}+\frac{3}{2}\sin^{-1}(\frac{1}{3}) \qquad \text{and} \qquad P_c(3,\infty,IAC)=\frac{15}{16}$$

Some representations for $P^{\lambda}_{\text{SgBP}}(3,\infty,\hbar)$ are provided by Gehrlein and Fishburn (1978a),Tataru and Merlin (1997) and Cervone et al. (2005). Now, all we have to do is to find $P(a;b\mathbf{M}a)$.

3.1.1 Representation under IC

The representation of the conditional probability of the Strong Borda Paradox provided by Gehrlein and Fishburn (1978a) under IC is as follows:

$$P^{\lambda}_{\text{SgBP}}(3,\infty,IC)=\frac{3\Phi_4(R)}{P_c(3,\infty,IC)} \tag{4}$$

where

$$\Phi_4(R)=\frac{1}{9}-\frac{1}{4\pi}\left(\sin^{-1}\left(\sqrt{\frac{2}{3z}}\right)+\sin^{-1}\left(\sqrt{\frac{1}{6z}}\right)\right)+\frac{1}{4\pi^2}\left\{\left(\sin^{-1}\left(\sqrt{\frac{2}{3z}}\right)\right)^2-\left(\sin^{-1}\left(\sqrt{\frac{1}{6z}}\right)\right)^2 - \int_0^1\sqrt{\frac{1}{36-(3-t)^2}}\cos^{-1}\left(\frac{6tz-g(t,z)}{2g(t,z)}\right)dt\right\}$$

with $z=1-(\lambda(1-\lambda)$ and $g(t,z)=4(3z-2)^2-(3z-2-tz)^2+6(3z-2)$.

Another representation is provided by Tataru and Merlin (1997) as follows

$$P^{\lambda}_{\text{SgBP}}(3,\infty,IC)=\frac{3}{\pi^2P_c(3,\infty,IC)}\int_0^{2\lambda-1}\left[\frac{2t\cos^{-1}\left(\frac{\sqrt{9t^2+3}}{\sqrt{(t^2+3)(4t^2+1)}}\right)}{(t^2+3)\sqrt{6t^2+2}}+\frac{t\cos^{-1}\left(\frac{\sqrt{3}(1-t^2)}{\sqrt{(3t^2+1)(t^2+3)(4t^2+1)}}\right)}{(t^2+3)\sqrt{6t^2+14}}\right]dt \tag{5}$$

Let us find $P(a;b\mathbf{M}a)$ following the technique of Gehrlein and Fishburn (1976). To do so, we consider Eq. 1 and define the following three discrete variables:

$$X_1=\begin{array}{ll}1-\lambda & :p_1\\ 1 & :p_2\\ -1+\lambda & :p_3\\ -1 & :p_4\\ \lambda & :p_5\\ -\lambda & :p_6\end{array} \qquad X_2=\begin{array}{ll}1 & :p_1\\ 1-\lambda & :p_2\\ \lambda & :p_3\\ -\lambda & :p_4\\ -1+\lambda & :p_5\\ -1 & :p_6\end{array} \qquad X_3=\begin{array}{ll}-1 & :p_1\\ -1 & :p_2\\ 1 & :p_3\\ 1 & :p_4\\ -1 & :p_5\\ 1 & :p_6\end{array}$$

where p_i is the probability that a voter who is randomly selected from the electorate is associated with the i^{th} ranking of Table 1. Under IC, $p_i=\frac{1}{6}$. For $X_j>0$, this indicates that the j^{th} inequality of Eq. 1 is satisfied. With n voters, Eq. 1 fully describes the Weak Borda effect when the average value of each of the X_j are positive. According the Gehrlein and Fishburn (1978b), $P(a;b\mathbf{M}a)$ is equal to the joint probability that $\overline{X}_1>0$, $\overline{X}_2>0$ and $\overline{X}_3>0$; when $n\to\infty$, it is equivalent to the trivariate normal positive orthant probability $\Phi_3(R')$ such that $\overline{X}_j\sqrt{n}\geq E(\overline{X}_j\sqrt{n})$ and R' is a correlation matrix between the variables X_j. Thus $P(a;b\mathbf{M}a)=\Phi_3(R')$. In our case, R' is as follows

$$R'=\begin{bmatrix}1 & \frac{1}{2} & -\sqrt{\frac{2}{3z}}\\ & 1 & -\sqrt{\frac{1}{6z}}\\ & & 1\end{bmatrix}$$

