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Preface

GENERALIZED FUNCTIONS,
LINEAR AND NONLINEAR PROBLEMS, I

This volume is the first of two dedicated to the proceedings of the International Conference GF 2011 on Generalized Functions, Linear and Nonlinear Problems, held from Monday 18 to Friday 22 April 2011 on the Campus de Schoelcher, Martinique, French West Indies. It was organized by the research laboratory CEREGMIA of the Université des Antilles et de la Guyane.

The development of mathematical research at the University of French West Indies and Guyana (UAG), and repeated encouragement by colleagues from many countries all over the world (Austria, Brazil, Japan, Korea, Russia, Serbia, UK, USA, . . .) led the group "Nonlinear Algebraic Analysis" of CEREGMIA at UAG to decide to organize an international conference in Martinique. The event is part of a series of international symposia on generalized functions and non-linear differential problems. Earlier instances have been organized in Guadeloupe (France, 2000), Novi Sad (Serbia, 2004), Bedlewo (Poland, 2007) and Vienna (Austria, 2009).

Like many other mathematicians we met or with whom we have effective collaborations, we work in the framework of generalized functions in a very large sense, including distributions and hyperfunctions, in order to solve linear and nonlinear problems. This involves a wide range of ideas, theories, methods and techniques that were the subject of the conference.

The spectrum of relationships among the themes of the conference with other mathematical fields has increased significantly. It includes, but is not limited to, the theory of distributions, hyperfunctions and algebras of generalized functions, linear and nonlinear differential problems, the concept of regularity and functoriality in connection with sheaf theory, local and microlocal analysis, pseudodifferential and Fourier-integral operators, applications to geometry and mathematical physics, and other related subjects.

The contributions collected in this volume represent investigations in functional analysis or PDE theory using several interesting approaches and various methods and techniques: approximation of singular parts, multi-scale analysis, ultradistributional boundary values, cohomogical construction of special sheaves, regularized semi-groups for stochastic problems, specific classes of sequences in questions of convolution, and weak asymptotic methods in systems of conservation laws.

We thank the officials and editorial board of the *Rendiconti* for accepting to dedicate two issues of this journal to a collection of selected publications corresponding to talks given at the GF 2011 conference, and for their kind collaboration in the process of publishing these.

The Editors and Head of the GF 2011 Organizing Committee

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SCHRÖDINGER OPERATOR OF THE FORM $-\Delta u + a\delta u + b\frac{\partial\delta}{\partial x_1}u$

Abstract. The paper is devoted to the study of the formal differential expression of the form

$$Lu = -\Delta u + a\delta u + b\frac{\partial\delta}{\partial x_1}u$$

with generalized coefficients. Approximations of the singular part by means of a family of finite range operators are constructed and resolvent convergence of the approximations is investigated.

1. Introduction

The stationary Schrödinger operator with singular potential, symbolically written as

$$(1) \qquad -\Delta u + a\,\delta u,$$

where δ is the Dirac δ-function, and a is the so-called coupling constant, models scattering on a particle located at the origin of coordinates.

The mathematical difficulties that appear during the investigation of expression (1) are related to the fact that the product $\delta \cdot u$ in (1) is not defined in the classical theory of distributions. Therefore, giving sense to the expression (1) as a self-adjoint operator in the space $L^2(\mathbb{R}^3)$ (which is usually necessary in quantum theory) requires overcoming some obstacles.

A mathematical interpretation of the expression (1) was given by F. Berezin and L. Faddeev in [4]. It looks as follows. Let $\mathring{L}$ be the restriction of the Laplace operator $-\Delta$ on the domain

$$D(\mathring{L}) = \{u \in H^2(\mathbb{R}^3), \ u(0) = 0\},$$

where $H^2(\mathbb{R}^3)$ is the Sobolev space. Then $\mathring{L}$ is a symmetric, but non-self-adjoint operator on $L^2(\mathbb{R}^3)$. All self-adjoint extensions $L^{(\alpha)}$ of the operator $\mathring{L}$ can be considered as possible perturbations of the Laplace operator by potentials, supported at zero. These self-adjoint extensions $L^{(\alpha)}$ are naturally parameterized by a single real parameter $\alpha \in (-\infty, +\infty]$, the value $\alpha = +\infty$ corresponds to the Laplace operator, i.e. $\alpha = +\infty$ if the perturbation does not influence the operator.

