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# TOWARDS POINTVALUE CHARACTERIZATIONS IN MULTI-PARAMETER ALGEBRAS

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**Abstract.** We extend classical results from the Colombeau algebra, concerning point-value characterizations of generalized functions, to the more general case of multi-parameter (C,E,P)–algebras. Our investigations include considerations of the different definitions of subspaces related to tempered generalized functions.

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## 1. Introduction

Pointvalue characterizations are useful for proving existence and uniqueness of solutions to various differential problems, and the well-definedness of (generalized) point-values is also relevant for considerations of the possibiliy of composition of generalized functions.

We extend the known results in two ways: We consider multi-indices as regularisation parameters. and scales other than the polynomial scale, in particular those generated by a given set of nets. This setting allows a fine analysis of the singular spectrum of solutions to a given problem, and to clearly identify the contribution of the different sources, like initial data or irregular coefficients [13, 8].

The results extend, mutatis mutandis, the known results from the usual Colombeau algebra [9, 2, 15], which are reproduced in the corresponding case. Nevertheless, the consideration of several parameters and non-polynomial scales is not always completely straightforward. Asymptotic bounds usually given explicitely in terms of $\varepsilon \in (0, 1]$, *e.g.*, in the notion of slow scale nets, do not make manifest in how far they correspond to the regularization and to what extent they are related to the choice of the polynomial scale. Since in our approach the parameters themselves cannot be used as a numerical value, the relation with the asymptotic scale is necessarily made manifest in an explicit manner.

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## 2. $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebras

We consider the setting of $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebras [12, 13], which is a special case of the asymptotic extension of topological algebras as described in [10].

**Definition 2.1.** Assume given a filter base $\mathcal{B}_\Lambda$ on a set of indices $\lambda\in\Lambda$. For $x,y\in\mathbb{K}^\Lambda$ with $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, the notation $x=O(y)$ (resp. $x=o(y)$) means that there is (resp. for all) $c>0$ and some $\Lambda'\in\mathcal{B}_\Lambda$, $|x_\lambda|\le c\,|y_\lambda|$ for all $\lambda\in\Lambda'$. Then for any **solid** subring $S\subset\mathbb{K}^\Lambda$, *i.e.*, a subring such that

$$\forall(x,s)\in\mathbb{K}^\Lambda\times S : x=O(s)\Rightarrow x\in S\ , \tag{1}$$

and any semi-normed $\mathbb{K}$–vector space $(\mathcal{E},\mathcal{P})$, we define

$$\mathcal{H}_{(S,\mathcal{E},\mathcal{P})}=\left\{f\in\mathcal{E}^\Lambda\mid\forall p\in\mathcal{P}:\ p(f)\in S\right\}\ , \tag{2}$$

where $p(f)=(p(f_\lambda))_{\lambda\in\Lambda}\in\mathbb{R}_+^\Lambda\subset\mathbb{K}^\Lambda$. We will also consider $\mathcal{H}_{(S,K,\mathcal{P})}$ for any subset $K\subset\mathcal{E}$ which does not need to be a vector (sub)space.

*Example* 1. A left filtering partial order $\prec$ on $\Lambda$ induces the base of filter $\mathcal{B}_\Lambda=\{\Lambda_\lambda;\lambda\in\Lambda\}$ with $\Lambda_\lambda=\{\lambda'\in\Lambda\mid\lambda'\prec\lambda\}$. Examples for $(\Lambda,\prec)$ are $(\mathbb{N},\ge)$ and $((0,1]\,,\le)$. In practical applications it can be useful to have several independent parameters, $\lambda=(\varepsilon,\eta,...)$, which may correspond to different processes of regularization, requiring different respective scales [14, 8]. We may also consider more complex types of parameters, e.g. $\lambda=(\varepsilon,\varphi)\in(0,1]\times\mathcal{D}(\Omega)$, where $\mathcal{D}(\Omega)$, the space of compactly supported smooth functions, would be equipped with an appropriate filter.

*Example* 2. The set of complex nets of at most polynomial growth indexed by $(0,1]$ can be written as $A=\left\{x\in\mathbb{C}^{(0,1]}\mid\limsup|x_\varepsilon|^{1/|\log\varepsilon|}<\infty\right\}$ [3]. For $\mathcal{E}=\mathcal{C}^\infty(\mathbb{R}^n)$ with the usual family of seminorms $\mathcal{P}=\{\,p_{K,\alpha}:f\mapsto\|\partial^\alpha f\|_{L^\infty(K)};\ K\Subset\mathbb{R}^n,\ \alpha\in\mathbb{N}^n\,\}$, this yields $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}=\mathcal{E}_M$, Colombeau's moderate nets.