Given the form of R', we can easily derive $\Phi_3(R')$ from the work of David and Mallows (1961):

$$\Phi_3(R')=\frac{1}{6}-\frac{1}{4\pi}\left(\sin^{-1}\left(\sqrt{\frac{2}{3z}}\right)+\sin^{-1}\left(\sqrt{\frac{1}{6z}}\right)\right) \tag{6}$$

Following Eq. 3, we derive Proposition 1.

Proposition 1. *For three-candidate elections and a scoring rule w_λ,*

$$\begin{aligned} P^\lambda_{WBE}(3,\infty,IC) &= 6\Phi_3(R') - 3\Phi_4(R) \\ &= 1 - P_c(3,\infty,IC) \times P^\lambda_{CE}(3,\infty,IC) \end{aligned}$$

$$P^\lambda_{BE}(3,\infty,IC) = 1 - P_c(3,\infty,IC)\Big(P^\lambda_{CE}(3,\infty,IC) - P^\lambda_{SgBP}(3,\infty,IC)\Big)$$

with $P^\lambda_{CE}(3,\infty,IC)$ the conditional probability that the winner is the Condorcet winner given that a Condorcet winner exists.

According to Proposition 1, the representations for the limiting probability of the Weak Borda and that of the Borda Effect under IC can be deduced from those of the Condorcet efficiency and of the Strong Borda Paradox wish are well documented in the literature.

Given that z is symmetric about $\lambda = 0.5$, it follows that $P^\lambda_{\text{WBE}}(3,\infty,IC) = P^{1-\lambda}_{\text{WBE}}(3,\infty,IC)$ and $P^\lambda_{\text{BE}}(3,\infty,IC) = P^{1-\lambda}_{\text{BE}}(3,\infty,IC)$. We report in Table 3, the computed values of the limiting probability of the (Weak) Borda Effect for $\lambda = 0(0.1)1$. For $0 \le \lambda \le \frac{1}{2}$, the probability tends to decrease and it increases for $\frac{1}{2} \le \lambda \le 1$. We find that the limiting probability is minimized by the Borda rule ($\lambda = \frac{1}{2}$) and it is maximized by the Plurality rule ($\lambda = 0$) and the Antiplurality rule ($\lambda = 1$).

Table 3 Computed values of the Borda effect under one-shot and scoring runoff rules

	One-shot rules						Runoff rules	
	Strong Borda effect		Weak Borda effect		Borda effect		Borda effect	
λ	IC	IAC	IC	IAC	IC	IAC	IC	IAC
0	0.0338	0.0277	0.3092	0.1736	0.3431	0.2014	0.1216	0.0920
0.1	0.0217	0.0180	0.2766	0.1582	0.2983	0.1762	0.1095	0.0834
0.2	0.0115	0.0098	0.2432	0.1447	0.2547	0.1545	0.0992	0.0752
0.3	0.0042	0.0039	0.2119	0.1350	0.2161	0.1389	0.0919	0.0683
0.4	0.0006	0.0006	0.1876	0.1328	0.1883	0.1336	0.0884	0.0637
0.5	0.0000	0.0000	0.1779	0.1458	0.1779	0.1458	0.0877	0.0625
0.6	0.0006	0.0012	0.1876	0.1825	0.1882	0.1837	0.0884	0.0632
0.7	0.0042	0.0057	0.2119	0.2335	0.2161	0.2392	0.0919	0.0664
0.8	0.0115	0.0127	0.2432	0.2913	0.2547	0.3040	0.0992	0.0724
0.9	0.0217	0.0209	0.2766	0.3513	0.2983	0.3722	0.1095	0.0805
1	0.0338	0.0295	0.3092	0.4097	0.3431	0.4392	0.1216	0.0903