The expression (1) by itself does not contain the information as to what self-adjoint extension $L^{(\alpha)}$ corresponds to the concrete situation. In application, the expression (1) arises as a formal limit (as $\varepsilon \to 0$) of some family of operators L_ε. For example let

$$(2) \qquad L_\varepsilon u = -\Delta u + q_\varepsilon(x)u,$$

where the potential $q_\varepsilon(x)$ is supported at ε-neighborhood of zero. Under the conditions

$$a(\varepsilon) = \int q_\varepsilon(x)dx \neq 0,$$

$$\int |q_\varepsilon(x)|dx \leq Ca(\varepsilon)$$

we have

$$\frac{1}{a(\varepsilon)}q_\varepsilon(x) \to \delta$$

and the family of potentials q_ε can be symbolically written as $a(\varepsilon)\delta$. Therefore the family (2) can be considered as an approximation of the formal expression (1).

The problem is to bring to light what self-adjoint extension corresponds to given approximation L_ε. As a rule, in usual sense the limit of L_ε does not exist and the resolvent convergence is considered here. Recall that one says that $L_\varepsilon \to L^{(\alpha)}$ in *resolvent sense*, if

$$(3) \qquad\qquad \lim_{\varepsilon \to 0}(L_\varepsilon - \lambda I)^{-1} = (L^{(\alpha)} - \lambda I)^{-1}.$$

Different approximations of (1) were investigated in many papers (see [1, 2, 6] and references in [1]).

The main result looks as follows: if $a(\varepsilon) = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$, the limit (3) exists and defines an operator $L^{(\alpha)}$; this limit is a non-trivial extension ($\alpha \neq \infty$) only in the so-called *resonance cases*, when $a(\varepsilon) = a_1\varepsilon + a_2\varepsilon^2 + \cdots$ and the number a_1 belong to a discrete set Λ from $\mathbb{R}$, where Λ depends on the given approximation.

In more general cases the family of potentials q_ε can be symbolically written as

$$(4) \qquad\qquad q_\varepsilon = a(\varepsilon)\delta + \sum_k b_k(\varepsilon)\frac{\partial\delta}{\partial x_k},$$

and then the family L_ε is an approximation of the formal expression

$$(5) \qquad\qquad Lu = -\Delta u + a\delta u + \sum_k b_k\frac{\partial\delta}{\partial x_k}u$$

Expressions of the form (5) were investigated early in the one-dimensional case [5, 8, 7].

In the present paper we consider some approximations of (5) in $L_2(\mathbb{R}^3)$ and calculate the limits (3). A new effect is discovered: *strong resonance cases* arise, when the limit (3) does not exist and the family L_ε cannot be interpreted as an operator in $L_2(\mathbb{R}^3)$.

2. Approximation using a family of finite rank operators

Let us consider the most simple approximation of the formal expression

$$(6) \qquad Lu = -\Delta u + a\delta u + b\frac{\partial \delta}{\partial x_1}u.$$

Let $\varphi_1, \varphi_2 \in D(\mathbb{R}^3)$ such that $\varphi_1(x), \varphi_2(x) \in \mathbb{R}$ and $\int \varphi_i(x)\,dx = 1$, $i = 1, 2$. The family of smooth functions

$$\varphi_{i,\varepsilon}(x) = \frac{1}{\varepsilon^3}\varphi_i\left(\frac{x}{\varepsilon}\right)$$

gives an approximation of δ as an element from the space of distributions $D'(\mathbb{R}^3)$. The family of linear functionals

$$\Phi_{k,\varepsilon}(u) = \int \varphi_{k,\varepsilon}(y)u(y)\,dy$$

gives an approximation of δ as a linear functional, since for smooth u

$$\Phi_{1,\varepsilon}(u)\varphi_{1,\varepsilon} \to u(0)\delta = \delta u,$$

the family of rank one operators

$$\Phi_{1,\varepsilon}(u)\varphi_{1,\varepsilon}$$

is an approximation of the operator of multiplication by δ. Let

$$\psi_\varepsilon(x) = \frac{\partial \varphi_{2,\varepsilon}(x)}{\partial x_1} = \frac{1}{\varepsilon^4}\frac{\partial \varphi_2}{\partial x_1}\left(\frac{x}{\varepsilon}\right).$$

In order to have below a uniform expression, we will use the notation

$$\varphi_3 = \frac{\partial \varphi_2(x)}{\partial x_1}, \quad \varphi_{3,\varepsilon} = \frac{1}{\varepsilon^3}\varphi_3\left(\frac{x}{\varepsilon}\right).$$

Then

$$\psi_\varepsilon = \frac{1}{\varepsilon}\varphi_{3,\varepsilon}.$$

The family of smooth functions ψ_ε gives an approximation of $\partial\delta/\partial x_1$ as an element from the space of distributions $D'(\mathbb{R}^3)$, the family of linear functionals

$$\Psi_\varepsilon(u) = \int \psi_\varepsilon(y)u(y)\,dy$$

gives an approximation of $\partial\delta/\partial x_1$ as a linear functional.