**Proposition 2.2.** *Consider $\Lambda$ and $(\mathcal{E},\mathcal{P})$ as in the above Definition 2.1.*

1. *If $A$ is a solid subring of $\mathbb{K}^\Lambda$, then $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ is an $A$–module for componentwise multiplication, and an $A$–algebra if $\mathcal{E}$ is a topological algebra.*
2. *If $I$ is a solid ideal of $A$, then $\mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$ is an $A$–linear subspace of $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$, and an ideal of $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ if $\mathcal{E}$ is a topological algebra.*
3. *As a consequence, the factor space $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$ is again an $A$–module, but also an $A/I$–module (and an algebra, if $\mathcal{E}$ is a topological algebra).*
4. *For $(\mathcal{E},\mathcal{P})=(\mathbb{K},\{|\cdot|\})$, we get $\mathcal{H}_{(A,\mathbb{K},|\cdot|)}/\mathcal{H}_{(I,\mathbb{K},|\cdot|)}=A/I$.*

**Definition 2.3.** Consider $(\mathcal{E},\mathcal{P})$ and $A$, $I$ as in the above Proposition 2.2. The factor ring $\mathcal{C}=A/I$ is called the ring of generalized numbers associated to $A$ and $I$, and the $\mathcal{C}$–algebra $\mathcal{A}_\mathcal{C}(\mathcal{E},\mathcal{P}):=\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$ is called the

$(\mathcal{C},\mathcal{E},\mathcal{P})$–algebra of generalized functions. If $(\mathcal{E},\mathcal{P})$ is a sheaf of $\mathbb{K}$–algebras over a topological space $X$, then we let (for open sets $\Omega\subset X$)

$$\mathcal{A}_{\mathcal{C}}(\mathcal{E},\mathcal{P}) \;:=\; \Omega\mapsto\mathcal{A}_{\mathcal{C}}(\mathcal{E}(\Omega),\mathcal{P}(\Omega))\;. \tag{3}$$

*Example* 3. Assume that for all $a\in A$ there is $\bar a\in A^*$ with $a=O(\bar a)$, where $A^*$ is the set of invertible elements of the ring $A$. Then we have the "canonically" associated ideal $I_A:=\{x\in A\mid\forall a\in A^*: x=o(a)\}$, which is solid if $A$ is:
For $A$ as in Example 2, we get nets going to zero faster than any power, $I_A=\big\{x\in\mathbb{C}^{]0,1]}\mid\lim|x_\varepsilon|^{1/|\log\varepsilon|}=0\big\}$. With $\mathcal{E},\mathcal{P}$ as before, this yields Colombeau's simplified (or "special" [9]) algebra $\mathcal{G}_s(\mathbb{R}^n)$ over the generalized numbers $\overline{\mathbb{C}}$.

*Example* 4 (("Overgenerated" $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebras.)). For any nonempty subset $B_0\subset(\mathbb{R}_+^*)^\Lambda$, let $B=\langle B_0\rangle$ be the closure of $B_0$ under addition and division (consisting of rational fractions of "linear combinations" with *positive* integer (or rational) coefficients of products of elements of $B_0$.) Then $A=A_{B_0}=\big\{x\in\mathbb{K}^\Lambda\mid\exists b\in B: x=O(b)\big\}$ is a solid ring, and $\mathcal{C}=\mathcal{C}_{B_0}=A/I_A$ is said to be generated by the set $B_0$, and $\mathcal{A}_{\mathcal{C}_{B_0}}(\mathcal{E},\mathcal{P})$ the $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebra generated by $B_0$. (In earlier publications, the term "overgenerated" had been used to describe this construction.) In practical applications, this construction is useful to construct the adequate algebra for a given differential problem [5, 6, 7]. For $B_0=\big\{(\varepsilon)_{\varepsilon\in(0,1]}\big\}$ we get back Colombeau's polynomial scale. Sometimes we use the fact that $B$ is countable whenever $B_0$ is countable or finite. (Actually every $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebra whose ideal is $I_A$ as given in Example 3, is generated by the set $B_0=A^*\cap\mathbb{R}_+^\Lambda$, but this set is uncountable except for pathological cases.)

*Remark* 2.4. The assignment $f\mapsto(f)_{\lambda\in\Lambda}+\mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$ defines a map $\mathbf{i}:\mathcal{E}\to\mathcal{A}_{\mathcal{C}}(\mathcal{E})$ iff $\mathbb{1}=(1)_{\lambda\in\Lambda}\in A$, or equivalently, if $A$ contains at least one (and thus any) nonzero constant sequence. Then this map is injective iff $(\mathcal{E},\mathcal{P})$ is Hausdorff and $\mathbb{1}\notin I$ ($\iff I\neq A$). **We shall assume these three conditions to hold throughout the sequel of this paper.** (The condition $(x_\lambda)_{\lambda\in\Lambda}\in I\Rightarrow\lim(x_\lambda)_{\lambda\in\Lambda}=0$ is sufficient but not necessary to have $\mathbb{1}\notin I$; and for $A=A_{B_0}$ and $I_A$ as in Example 4, all these conditions on $A$ an $I$ are satisfied for arbitrary sets $B_0$.)

**Proposition 2.5.** *If $(\mathcal{E},\mathcal{P})$ is a presheaf of semi-normed $\mathbb{K}$–algebras over a topological space $X$, i.e.,*

1. *for any open $\Omega\subset X$, the algebra $\mathcal{E}(\Omega)$ is endowed with the set $\mathcal{P}(\Omega)$ of seminorms such that, if $\Omega_1\subset\Omega_2\subset\Omega$ and $\rho_1^2$ is the restriction from $\Omega_2$ to $\Omega_1$, then for each $p\in\mathcal{P}(\Omega_1)$, we have $p\circ\rho_1^2\in\mathcal{P}(\Omega_2)$.*
2. *for any open covering $(U_i)_i$ of an open set $\Omega\subset X$ and each $p\in\mathcal{P}(\Omega)$, there is a finite subfamily $(U_{i_1},\dots,U_{i_n})$ of $(U_i)_i$ and $p_1\in\mathcal{P}(U_{i_1})$, ..., $p_n\in\mathcal{P}(U_{i_n})$ such that for all $u\in\mathcal{E}(\Omega)$, $p(u)\le p_1(u|_{U_{i_1}})+\dots+p_n(u|_{U_{i_n}})$ ,*

*then $\mathcal{A}_{\mathcal{C}}(\mathcal{E},\mathcal{P})$ defined in (3) is again a presheaf.*
*Moreover, if $\mathcal{E}$ is a fine sheaf, then $\mathcal{A}_{\mathcal{C}}(\mathcal{E},\mathcal{P})$ also is a fine sheaf.*

The proof is given in [12], and, for the last statement, in [10].