3.1.2 Representation under IAC

Under IAC, when $n \to \infty$, we deduce $P_c(3,\infty,IAC) \times P^\lambda_{\text{SgBP}}(3,\infty,IAC)$ from the results of Cervone et al. (2005):

$$P_c(3,\infty,IAC) \times P^\lambda_{\text{SgBP}}(3,\infty,IAC) = \begin{cases} \frac{(2\lambda-1)^3(12-9\lambda-2\lambda^2)}{432(\lambda-1)^3} & \text{for } \ 0 \le \lambda \le \frac{1}{2} \\ \frac{(2\lambda-1)^3(2-53\lambda+331\lambda^2-88\lambda^3+12\lambda^4)}{1728\lambda^3(3\lambda-1)(\lambda+1)} & \text{for } \ \frac{1}{2} \le \lambda \le 1 \end{cases} \tag{7}$$

In order to find $P(a;b\mathbf{M}a)$, let us denote by $\mathcal{V}^\lambda_{ab}$ the set of all voting situations at which a is the winner given λ and he is majority dominated only by b. A profile $\pi \in \mathcal{V}^\lambda_{ab}$ implies that the inequalities of Eq. 1 are satisfied. Notice that as $n \to \infty$, $P(a;b\mathbf{M}a) = vol\,(P_{ab})$ the 5-dimensional volume of the polytope P_{ab} is obtained from the characterization of $\mathcal{V}^\lambda_{ab}$ just by replacing each n_j by $p_j = \frac{n_j}{n}$

in the simplex $\mathfrak{S} = \{(p_1, p_2, \ldots, p_6) : \sum_{t=1}^{6} p_j = 1 \text{ with } p_j \geq 0, j = 1, 2, \ldots, 6\}$. Given $0 \leq \lambda \leq 1$, computing $vol\,(P_{ab})$ leads to what follows[3]:

$$P(a; b\mathbf{M}a) = \begin{cases} \frac{58-221\lambda+276\lambda^2+29\lambda^3-328\lambda^4+213\lambda^5-20\lambda^6-8\lambda^7}{864(\lambda+1)(\lambda-1)^3(\lambda-2)} & \text{for } 0 \leq \lambda \leq \frac{1}{2} \\ \\ \frac{-2+37\lambda-318\lambda^2+890\lambda^3-910\lambda^4-246\lambda^5+280\lambda^6+16\lambda^7}{1728\lambda^3(\lambda-2)(\lambda+1)} & \text{for } \frac{1}{2} \leq \lambda \leq 1 \end{cases} \tag{8}$$

Following Eq. 3, we get Proposition 2.

Proposition 2. *For three-candidate elections and a scoring rule w_λ,*

$$\begin{aligned} P^{\lambda}_{WBE}(3, \infty, IAC) &= \begin{cases} \frac{150-513\lambda+529\lambda^2+194\lambda^3-692\lambda^4+367\lambda^5-28\lambda^6-8\lambda^7}{432(\lambda+1)(\lambda-2)(\lambda-1)^3} & \textit{for } \ 0 \leq \lambda \leq \frac{1}{2} \\ \\ \frac{8-126\lambda+1163\lambda^2-4939\lambda^3+8882\lambda^4-2416\lambda^5-11580\lambda^6+5984\lambda^7+192\lambda^8}{1728\lambda^3(3\lambda-1)(\lambda+1)(\lambda-2)} & \textit{for } \ \frac{1}{2} \leq \lambda \leq 1 \end{cases} \\ &= 1 - P_c(3, \infty, IAC) \times P^{\lambda}_{CE}(3, \infty, IAC) \\ &= \\ P^{\lambda}_{BE}(3, \infty, IAC) &= 1 - P_c(3, \infty, IAC)\Big(P^{\lambda}_{CE}(3, \infty, IAC) - P^{\lambda}_{SgBP}(3, \infty, IAC)\Big) \end{aligned}$$