For a smooth function u, by definition

$$\frac{\partial \delta}{\partial x_1}u = -\frac{\partial u}{\partial x_1}(0)\delta + u(0)\frac{\partial \delta}{\partial x_1} = \langle\delta';u\rangle\delta + \langle\delta;u\rangle\delta'$$

and the family of rank two operators $\Psi_\varepsilon(u)\varphi_{2,\varepsilon}(x) + \Phi_{2,\varepsilon}(u)\psi_\varepsilon(x)$ is an approximation of the operator of multiplication by $\partial\delta/\partial x_1$.

Therefore the family of operators

(7) $$L_\varepsilon(u) = -\Delta u + T_\varepsilon u,$$

where

(8)
$$T_\varepsilon u = a(\varepsilon)\varphi_{1,\varepsilon}(x) \int u(y)\varphi_{1,\varepsilon}(y)\,dy$$
$$+ b(\varepsilon) \left[\varphi_{2,\varepsilon}(x) \int u(y)\psi_\varepsilon(y)\,dy + \psi_\varepsilon(x) \int u(y)\varphi_{2,\varepsilon}(y)\,dy \right],$$

is an approximation of the formal expression (6).

The problem is to find the limit of these approximations in the sense of resolvent convergence.

For fixed $\varepsilon > 0$ the resolvent $R(\lambda, \varepsilon) = (L_\varepsilon - \lambda I)^{-1}$ can be constructed in explicit form by using results from [3].

Let

$$A(\varepsilon) = \begin{pmatrix} a(\varepsilon) & 0 & 0 \\ 0 & 0 & b(\varepsilon) \\ 0 & b(\varepsilon) & 0 \end{pmatrix}$$

be a matrix, generated by the coefficients $a(\varepsilon)$ and $b(\varepsilon)$. The inverse matrix is

$$A^{-1}(\varepsilon) = \begin{pmatrix} \frac{1}{a(\varepsilon)} & 0 & 0 \\ 0 & 0 & \frac{1}{b(\varepsilon)} \\ 0 & \frac{1}{b(\varepsilon)} & 0 \end{pmatrix}.$$

Let us introduce the fundamental solution

$$E_\lambda(x) = \frac{1}{4\pi\|x\|} e^{-\mu\|x\|},$$

where $\mu^2 = -\lambda$, $\operatorname{Re}\mu > 0$ and a vector function

$$\bar{E}(\varepsilon) = (E_1(\varepsilon); E_2(\varepsilon); E_3(\varepsilon)), \quad E_k(\varepsilon) \in L_2(\mathbb{R}^3),$$

where

$$E_1(\varepsilon) = E_\lambda * \varphi_{1,\varepsilon}, \quad E_2(\varepsilon) = E_\lambda * \varphi_{2,\varepsilon}, \quad E_3(\varepsilon) = E_\lambda * \psi_\varepsilon.$$

Denote

$$\langle u, v \rangle = \int u(x)v(x)\,dx,$$

$$\bar{F}(\varepsilon) = (f_1(\varepsilon); f_2(\varepsilon); f_3(\varepsilon)), \quad f_k(\varepsilon) \in \mathbb{C},$$

where

$$f_1(\varepsilon) = \langle \varphi_{1,\varepsilon}, E_\lambda * f \rangle, \quad f_2(\varepsilon) = \langle \varphi_{2,\varepsilon}, E_\lambda * f \rangle, \quad f_3(\varepsilon) = \langle \psi_\varepsilon, E_\lambda * f \rangle.$$