## 3. Multiparameter algebras of tempered generalized functions

We first study the relations between two closely related definitions of spaces of tempered generalized functions, which generalize the "simplified" version $\mathcal{G}_\tau(\Omega)$ of the corresponding space introduced by Colombeau in [1]. An important property of functions in $\mathcal{G}_\tau(\Omega)$ is that their point-values in (not necessarily compactly supported) generalized points are well-defined. This is also relevant when considering the possibility of composition of generalized functions. In the previously introduced framework it is most natural to consider

$$\mathcal{A}_\mathcal{C}(\mathcal{O}_M)(\Omega) := \mathcal{A}_\mathcal{C}(\mathcal{O}_M(\Omega), \mathcal{P}_\tau(\Omega)) \ ,$$

the $\mathcal{C}$–extension of Schwartz's space $\mathcal{O}_M(\Omega)$ of "multipliers" or slowly increasing functions, with topology given by the family of semi-norms

$$\mathcal{P}_\tau(\Omega) = \{\, p_{\varphi,\alpha} : f \mapsto \|\varphi \cdot \partial^\alpha f\|_{L^\infty(\Omega)} \ ;\ \varphi \in \mathcal{S}(\Omega),\ \alpha \in \mathbb{N}^n \,\} \ .$$

Elements of $\mathcal{O}_M(\Omega)$ are smooth functions for which all these seminorms are finite,

$$\mathcal{O}_M(\Omega) = \{ f \in \mathcal{C}^\infty(\Omega) \mid \forall \alpha \in \mathbb{N}^n \ \ \forall \varphi \in \mathcal{S}(\Omega) : p_{\varphi,\alpha}(f) < \infty \} \ .$$

For the sequel, it is also important to note that $\mathcal{O}_M(\Omega)$ is a topological algebra, which is trivial if $\Omega$ is bounded, but else (and in particular for $\Omega = \mathbb{R}^n$) requires a rather lengthy proof of Lemma 4 of [2] (personal communication by A. Delcroix).

It is well known [11] that for $\Omega = \mathbb{R}^n$, we have $\mathcal{O}_M(\mathbb{R}^n) = \mathcal{O}_M^{\mathrm{g}}(\mathbb{R}^n)$, where

$$\mathcal{O}_M^{\mathrm{g}}(\Omega) = \{ f \in \mathcal{C}^\infty(\Omega) \mid \forall \alpha \in \mathbb{N}^n \ \ \exists r \in \mathbb{N} : q_{-r,\alpha}(f) < \infty \} \ ,$$

with $q_{r,\alpha} : f \mapsto \sup\big\{\, |(1 + \|x\|)^r\, \partial^\alpha f(x)| \,;\ x \in \Omega \,\big\}$. We do not claim, however, that the topologies induced by $\mathcal{P}_\tau(\mathbb{R}^n)$ resp. $\mathcal{Q}_\tau = \{q_{r,\alpha}\}$ are the same. Actually, the $q_{r,\alpha}$ are not seminorms on the whole of $\mathcal{O}_M^{\mathrm{g}}(\Omega)$, which could be written as projective limit of the inductive limit of the spaces $E_{r,\alpha}$ on which these seminorms are finite. For the same reason, the corresponding factor algebra

$$\mathcal{G}_{\tau,\mathcal{C}}(\Omega) = \mathcal{M}_{\tau,A}(\Omega) / \mathcal{M}_{\tau,I}(\Omega) \tag{4}$$

where for any $S \subset \mathbb{K}^\Lambda$,

$$\mathcal{M}_{\tau,S}(\Omega) = \big\{ f \in (\mathcal{O}_M^{\mathrm{g}}(\Omega))^\Lambda \mid \forall \alpha \in \mathbb{N}^n \ \ \exists r \in \mathbb{N} : (q_{-r,\alpha}(f_\lambda))_\lambda \in S \big\} \ , \tag{5}$$

does not fit in the framework of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$–algebras as defined in Def. 2.3. (It is included, however, in the more general concept reviewed in [3].) Since we will not apply the construction of Def. 2.3 with this space, we do not need to know whether $\mathcal{O}_M^{\mathrm{g}}(\Omega)$ is a topological algebra. The obvious estimates using the $q_{r,\alpha}$ are sufficient to establish $\mathcal{M}_{\tau,A}(\Omega)$ as an algebra and $\mathcal{M}_{\tau,I}(\Omega)$ as an ideal thereof.

*Remark* 3.1. In the above definition, the integer $r \in \mathbb{N}$ must not depend on $\lambda \in \Lambda$, *i.e.*, for any representative $u \in \mathbf{u}$, the whole net $u = (u_\lambda)_\lambda$ must lie in a subspace of $\mathcal{C}^\infty(\Omega)$ on which some $q_{-r,\alpha}$ is finite, for given $\alpha \in \mathbb{N}$.