Proposition 2 tells us that under IAC, representations for the limiting probabilities of the Weak Borda Effect and that of the Borda Effect can also be deduced from those of the Condorcet efficiency and the Strong Borda Paradox. We then derive the values provided in Table 3. Notice that under IAC, the likelihood of the Weak Borda Effect is minimized at $\lambda^\star = \frac{16709}{44883} \approx 0.3723$ where the probability is 0.1324; the likelihood of the Borda effect is minimized at $\lambda^\star = \frac{5063}{13009} \approx 0.3892$ where the probability is 0.1335. For both the Weak Borda Effect and the Borda effect, as λ grows from 0 to $\lambda^\star$, the probability of the effect tends to decrease and it increases when λ grows from $\lambda^\star$ to 1.

3.2 Representations for the limiting probability for scoring runoff rules

The Borda effect can also be observed with runoff scoring rules. Nonetheless, notice that only the Weak Borda Effect can be observe; it is obvious that this cannot be the case for the Strong Borda Effect. So, with runoff scoring rules, the Weak Borda Effect is equivalent to the Borda Effect.

Let us now provide representation of the limiting probability of the Borda Effect for all the scoring runoff rules both under IC and IAC. Without loss of generality, the following inequalities characterize a voting situation exhibiting the Weak Borda Effect.

$$\begin{cases} a\mathbf{S}_\lambda c \\ a\mathbf{S}_\lambda c \\ a\mathbf{M}b \\ c\mathbf{M}a \end{cases} \Leftrightarrow \begin{cases} n_1 + (1-\lambda)n_2 + \lambda n_3 - \lambda n_4 + (\lambda-1)n_5) - n_6 > 0 \\ \lambda n_1 - \lambda n_2 + n_3 + (1-\lambda)n_4 - n_5) + (\lambda-1)n_6 > 0 \\ n_1 + n_2 - n_3 - n_4 + n_5 - n_6 > 0 \\ -n_1 - n_2 - n_3 + n_4 + n_5 + n_6 > 0 \end{cases} \tag{9}$$

Remark 1. *We notice that under the Borda runoff, the Borda effect can only occur in case of a majority cycle. This is because if the inequalities of Eq. 9 are satisfied and that c**M**b, this indicates that the Condorcet winner c is ranked last by the Borda rule: we know that it is not possible. So, in three-candidate elections, it is only in case of a majority cycle that the Borda runoff can produce the Borda effect.*

[3] The computer program we used is available upon request.

3.2.1 Representation under IC

One can get a representation of the limiting probability of the Borda effect by following the technique of David and Mallows based on quadrivariate normal positive orthant probabilities as we did in Section 3.1.1. So, we consider Eq. 9 and define the following four discrete variables:

$$X_1 = \begin{array}{ll} 1 & : p_1 \\ 1-\lambda & : p_2 \\ \lambda & : p_3 \\ -\lambda & : p_4 \\ -1+\lambda & : p_5 \\ -1 & : p_6 \end{array} \quad X_2 = \begin{array}{ll} \lambda & : p_1 \\ -\lambda & : p_2 \\ 1 & : p_3 \\ 1-\lambda & : p_4 \\ -1 & : p_5 \\ -1+\lambda & : p_6 \end{array} \quad X_3 = \begin{array}{ll} 1 & : p_1 \\ 1 & : p_2 \\ -1 & : p_3 \\ -1 & : p_4 \\ 1 & : p_5 \\ -1 & : p_6 \end{array} \quad X_4 = \begin{array}{ll} -1 & : p_1 \\ -1 & : p_2 \\ -1 & : p_3 \\ 1 & : p_4 \\ 1 & : p_5 \\ 1 & : p_6 \end{array}$$