THEOREM 1. *Let* $\varepsilon > 0$ *and suppose that* $a(\varepsilon) \in \mathbb{R}$, $a(\varepsilon) \neq 0$, $b(\varepsilon) \in \mathbb{R}$, $b(\varepsilon) \neq 0$. *The resolvent* $R(\lambda, \varepsilon)$ *is determined for* $\mathrm{Re}\ \lambda \neq 0$ *and can be given by the expression*

$$(9) \qquad R(\lambda, \varepsilon)f = f * E_\lambda - \left\langle \left[A^{-1}(\varepsilon) + B(\varepsilon, \lambda) \right]^{-1} \bar{F}(\varepsilon),\ \bar{E}(\varepsilon) \right\rangle,$$

where

$$B(\varepsilon, \lambda) = \begin{pmatrix} \langle \varphi_{1,\varepsilon}; E_1(\varepsilon) \rangle & \langle \varphi_{1,\varepsilon}; E_2(\varepsilon) \rangle & \langle \varphi_{1,\varepsilon}; E_3(\varepsilon) \rangle \\ \langle \varphi_{2,\varepsilon}; E_1(\varepsilon) \rangle & \langle \varphi_{2,\varepsilon}; E_2(\varepsilon) \rangle & \langle \varphi_{2,\varepsilon}; E_3(\varepsilon) \rangle \\ \langle \psi_\varepsilon; E_1(\varepsilon) \rangle & \langle \psi_\varepsilon; E_2(\varepsilon) \rangle & \langle \psi_\varepsilon; E_3(\varepsilon) \rangle \end{pmatrix}.$$

If $a(\varepsilon) \equiv 0$, $b(\varepsilon) \in \mathbb{R}$, $b(\varepsilon) \neq 0$, *then*

$$(10) \qquad R(\lambda, \varepsilon)f = f * E_\lambda - \left\langle \left[A^{-1}(\varepsilon) + B(\varepsilon, \lambda) \right]^{-1} \bar{F}(\varepsilon), \bar{E}(\varepsilon) \right\rangle,$$

where

$$A(\varepsilon) = \begin{pmatrix} 0 & b(\varepsilon) \\ b(\varepsilon) & 0 \end{pmatrix}, \quad B(\varepsilon, \lambda) = \begin{pmatrix} \langle \varphi_{2,\varepsilon}; E_2(\varepsilon) \rangle & \langle \varphi_{2,\varepsilon}; E_3(\varepsilon) \rangle \\ \langle \psi_\varepsilon; E_2(\varepsilon) \rangle & \langle \psi_\varepsilon; E_3(\varepsilon) \rangle \end{pmatrix},$$

$$\bar{F}(\varepsilon) = (f_2(\varepsilon); f_3(\varepsilon)), \quad \bar{E}(\varepsilon) = (E_2(\varepsilon); E_3(\varepsilon)).$$

3. Resolvent convergence of approximations

According to (9), the behavior of the resolvent $R(\lambda, \varepsilon)$ depends on the behavior of the matrices A^{-1}, $B(\varepsilon, \lambda)$, and on the behavior of the vectors $\bar{E}(\varepsilon)$ and $\bar{F}(\varepsilon)$.

Let us denote

$$D(\varepsilon, \lambda) = A^{-1}(\varepsilon) + B(\varepsilon, \lambda)$$

and let

$$D^{-1}(\varepsilon, \lambda) = (d_{ij}).$$

Then

$$\left\langle \left[A^{-1}(\varepsilon) + B(\varepsilon, \lambda) \right]^{-1} \bar{F}(\varepsilon), \bar{E}(\varepsilon) \right\rangle = \left\langle D^{-1}(\varepsilon, \lambda) \bar{F}(\varepsilon), \bar{E}(\varepsilon) \right\rangle$$

$$= [d_{11}(\varepsilon)f_1(\varepsilon) + d_{12}(\varepsilon)f_2(\varepsilon) + d_{13}(\varepsilon)f_3(\varepsilon)] E_1(\varepsilon)$$

$$+ [d_{21}(\varepsilon)f_1(\varepsilon) + d_{22}(\varepsilon)f_2(\varepsilon) + d_{23}(\varepsilon)f_3(\varepsilon)] E_2(\varepsilon)$$

$$+ [d_{31}(\varepsilon)f_1(\varepsilon) + d_{32}(\varepsilon)f_2(\varepsilon) + d_{33}(\varepsilon)f_3(\varepsilon)] E_3(\varepsilon).$$

Let us consider the behavior of the vectors $\bar{E}(\varepsilon)$ and $\bar{F}(\varepsilon)$ as $\varepsilon \to 0$.