**Theorem 3.2.** *(i) Consider $\mathcal{C} = A/I$ as in Def. 2.3. Then, for $S = A$ and $S = I$, we have $\mathcal{M}_{\tau,S}(\mathbb{R}^n) \subset \mathcal{H}_{(S,\mathcal{O}_M,\mathcal{P}_\tau)}(\mathbb{R}^n)$.*
*(ii) Assume the additional hypothesis that the base of filter $\mathcal{B}_\Lambda$ is countable, and that $A$ and $I$ are given as*

$$A = \left\{ x \in \mathbb{K}^\Lambda \mid \exists \ell \in \mathbb{Z} : x = O(b^{(\ell)}) \right\}, I = \left\{ x \in \mathbb{K}^\Lambda \mid \forall \ell \in \mathbb{Z} : x = o(b^{(\ell)}) \right\}$$

*in terms of a countable set $\left\{ b^{(k)} \; ; \; k \in \mathbb{Z} \right\} \subset \mathbb{R}_+^\Lambda$ such that $\forall k, \ell \in \mathbb{Z} : k < \ell \Rightarrow b^{(k)} = o(b^{(\ell)})$. Then we have: $\mathcal{M}_{\tau,A}(\mathbb{R}^n) = \mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)}(\mathbb{R}^n)$ and therefore $\mathcal{A}_\mathcal{C}(\mathcal{O}_M)(\mathbb{R}^n)$ can be seen as $\mathcal{G}_{\tau,\mathcal{C}}(\mathbb{R}^n)$ modulo the canonical image, in $\mathcal{G}_{\tau,\mathcal{C}}(\mathbb{R}^n)$, of the larger ideal $\mathcal{H}_{(I,\mathcal{O}_M,\mathcal{P}_\tau)}(\mathbb{R}^n)$.*

*Remark* 3.3. Such a countable set $\left\{ b^{(\ell)} \right\}$ exists for asymptotic algebras [4] and thus in the Colombeau case. In most applications, when $A/I$ is generated by a finite number of nets, we can choose a subset of $B$ (*c.f.* Example 4) with the required property. The hypothesis on $\left\{ b^{(\ell)} \right\}$ could be relaxed, but for the scope of this short paper, we have to confine ourselves to this somehow limited framework, leaving a more general treatment for a future work.

To prove the above theorem, we will use the following Lemma:

**Lemma 3.4.** *Consider $f \in \mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)}(\Omega)$, with $A$ as in Theorem 3.2. We have $f \in \mathcal{M}_{\tau,A}(\Omega)$ if, and only if,*
(6)

$$\forall \alpha \in \mathbb{N}^n \;\; \exists \ell, r \in \mathbb{N} \;\; \exists K \Subset \Omega \;\; \exists \Lambda' \in \mathcal{B}_\Lambda \;\; \forall \lambda \in \Lambda' \;\; \forall x \notin K : \frac{(1 + \|x\|)^\ell}{(\partial^\alpha f_\lambda(x))_\lambda} \leq b_\lambda^{(r)} \; .$$

*Proof.* From the definition (5) of $\mathcal{M}_{\tau,A}(\Omega)$, it is clear that (6) is satisfied for any $f \in \mathcal{M}_{\tau,A}(\Omega)$, with *any* $K \Subset \Omega$, and $\ell = p$, $r = \ell'$, where $b^{(\ell')}$ is dominating $q_{-p,\alpha}(f)$ in (5). Conversely, assume that (6) holds for some $f \in \mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)}$. We have to show that for each $\alpha \in \mathbb{N}^n$, there is $r''$ such that the analogous relation is verified also inside $K$. For this, it is sufficient to consider the definition of $\mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)}$ with the seminorm $p_{\varphi,\alpha}$ for $\varphi \in \mathcal{D}(\Omega) \subset \mathcal{S}(\Omega)$ equal to 1 on $K$: This implies that $p_{\varphi,\alpha}(f)$ is an element of $A$, which by the hypothesis is dominated by some $b^{(r')}$.
Multiplying by $(1+\|x\|)^{-\ell}$ and restricting $x$ to $K$ makes the left-hand side only smaller. Thus, for $\ell'' = \max\{\ell, \ell'\}$, choosing $r''$ such that $b^r + b^{r'} = O(b^{r''})$ we have the inequality in (6) for all $x \in \Omega$, *i.e.*, $f \in \mathcal{M}_{\tau,A}(\Omega)$. □

*Proof. of Theorem 3.2.* (i) From the definitions (of $\mathcal{S}$ in particular), we have $\mathcal{M}_{\tau,X}(\Omega) \subset \mathcal{H}_{(X,\mathcal{O}_M,\mathcal{P}_\tau)}(\Omega)$ for $X = A$ and $X = I$: For any $\alpha$, if such $p$ exists in (5), then, since any $\varphi \in \mathcal{S}$ decreases faster than $(1 + \|x\|)^{-p}$, one has $p_{\varphi,\alpha} \leq C\, q_{-p,\alpha}$ (with $C = \sup|(1 + \|x\|)^p \varphi(x)|$), and since $X$ is solid, $q_{-p,\alpha}(f) \in X \Rightarrow p_{\varphi,\alpha}(f) \in X$.