With n voters, Eq. 9 fully describes the Weak Borda effect when the average value of each of the X_j are positive. According to Gehrlein and Fishburn (1978b), $P^{\lambda^r}_{WBE}(3,\infty,IC)$ the limiting probability of the Weak Borda effect is equal to the joint probability that $\overline{X}_1 > 0$, $\overline{X}_2 > 0$, $\overline{X}_3 > 0$ and $\overline{X}_4 > 0$; when $n \to \infty$, it is equivalent to the quadrivariate normal positive orthant probability $\Phi_4(R")$ such that $\overline{X}_j\sqrt{n} \geq E(\overline{X}_j\sqrt{n})$ and where $R"$ is a correlation matrix between the variables X_j. The matrix $R"$ is as follows

$$R" = \begin{bmatrix} 1 & \frac{1}{2} & \sqrt{\frac{1}{6z}} & -\sqrt{\frac{2}{3z}} \\ & 1 & -\sqrt{\frac{1}{6z}} & -\sqrt{\frac{1}{6z}} \\ & & 1 & -\frac{1}{3} \\ & & & 1 \end{bmatrix}$$

By taking a look on the results of David and Mallows (1961) and the related literature, the matrix $R"$ does not look at all close to any special form that we are familiar with; finding a representation for $\Phi_4(R")$ seems to be a tricky task[4]. Fortunately, Gehrlein (1979) (see also Gehrlein (2017)) developed a general representation to obtain numerical values of $\Phi_4(R")$ as a function of a series of bounded integrals over a single variable. Using the formula suggested by Gehrlein (1979), we get $\Phi_4(R")$ and then we derive Proposition 3.

Proposition 3. *For three-candidate elections and a scoring runoff rule w_{λ^r},*

$$\begin{aligned} P^{\lambda^r}_{WBE}(3,\infty,IC) &= 6\Phi_4(R") \\ &= \frac{1}{2} + \frac{3}{2\pi^2}\Bigg[-\left(\frac{2}{3z-2}\right)^{\frac{1}{2}}\int_0^1 \cos^{-1}\left(\frac{F_1(z,t)}{N_1(z,t)\times N_2(z,t)}\right)dt \\ &\quad -\left(\frac{1}{6z-1}\right)^{\frac{1}{2}}\int_0^1 \cos^{-1}\left(\frac{F_2(z,t)}{N_2(z,t)\times N_3(z,t)}\right)dt - \frac{\sqrt{2}}{4}\int_0^1 \cos^{-1}\left(\frac{F_3(z,t)}{N_1(z,t)\times N_3(z,t)}\right)dt\Bigg] \\ &= 1 - \left(P_c(3,\infty,IC)\times P^{\lambda^r}_{CE}(3,\infty,IC)\right) \end{aligned}$$

where

$$F_1(z,t) = (6z)^{-\frac{3}{2}}(9z-6t^2); \qquad N_1(z,t) = \tfrac{1}{3}\left(9-t^2-\tfrac{1}{z}(4t^2+\tfrac{3}{2})\right)^{\frac{1}{2}}$$

$$F_2(z,t) = (6z)^{-\frac{3}{2}}(3zt^2-9z-3t^2); \qquad N_2(z,t) = \tfrac{1}{2}\left(\tfrac{3z-2t^2}{z}\right)^{\frac{1}{2}}$$

$$F_3(z,t) = \tfrac{zt^2-9z+7t^2-3}{18z}; \qquad N_3(z,t) = \tfrac{1}{3}\left(\tfrac{-2zt^2+18z-5t^2-3}{2z}\right)^{\frac{1}{2}}$$

[4] Thanks to Bill Gehrlein for pointing this out and for his help.

According to Proposition 3, one can derive the representation of the Borda Effect for runoff scoring rules from that of the Condorcet efficiency. Up to our knowledge, only the representation of the Condorcet efficiency of the Plurality runoff, the Antiplurality runoff and the Borda runoff are provided in the literature (see for example Gehrlein and Lepelley (2011)). So, from Proposition 3, the reader can get the overall Condorcet efficiency of scoring runoff rules.

The computed values of $P_{WBE}^{\lambda^r}(3,\infty,IC)$ are provided in Table 3. It comes that for all λ, we get $8.7\% < P_{WBE}^{\lambda^r}(3,\infty,IC) < 12.2\%$. It tends to decrease for $0 \leq \lambda \leq \frac{1}{2}$ and it increases for $\frac{1}{2} \leq \lambda \leq 1$. The probability is minimized by the Borda runoff ($\lambda = \frac{1}{2}$) and maximized by the Plurality runoff ($\lambda = 0$) and the Antiplurality runoff ($\lambda = 1$).