It follows from the properties of functions $\varphi_{i,\varepsilon}$ that the limits

$$\lim_{\varepsilon \to 0} E_1(\varepsilon) = \lim_{\varepsilon \to 0} E_2(\varepsilon) = E_\lambda$$

exist in the space $L_2(\mathbb{R}^3)$, and for any $f \in L_2(\mathbb{R}^3)$ there exist the limits

$$\lim_{\varepsilon \to 0} f_1(\varepsilon) = \lim_{\varepsilon \to 0} f_2(\varepsilon) = (f * E_\lambda)(0).$$

In the distribution space $D'(\mathbb{R}^3)$ we have

$$\psi_\varepsilon \to \frac{\partial \delta}{\partial x_1}, \quad E_3(\varepsilon) = E_\lambda * \psi_\varepsilon \to \frac{\partial E_\lambda}{\partial x_1}.$$

But in the space $L_2(\mathbb{R}^3)$

$$\|\psi_\varepsilon\| = \sqrt{\int \left(\frac{1}{\varepsilon^2}\psi\left(\frac{x}{\varepsilon}\right)\right)^2 dx} = \left(\int |\psi(t)|^2 dt\right)^{\frac{1}{2}} \frac{1}{\varepsilon\sqrt{\varepsilon}}$$

the norm $\|E_3(\varepsilon)\|$ is increasing as $1/\varepsilon\sqrt{\varepsilon}$ and $E_3(\varepsilon)$ do not have a limit in the space $L_2(\mathbb{R}^3)$.

Similarly, it can be that for $f \in L_2(\mathbb{R}^3)$ the value $f_3(\varepsilon)$ increases and does not have a limit, but always $f_3(\varepsilon) = o(1/\varepsilon\sqrt{\varepsilon})$.

Therefore the finite limit of the resolvent (9) exists only if the elements $d_{13}(\varepsilon)$, $d_{23}(\varepsilon)$, $d_{33}(\varepsilon)$, as well as $(d_{31}(\varepsilon)f_1(\varepsilon) + d_{32}(\varepsilon)f_2(\varepsilon) + d_{33}(\varepsilon)f_3(\varepsilon))$ are small, namely

$$(11) \qquad d_{13}(\varepsilon) \sim o\left(\varepsilon^{\frac{3}{2}}\right); \quad d_{31}(\varepsilon)f_1(\varepsilon) + d_{32}(\varepsilon)f_2(\varepsilon) + d_{33}(\varepsilon)f_3(\varepsilon) \sim o\left(\varepsilon^{\frac{3}{2}}\right).$$

It follows that it is not enough to find the limit of the family of inverse matrices $D^{-1}(\varepsilon,\lambda)$, but it is also necessary to check subsequent terms (not only the main term) of the expansion of the matrix $D^{-1}(\varepsilon,\lambda)$, on which the behavior of expressions (11) depends.

Let us consider the behavior of d_{ij} as $\varepsilon \to 0$.

LEMMA 1. *The functions*

$$b_{lj}(\varepsilon,\lambda) = \langle \varphi_{l,\varepsilon}, E_\lambda * \varphi_{j,\varepsilon} \rangle$$

are analytic functions of two variables ε, μ $(\lambda \in \mathbb{C} \setminus \mathbb{R}_0^+,\ \lambda = -\mu^2,$ *where* $\operatorname{Re}\mu > 0$, $\mu = (-\lambda)^{\frac{1}{2}}$, *a continuous branch of the function* $(-\lambda)^{\frac{1}{2}}$), *and admit an expansion*

$$b_{lj}(\varepsilon,\lambda) \equiv b_{lj}(\varepsilon,-\mu^2) = \frac{1}{\varepsilon}\sum_{k=0}^{\infty}(-1)^k(\varepsilon\mu)^k M_{lj}^{(k-1)} = \sum_{k=-1}^{\infty}\varepsilon^k(-\mu)^{k+1}M_{lj}^{(k)},$$

where

$$M_{lj}^{(k)} = \frac{1}{4\pi(k+1)!}\int (\varphi_l * \varphi_j)|x|^k\, dx.$$

In particular, according to the properties of functions $\varphi_i(x)$,

$$M_{11}^{(0)} = M_{12}^{(0)} = M_{21}^{(0)} = M_{22}^{(0)} = \frac{1}{4\pi},$$

$$M_{13}^{(0)} = M_{23}^{(0)} = M_{33}^{(0)} = M_{31}^{(0)} = M_{32}^{(0)} = 0.$$

THEOREM 2. *Let*

$$a(\varepsilon) = \varepsilon a_1 + \varepsilon^2 a_2 + \varepsilon^3 a_3 + \dots, \quad b(\varepsilon) = \varepsilon^p b_p + \varepsilon^{p+1} b_{p+1} + \cdots,$$

where $a_1 \neq 0$, $b_p \neq 0$.