(ii) For the converse inclusion with $X = A$ and $\Omega = \mathbb{R}^n$, we assume that $\mathcal{B}_\Lambda$ has an equivalent countable base $\Lambda_1 \supset \Lambda_2 \supset ....$ Then, in view of Lemma 3.4, if $f \notin \mathcal{M}_{\tau,A}(\Omega)$ then

$$\exists\alpha \in \mathbb{N}^n \;\; \forall\ell, r \in \mathbb{N} \;\; \forall K \Subset \Omega \;\; \forall\Lambda' \in \mathcal{B}_\Lambda \;\; \exists\lambda \in \Lambda' \;\; \exists x \notin K : \frac{|\partial^\alpha f_\lambda(x)|}{(1 + \|x\|)^\ell} > b_\lambda^{(r)} \;.$$

For $\Omega = \mathbb{R}^n$, this allows to construct, for some $\alpha \in \mathbb{N}^n$, the sequence $(x_\ell)_{\ell\in\mathbb{N}}$ and $(\lambda_\ell)_{\ell\in\mathbb{N}}$ such that $\|x_{\ell+1}\| \geq \|x_\ell\| + 2$, $\lambda_\ell \in \Lambda_\ell$ and $(1 + \|x_\ell\|^2)^{-\ell} |\partial^\alpha f_{\lambda_\ell}(x_\ell)| \geq b_{\lambda_\ell}^{(r)}$ for all $\ell \in \mathbb{N}$. Let us consider the element $\varphi \in \mathcal{S}$ which consists of "bumps" of height 1 centered in these $x_\ell$,

$$\varphi(x) = \sum_{\ell\in\mathbb{N}} (1 + \|x_\ell\|^2)^{-\ell} \rho(x - x_\ell), \rho \in \mathcal{D}(\mathbb{R}^n), \operatorname{supp}\rho \subset B_1(o), 0 \leq \rho \leq 1 = \rho(o).$$

Obviously, it is such that $p_{\varphi,\alpha}(f_{\lambda_\ell}) \geq b_{\lambda_\ell}^{(\ell)}$ for every $\ell$, therefore $(p_{\varphi,\alpha}(f_\lambda))_\lambda$ is not dominated by any $a \in A$ and thus $f \notin \mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)}$. □

We have the following characterization of the ideal $\mathcal{H}_{(I,\mathcal{O}_M,\mathcal{P}_\tau)}$:

**Lemma 3.5.** *Under the same hypotheses as in part (ii) of Theorem 3.2,*

$$\mathcal{H}_{(I,\mathcal{O}_M,\mathcal{P}_\tau)}(\mathbb{R}^n) = \\ = \left\{ u \in \mathcal{O}_M(\mathbb{R}^n)^\Lambda \mid \forall\alpha \in \mathbb{N}^n \; \forall\ell \in \mathbb{Z} \; \exists p \in \mathbb{N} : q_{-p,\alpha}(u_\lambda) = o(b_\lambda^{(\ell)}) \right\}.$$

*Proof.* With the quantifiers and asymptotics exchanged, the proof of the non-trivial inclusion is here the same as for $\mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)} \subset \mathcal{M}_{\tau,A}$ in the preceding theorem. □

## 4. Generalized points and point values of generalized functions

Here we generalize classical results concerning point values in the Colombeau algebra, as given, *e.g.*, in [9], to the multiparametric algebras introduced above.

**Definition 4.1.** For a given ring of generalized numbers $\mathcal{C} = A/I$, the **generalized points** in $\Omega \subset \mathbb{R}^n$, $\widetilde{\Omega} = \Omega_A / \sim$, are equivalence classes of $A$–moderate sequences $x \in \Omega_A = \mathcal{H}_{(A,\Omega,\|\cdot\|)} = \left\{ x \in \Omega^\Lambda \mid (\|x_\lambda\|)_\lambda \in A \right\}$ modulo the equivalence relation

$$x \sim y \iff (\|x_\lambda - y_\lambda\|)_\lambda \in I \iff x - y \in \mathcal{H}_{(I,\mathbb{R}^n,\|\cdot\|)} \;.$$

The **compactly supported** points in $\widetilde{\Omega}$ are those having a representative in a compact set, $\widetilde{\Omega}_c = \widetilde{\Omega} \cap \left\{ \widetilde{x} \;;\; x \in K^\Lambda,\; K \Subset \Omega \right\}$, or equivalently, those having a compact **support**, defined as the set of cluster points of any representative.

*Remark* 4.2. Since an open set $\Omega \subsetneq \mathbb{R}^n$ is not a vector space, we cannot write $\widetilde{\Omega}$ as quotient vector space, but have to use the set-theoretic formulation modulo an equivalence relation. However, for applications (where we are only interested in the behavior for "$\lambda$ small enough"), it amounts to the same to consider points of $\widetilde{\mathbb{R}^n} = \mathcal{A}_{\mathcal{C}}(\mathbb{R}^n, \|\cdot\|)$ which have a representative in $\Omega^\Lambda$. Since elements of $I$ have zero limit, this implies that, for open $\Omega$, *all* representatives of such points lie in $\Omega$ for $\lambda$ small enough. (However, for some values of $\lambda$, we may have $x_\lambda \notin \Omega$. Then, an expression $f(x_\lambda)$ is not defined for these $\lambda$, if the domain of $f$ is $\Omega$.)