Remark 2. *Our formula is in line with Remark 1 since we find for the Borda runoff that $P_{WBE}^{\lambda^r}(3,\infty,IC)$ is equal to the probability of majority cycle under IC which is well documented in the literature.*

3.2.2 Representation under IAC

Following the same scheme as in Section 3.1.2, we compute the volume and get Proposition 4.

Proposition 4. *For three-candidate elections and a scoring runoff rule w_{λ^r},*

$$P_{WBE}^{\lambda^r}(3,\infty,IAC) = \begin{cases} \frac{96\lambda^7+176\lambda^6+1028\lambda^5-6420\lambda^4+11138\lambda^3-9157\lambda^2+3777\lambda-636}{1728(\lambda-2)(3\lambda-2)(\lambda-1)^3} & \text{for} \quad 0 \leq \lambda \leq \frac{1}{2} \\ \frac{-16\lambda^5+128\lambda^4-133\lambda^3+68\lambda^2-7\lambda-1}{432\lambda^3} & \text{for} \quad \frac{1}{2} \leq \lambda \leq 1 \end{cases}$$

$$= 1 - \left(P_c(3,\infty,IAC) \times P_{CE}^{\lambda^r}(3,\infty,IAC) \right)$$

The computed values of $P_{WBE}^{\lambda^r}(3,\infty,IAC)$ are provided in Table 3. We notice that the probabilities are lower that those obtained under IC for all λ with $6.4\% < P_{WBE}^{\lambda^r}(3,\infty,IAC) < 9.3\%$. It tends to decrease for $0 \leq \lambda \leq \frac{1}{2}$ and it increases for $\frac{1}{2} \leq \lambda \leq 1$. The probability is minimized by the Borda runoff ($\lambda = \frac{1}{2}$) and maximized by the Plurality runoff ($\lambda = 0$). Remark 2 also holds here.

4 Concluding remarks

The *Borda Effect* is among the declensions of the Borda Paradox and it was first introduced and defined by Colman and Poutney (1978). Colman and Poutney (1978) distinguished the *Strong Borda Effect* and the *Weak Borda Effect*: the *Strong Borda Effect* describes a situation in which the Plurality rule elect the Condorcet loser while the *Weak Borda Effect* (WBE) is related to a situation under which the Plurality winner is majority dominated by only one of the Plurality losers. The results of Colman and Poutney (1978, p.17) are quite limited in the scope as they only dealt with the Plurality rule while this phenomenon can also affect all the scoring rules and scoring runoffs. In this paper we showed that the representation of the (Weak) Borda Effect for general weighted scoring rules and scoring runoff rules can be deduced from those of the Condorcet efficiency and the Strong Borda Paradox. For one-shot rules, we found under assumption $\hbar$ (IC or IAC) that

$$P_{\text{WBE}}^{\lambda}(3,\infty,\hbar) = 1 - P_c(3,\infty,\hbar) \times P_{\text{CE}}^{\lambda}(3,\infty,\hbar)$$

$$P_{\text{BE}}^{\lambda}(3,\infty,\hbar) = 1 - P_c(3,\infty,\hbar)\left(P_{\text{CE}}^{\lambda}(3,\infty,\hbar) - P_{\text{SgBP}}^{\lambda}(3,\infty,\hbar) \right)$$

These relations teach us that the Condorcet efficiency of a scoring rule impacts his vulnerability to the Borda Effect: the more it is likely to select the Condorcet winner when it exists, the less it is susceptible to produce the Borda Effect. On the Contrary, the more a scoring rule is likely to select the Condorcet loser when it exists, the more it is likely to exhibit the Borda Effect. The first relation holds for scoring runoff rules. We also noticed that the likelihood of the Weak Borda Effect is quite low under the runoff rules than with the one-shot rules.

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