I. *If $a_1 \neq -1/M_{11}^{(-1)}$, then the resolvents (9) converge to the resolvent of the Laplace operator.*

II. *Suppose that the resonance condition $a_1 = -1/M_{11}^{(-1)}$ holds.*

- *If $p \geq 4$, the resolvents (9) converge to the resolvent of the operator A^α, where $\alpha = -a_2 \big(M_{11}^{(-1)}\big)^2$.*

- *If $p = 3$ the resolvents (9) converge to the resolvent of the operator A^α*

$$R_\alpha(\mu)f = f * E_\lambda - \frac{4\pi}{4\pi\alpha - \mu}\left[(f * E_\lambda(0))\right]E_\lambda$$

where $\alpha = -a_2\big(M_{11}^{(-1)}\big)^2 - b_3\big(M_{12}^{(-1)}M_{32}^{(-2)} + M_{21}^{(-1)}M_{13}^{(-2)}\big).$

- *If $p \leq 2$ the limit of the family of resolvents (9) does not exist.*

Proof. The matrix $D(\varepsilon, \lambda)$ can be written in the form

$$\begin{pmatrix} \frac{1}{a(\varepsilon)} + \frac{1}{\varepsilon}M_{11}^{(-1)} - \frac{\mu}{4\pi} + \dots & \frac{1}{\varepsilon}M_{12}^{(-1)} - \frac{\mu}{4\pi} + \dots & \frac{1}{\varepsilon^2}M_{13}^{(-1)} + \dots \\ \frac{1}{\varepsilon}M_{21}^{(-1)} - \frac{\mu}{4\pi} + \dots & \frac{1}{\varepsilon}M_{22}^{(-1)} - \frac{\mu}{4\pi} + \dots & \frac{1}{b(\varepsilon)} + \frac{1}{\varepsilon^2}M_{23}^{(-1)} + \dots \\ \frac{1}{\varepsilon^2}M_{31}^{(-1)} + \dots & \frac{1}{b(\varepsilon)} + \frac{1}{\varepsilon^2}M_{32}^{(-1)} + \dots & \frac{1}{\varepsilon^3}M_{33}^{(-1)} + \dots \end{pmatrix}.$$

Let us consider the most interesting case $p = 3$, when

$$\frac{1}{b(\varepsilon)} = \frac{1}{\varepsilon^3}b_{-3} + \frac{1}{\varepsilon^2}b_{-2} + \dots,$$

$$\frac{1}{a(\varepsilon)} = \frac{1}{\varepsilon}a_{-1} + \tilde{a}_0 + \dots,$$

where $b_{-3} = \frac{1}{b_3}$, $a_{-1} = \frac{1}{a_1}$. In this case the expansion of the matrix $D(\varepsilon, \lambda)$ is

$$\begin{pmatrix} \frac{1}{\varepsilon}(M_{11}^{(-1)} + a_{-1}) - \frac{\mu}{4\pi} + \dots & \frac{1}{\varepsilon}M_{12}^{(-1)} - \frac{\mu}{4\pi} + \dots & \frac{1}{\varepsilon^2}M_{13}^{(-1)} + \dots \\ \frac{1}{\varepsilon}M_{21}^{(-1)} - \frac{\mu}{4\pi} + \dots & \frac{1}{\varepsilon}M_{22}^{(-1)} - \frac{\mu}{4\pi} + \dots & \frac{1}{\varepsilon^3}b_{-3} + \frac{1}{\varepsilon^2}(M_{23}^{(-1)} + b_{-2}) + \dots \\ \frac{1}{\varepsilon^2}M_{31}^{(-1)} + \dots & \frac{1}{\varepsilon^3}b_{-3} + \frac{1}{\varepsilon^2}(M_{32}^{(-1)} + b_{-2}) + \dots & \frac{1}{\varepsilon^3}M_{33}^{(-1)} + \dots \end{pmatrix}.$$