We now prove the following generalization of Proposition 1.2.45 in [9]:

**Theorem 4.3.** *Let $\mathcal{C} = A/I$ be a ring of generalized numbers, $\mathcal{E}$ the space of $C^\infty$ functions on a connected open $\Omega \subset \mathbb{R}^n$, with topology given by the supremum norms of all derivatives on compact sets, $\mathcal{P} = \{p_{K,\alpha} : f \mapsto \|\partial^\alpha f\|_{L^\infty(K)}\}$. Then, for any $\mathbf{u} \in \mathcal{A}_{\mathcal{C}}(\mathcal{E},\mathcal{P})$ and $\tilde{x} \in \widetilde{\Omega}_c$, $\mathbf{u}(\widetilde{x})$ is a well defined element of $\mathcal{C} = \widetilde{\mathbb{K}}$.*

This means that the sequence $(u_\lambda(x_\lambda))_\lambda$ is an element of $A$, for any representatives $(u_\lambda)_\lambda$ resp. $(x_\lambda)_\lambda$ of $\mathbf{u}$ resp. $\tilde{x}$, and that its class modulo $I$ is independent of the choice of these representatives.

*Proof.* Consider representatives $(u_\lambda)_\lambda$, $(v_\lambda)_\lambda$ of $\mathbf{u}$ and $(x_\lambda)_\lambda$, $(y_\lambda)_\lambda$ of $\widetilde{x}$. Let us first show that $(u_\lambda(x_\lambda))_\lambda \in A$. Indeed, we can assume that for all "sufficiently small" $\lambda$, $x_\lambda$ lies in some compact $K$. Then, since for all compact sets $K$ and $\alpha \in \mathbb{N}^n$, $p_{K,\alpha}(u_\lambda) \in A$, we have that $(u_\lambda(x_\lambda))_\lambda \in A$. In the same way we have for any $j \in \mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$, $(j_\lambda(x_\lambda))_\lambda \in I$. We use this in

$$u_\lambda(x_\lambda) - v_\lambda(y_\lambda) = u_\lambda(x_\lambda) - u_\lambda(y_\lambda) + [u_\lambda(y_\lambda) - v_\lambda(y_\lambda)]$$

to see that the last part is an element of $I$. As to the first part, we use

$$\begin{aligned} u_\lambda(x_\lambda) - u_\lambda(y_\lambda) &= \int_{x_\lambda}^{y_\lambda} \operatorname{grad} u_\lambda(\xi) \cdot \mathrm{d}\xi \\ &= \int_0^1 \operatorname{grad} u_\lambda(x_\lambda + s(y_\lambda - x_\lambda)) \cdot (y_\lambda - x_\lambda)\ \mathrm{d}s\ . \end{aligned}$$

(Since we have $x_\lambda - y_\lambda \to 0$ following $\mathcal{B}_\Lambda$, all segments connecting $x_\lambda$ and $y_\lambda$ eventually lie in $u_\lambda$'s domain $\Omega$.) Thus

$$|u_\lambda(x_\lambda) - u_\lambda(y_\lambda)| \le \|y_\lambda - x_\lambda\| \int_0^1 \|\operatorname{grad} u_\lambda(x_\lambda + s(y_\lambda - x_\lambda))\|\,\mathrm{d}s\ ,$$

and using that $(\|y_\lambda - x_\lambda\|)_\lambda \in I$ and $(p_{K,\alpha}(u_\lambda))_\lambda \in A$ (for $|\alpha| = 1$ and some compact $K$ containing the segments $[x_\lambda, y_\lambda]$, which exists since both $x$ and $y$ are in $\widetilde{\Omega}_c$), we finally get $u_\lambda(x_\lambda) - v_\lambda(y_\lambda) \in I$, *i.e.*, the required independence of respective representatives. □

The following Lemma, which generalizes Theorem 1.2.3 in [9], will be used to prove Theorem 4.6:

**Lemma 4.4** ((Characterization of the ideal by 0-order estimate)). *Assume that $I = I_A$ (cf. Example 3) and for every $x \in A$ and $a \in A^*$, there is $b \in A^*$ such that $b\,x = o(a)$. Then we have $\mathcal{H}_{(I,\mathcal{E},\mathcal{P})} = \mathcal{H}_{(A,\mathcal{E},\mathcal{P})} \cap \mathcal{H}_{(I,\mathcal{E},\mathcal{P}_0)}$, where $\mathcal{P}_0 = \{p_{K,0}; K \Subset \Omega\}$, $p_{K,0} = \|\cdot\|_{L^\infty(K)}$. In other words, for $u \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ we have $u \in \mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$ iff for every $K \Subset \Omega$, $\left(\|u_\lambda\|_{L^\infty(K)}\right)_\lambda \in I$.*

*Remark* 4.5. The second assumption is satisfied whenever every $x \in A$ are dominated by some $y \in A^*$, thus in particular in algebras generated (as in Example 4) by a set $B_0$ having an element going to 0 or to infinity.

*Proof.* We only have to show the inclusion $\supset$. Consider $u = (u_\lambda) \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ such that $p_{K,0}(u) \in I$ for all $K \Subset \Omega$. It is enough to show that for any partial derivative $\partial_i$, we still have $p_{K,0}(\partial_i u) \in I$ for all $K \Subset \Omega$. Then, since $\partial_i u$ is still in $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$, the result holds for any derivative by immediate recurrence. Let $K \Subset \Omega$ and $a \in A^*$ be given. We will show that $p_{K,0}(\partial_i u) = o(a)$. As usual, we let $L = K + \overline{B}_{\delta/2}(0)$, where $\delta = \min(\mathrm{dist}(K, \partial\Omega), 1)$. We know that $\partial_i^2 u \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$, thus, by assumption, there exists $h \in A^* : p_{L,0}(h\,\partial_i^2 u) = o(a)$, and we can assume that $|h_\lambda| < \delta/2$ for all $\lambda \in \Lambda$. By Taylor's theorem, $\partial_i u(x) = h^{-1}\left(u(x + h\,e_i) - u(x)\right) - \frac{1}{2} h\,\partial_i^2 u(x + \tilde{h}\,e_i)$, with $\tilde{h}_\lambda \in [0, h_\lambda]$. From this we get, as required,

$$p_{K,0}(\partial_i u) \;\le\; \underbrace{|h^{-1}|}_{\in A}\; 2\; \underbrace{p_{L,0}(\hat{u})}_{\in I} + \frac{1}{2}\underbrace{|h|\,p_{L,0}(\partial_i^2 \hat{u})}_{=o(a)} \;=\; o(a).$$

□

**Theorem 4.6.** *Under the assumptions of Lemma 4.4, if $\mathbf{u} \in \mathcal{A}_\mathcal{C}(\mathcal{E},\mathcal{P})$, then*

$$\mathbf{u} = 0 \in \mathcal{A}_\mathcal{C}(\mathcal{E},\mathcal{P}) \iff \forall \tilde{x} \in \Omega_c : \mathbf{u}(\widetilde{x}) = 0 \in \mathcal{C}\,.$$

*Proof.* The implication "⇒" is a consequence of Theorem 4.3. Let us show "⇐" by contraposition: Assume $\mathbf{u} \neq 0$. This means that for some $K \Subset \Omega$ and some representative $(u_\lambda) \in \mathbf{u}$, $\left(p_{K,0}(u_\lambda)\right)_\lambda \notin I$ (using the preceding Lemma 4.4). Now, if we let $x_\lambda \in K$ such that $u_\lambda(x_\lambda) = \|u_\lambda\|_{L^\infty(K)}$, then $\tilde{x} \in \Omega_c$ and $\mathbf{u}(\tilde{x}) \neq 0$. □

The requirement of compactly supported points can be dropped if we confine ourselves to tempered generalized functions defined in (4), in analogy to Proposition 1.2.45 in [9].

**Theorem 4.7.** *For $\mathbf{u} \in \mathcal{G}_{\tau,\mathcal{C}}(\Omega)$ and $\tilde{x} \in \widetilde{\Omega}$, $\mathbf{u}(\tilde{x})$ is a well-defined element of $\mathcal{C}$.*

*Proof.* Let $u$ resp. $x$ be representatives of $\mathbf{u}$ resp. $\widetilde{x}$. We have that $r \in \mathbb{N}$ such that $a_\lambda = \sup_{\xi\in\Omega}(1 + |\xi|)^{-r}|u_\lambda(\xi)|$ defines an element $a = (a_\lambda)_\lambda$ of $A$, and $b = (\|x_\lambda\|)_\lambda$ is also in $A$. Replacing $\xi$ by $x_\lambda$, we get $|u_\lambda(x_\lambda)| \le (1 + b_\lambda)^r a_\lambda$, and since $A$ is a solid ring, we also have $\left(u_\lambda(x_\lambda)\right)_\lambda \in A$. As in the previous proof, $|u_\lambda(x_\lambda) - u_\lambda(y_\lambda)| \in I$ if $y$ is another representative of $\widetilde{x}$ and thus $x - y \in I$, and in the same way $|u_\lambda(x_\lambda) - v_\lambda(x_\lambda)| \in I$ for any other representative $v$ of $\mathbf{u}$, achieving the proof. □

The following Lemma generalizes Theorem 1.2.25 in [9, p.27]:

**Lemma 4.8** ((Characterization of $\mathcal{M}_{\tau,I}$ by 0-order estimates.))**.**
*Under the assumptions of Lemma 4.4, and the additional hypothesis that $\Omega$ is an $n$–dimensional box, we have* $\mathcal{M}_{\tau,I} = \mathcal{M}_{\tau,A} \cap \mathcal{M}_{\tau^*,I}$ *, where*

$$\mathcal{M}_{\tau^*,I} = \Big\{ f \in (\mathcal{C}^\infty(\Omega))^\Lambda \mid \exists p \in \mathbb{N} : \big(\sup_{x\in\Omega} |(1+\|x\|)^{-p} f_\lambda(x)|\big)_{\lambda\in\Lambda} \in I \Big\} \ .$$

*Proof.* For $u \in \mathcal{M}_{\tau,A} \cap \mathcal{M}_{\tau^*,I}$, we will show that $q_{-p,0}(\partial_i u) = o(a)$ for some $p \in \mathbb{N}$ and all $a \in A^*$. Let $p \in \mathbb{N}$ such that $q_{-p,0}(u) \in I$ and $q_{-p,0}(\partial_i^2 u) \in A$, and let $a \in A^*$ be given. Using the assumption, there is $h \in A^*$ such that $h\, q_{-p,0}(\partial_i^2 u) = o(a)$ (and we can assume that $h_\lambda \to 0$). Again, by Taylor's theorem, $\partial_i u(x) = h^{-1}\,(u(x+h\,e_i) - u(x)) - \frac{1}{2} h\, \partial_i^2 u(x+\tilde{h}\,e_i)$ , with $\tilde{h}_\lambda \in [0, h_\lambda]$. (Since $\Omega$ is a box, for each $e_i$ the sign of $h_\lambda$ can be chosen such that the segments $[x, x+h\,e_i]$ lie in $\Omega$.) In the expression of $q_{-p,0}$ we use $\|x\| \ge \|x+h_\lambda\, e_i\| - \|h_\lambda\|$ and $(1+\|x+h\,e_i\| - \|h\|)^{-p} = (1+\|x+h\,e_i\|)^{-p}(1+O(h))$ to get