For the inverse matrix, we have expression

$$D^{-1}(\varepsilon, \lambda) = \frac{1}{\det D(\varepsilon, \lambda)}D^{\#}(\varepsilon, \lambda),$$

where

$$
\begin{aligned}
\det D(\varepsilon,\lambda) \;=\; & -\frac{1}{\varepsilon^7}(b_{-3})^2(M_{11}^{(-1)}+a_{-1}) \\
& \frac{1}{\varepsilon^6}\left[\left(M_{12}^{(-1)}M_{31}^{(-1)}+M_{21}^{(-1)}M_{13}^{(-1)}-(M_{11}^{(-1)}+a_{-1})(M_{23}^{(-1)}+b_{-2})\right)b_{-3}\right. \\
& \left.-\left(-\frac{\mu}{4\pi}+\tilde{a}_0\right)(b_{-3})^2-(M_{11}^{(-1)}+a_{-1})(M_{23}^{(-1)}+b_{-2})b_{-3}\right]+\dots,
\end{aligned}
$$

$$
D^{\#}(\varepsilon,\lambda)=\begin{pmatrix} d_{11}^{\#} & d_{12}^{\#} & d_{13}^{\#} \\ d_{21}^{\#} & d_{22}^{\#} & d_{23}^{\#} \\ d_{31}^{\#} & d_{32}^{\#} & d_{33}^{\#} \end{pmatrix},
$$

$$
d_{11}^{\#}=-\frac{1}{\varepsilon^6}(b_{-3})^2+\dots,
$$

$$
d_{12}^{\#}=-\frac{1}{\varepsilon^5}b_{-3}M_{13}^{(-1)}+\dots,
$$

$$
d_{13}^{\#}=\frac{1}{\varepsilon^4}M_{12}^{(-1)}b_{-3}+\dots,
$$

$$
d_{21}^{\#}=-\frac{1}{\varepsilon^5}b_{-3}M_{31}^{(-1)}+\dots,
$$

$$
d_{22}^{\#}=\frac{1}{\varepsilon^4}\left(M_{33}^{(-1)}(M_{11}^{(-1)}+a_{-1})-M_{31}^{(-1)}M_{13}^{(-1)}\right)+\dots
$$

$$
d_{23}^{\#}=-\frac{1}{\varepsilon^4}b_{-3}(M_{11}^{(-1)}+a_{-1})+\dots
$$

$$
d_{31}^{\#}=\frac{1}{\varepsilon^4}M_{21}^{(-1)}b_{-3}+\dots
$$

$$
d_{32}^{\#}=-\frac{1}{\varepsilon^4}b_{-3}(M_{11}^{(-1)}+a_{-1})+\dots,
$$

$$
d_{33}^{\#}=\frac{1}{\varepsilon^2}\left(M_{22}^{(-1)}(M_{11}^{(-1)}+a_{-1})-M_{12}^{(-1)}M_{21}^{(-1)}\right)+\dots.
$$

If

$$
(M_{11}^{(-1)}+a_{-1})(b_{-3})^2\neq 0,
$$

then

$$
D^{-1}(\varepsilon,\lambda)\to 0,\ \ d_{13}=O(\varepsilon^3),\ \ d_{33}=O(\varepsilon^5),
$$

and the condition (11) fulfilled. It follows that the resolvents (9) converge to the resolvent of the Laplace operator.

The limit of the matrix $D^{-1}(\varepsilon,\lambda)$ can be non-zero, if *the resonance condition*

(12) $$(M_{11}^{(-1)} + a_{-1})(b_{-3})^2 = 0,$$

is fulfilled. This condition is equivalent to

$$a_1 = -1/M_{11}^{(-1)}$$

and the resonance is possible only if the coefficient $a(\varepsilon)$ admits an expansion

$$a(\varepsilon) = \varepsilon a_1 + \varepsilon^2 a_2 + o(\varepsilon^2),$$

where $a_1 = -1/M_{11}^{(-1)}$.