$$q_{-p,0}(\partial_i u) \ \le \ \underbrace{|h^{-1}|}_{\in A}\ \underbrace{q_{-p,0}(u)\,(2+O(h))}_{\in I} + \tfrac{1}{2}\ \underbrace{|h|\, q_{-p,0}(\partial_i^2 \hat{u})(1+O(h))}_{=o(a)} \ = \ o(a).$$

□

**Theorem 4.9.** *Under the hypothesis of Lemma 4.8 and assuming that $\mathcal{B}_\Lambda$ is cofinal to a countable filter base,* $\mathbf{u} \in \mathcal{G}_{\tau,\mathcal{C}}(\Omega)$ *is zero iff* $\mathbf{u}(\widetilde{x}) = 0 \in \mathcal{C}$ *for all* $\widetilde{x} \in \widetilde{\Omega}$.

*Remark* 4.10. As in [9, Thm. 1.2.50], the above holds for any moderate open set $\Omega$.

*Proof.* The sense ($\Rightarrow$) is a consequence of Theorem 4.7, e.g., by taking as representative of $\mathbf{u}$ the sequence identically equal to zero. Now consider ($\Leftarrow$), by contraposition. Assume that $\mathbf{u} \in \mathcal{G}_{\tau,\mathcal{C}} \setminus \{o\}$, *i.e.*, $(u_\lambda)_\lambda \in \mathcal{M}_{\tau,A} \setminus \mathcal{M}_{\tau^*,I}$ (using Lemma 4.8). By definition and assumptions made on $A$, $I_A$, this means that

$$\forall\alpha \in \mathbb{N}^d \ \ \exists p \in \mathbb{N} \ \ \exists a \in A \ \ \forall\Lambda' \in \mathcal{B}_\Lambda \ \ \exists\lambda \in \Lambda' : \sup_{x\in\Omega} |(1+\|x\|)^{-p}\, \partial^\alpha u_\lambda(x)| \le a_\lambda \qquad (*)$$

(where $a \in A$ can be taken invertible, $a \in A^*$, without loss of generality), and

$$\forall q \in \mathbb{N},\ \exists j \in A \setminus I,\ \forall\Lambda' \in \mathcal{B}_\Lambda,\ \exists\lambda \in \Lambda' : \sup_{x\in\Omega} |(1+\|x\|)^{-q}\, u_\lambda(x)| \ge j_\lambda \ , \quad (**)$$

$j \in A^*$, w.l.o.g., according to the assumption. Now take $\alpha = 0$ and $p \in \mathbb{N}$, $a \in A$ as in $(*)$, and $j \in A^*$ such that $(**)$ holds with $q = p+1$. Then, $(1+\|x\|)^{-p-1}\,|u_\lambda(x)| \le (1+\|x\|)^{-1}\, a_\lambda < j_\lambda$ whenever $\|x\| \ge a_\lambda\, j_\lambda^{-1}$, $\lambda \in \Lambda_0$. Thus, in view of $(**)$,

$$\forall\Lambda' \subset \Lambda_0,\ \exists\lambda \in \Lambda' : \sup_{\|x\|\le a_\lambda j_\lambda^{-1}} |u_\lambda(x)| \ge \sup_{\|x\|\le a_\lambda j_\lambda^{-1}} (1+\|x\|)^{-p-1}\,|u_\lambda(x)| \ge j_\lambda \ .$$

Thus there is a sequence $(\Lambda_k)_k$ which can be taken cofinal to $\mathcal{B}_\Lambda$, $(\lambda_k)_k$ with $\lambda_k \in \Lambda_k$, and $(x_k)_k \in \Omega^{\mathbb{N}}$ such that $\|x_k\| \le a_{\lambda_k}\, j_{\lambda_k}^{-1}$ and $|u_{\lambda_k}(x_k)| \ge \frac{1}{2} j_{\lambda_k}$. If we let $x_\lambda = x_k$ for $\lambda \in \Lambda_k \setminus \Lambda_{k+1}$, then $(\|x_\lambda\|)_\lambda \in A$, thus $\widetilde{x} \in \widetilde{\Omega}$, and $(u_\lambda(x_\lambda))_\lambda \notin I$, *i.e.*, $\mathbf{u}(\widetilde{x}) \ne 0 \in \mathcal{C}$, which achieves the proof of the "if" part of Theorem. □

We can establish the pointvalue characterizations in $\mathcal{A}_{\mathcal{C}}(\mathcal{O}_M, \mathcal{P}_\tau)$, already known for the simplified Colombeau case (H. Vernaeve, personal communication), using

**Definition 4.11.** A generalized point $\widetilde{x} \in \widetilde{\Omega}$ is of slow scale (c.f. [15]), iff

$$\exists a \in A^* \quad \forall n \in \mathbb{N} \quad |x_\lambda|^n = O(a_\lambda) \ . \tag{7}$$

The detailed theorems and proofs will be given in a forthcoming paper.

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