Under this condition

$$\det D(\varepsilon,\lambda) = \frac{1}{\varepsilon^6}\left[\left(M_{12}^{(-1)}M_{31}^{(-1)} + M_{21}^{(-1)}M_{13}^{(-1)}\right)b_{-3} - \left(-\frac{\mu}{4\pi} + \tilde{a}_0\right)(b_{-3})^2\right] + \dots$$

and $D^{-1}(\varepsilon,\lambda)$ is a matrix of the form

(13) $$\begin{pmatrix} \frac{-(b_{-3})^2}{\left(M_{12}^{(-1)}M_{31}^{(-1)}+M_{21}^{(-1)}M_{13}^{(-1)}\right)b_{-3}-\left(-\frac{\mu}{4\pi}+\tilde{a}_0\right)(b_{-3})^2} + \varepsilon(\cdots) & \varepsilon(\cdots) & \varepsilon^2(\cdots) \\ \varepsilon(\cdots) & \varepsilon^2(\cdots) & \varepsilon^2(\cdots) \\ \varepsilon^2(\cdots) & \varepsilon^2(\cdots) & \varepsilon^4(\cdots) \end{pmatrix}.$$

So

$$\lim_{\varepsilon\to 0} D^{-1}(\varepsilon,\lambda) = \begin{pmatrix} \frac{4\pi}{4\pi\alpha-\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\alpha = -a_2\left(M_{11}^{(-1)}\right)^2 - b_3\left(M_{12}^{(-1)}M_{32}^{(-1)} + M_{21}^{(-1)}M_{13}^{(-1)}\right).$$

It follows from (13) that the condition (11) are fulfilled and the limit of the family of resolvents (9) is the resolvent of the operator A^α:

$$R_\alpha(\mu)f = f * E_\lambda - \frac{4\pi}{4\pi\alpha - \mu}\left[(f * E_\lambda(0))\right]E_\lambda.$$

If $p \neq 3$ the calculations are similar. $\qquad\square$

We emphasize that a new effect arises here: it can be that the finite limit of the resolvents (9) does not exist. Let us demonstrate this effect in detail for the case $a(\varepsilon) = 0$.

THEOREM 3. *Let $a(\varepsilon) = 0$ and $b(\varepsilon) = \varepsilon^p b_p + \varepsilon^{p+1} b_{p+1} + \cdots$, where $b_p \neq 0$.*

- *If $p \geq 3$, then the limit of the family (4) in resolvent sense is the Laplace operator.*

- *If $p = 2$ and*

$$M_{33}^{(-3)} M_{22}^{(-1)} - \left(M_{23}^{(-2)} + R_2 \right) \left(M_{32}^{(-2)} + R_2 \right) \neq 0,$$

  then the limit of the family (4) in resolvent sense is the Laplace operator.

- *If $p = 2$ and resonance condition*

$$M_{33}^{(-3)} M_{22}^{(-1)} - \left(M_{23}^{(-2)} + R_2 \right) \left(M_{32}^{(-2)} + R_2 \right) = 0$$

  is fulfilled, then limit of the family of resolvents (6) does not exist.

Proof. If $a(\varepsilon) = 0$, then the matrix $D(\varepsilon, \lambda)$ is

$$D(\varepsilon, \lambda) = \begin{pmatrix} \frac{1}{\varepsilon} M_{22}^{(-1)} - \frac{\mu}{4\pi} + \ldots & \frac{1}{b(\varepsilon)} + \frac{1}{\varepsilon^2} M_{23}^{(-1)} + \ldots \\ \frac{1}{b(\varepsilon)} + \frac{1}{\varepsilon^2} M_{32}^{(-1)} + \ldots & \frac{1}{\varepsilon^3} M_{33}^{(-1)} + \ldots \end{pmatrix}.$$

Remark that

$$\frac{1}{b(\varepsilon)} = \frac{1}{\varepsilon^p} R_p + \frac{1}{\varepsilon^{p-1}} R_{p-1} + \ldots,$$

where $R_p = 1/b_p$.

If $p > 3$, then the main term in the expansion of the matrix $D(\varepsilon, \lambda)$ is the invertible matrix

$$\frac{1}{\varepsilon^p} \begin{pmatrix} 0 & R_p \\ R_p & 0 \end{pmatrix}$$

and $D^{-1}(\varepsilon, \lambda) \to 0$ as ε^p.

If $p = 3$, then the expansion of the matrix $D(\varepsilon, \lambda)$ begins from the invertible matrix

$$\frac{1}{\varepsilon^3} \begin{pmatrix} 0 & R_p \\ R_p & M_{33}^{(-1)} \end{pmatrix}$$

and thus $D^{-1}(\varepsilon, \lambda) \to 0$ as ε^3 when $\varepsilon \to 0$.

This means that if $p \geq 3$, then the conditions (11) are fulfilled and the limit of the family (7) in the resolvent sense is the Laplace operator.

If $p = 2$, then the matrix $D(\varepsilon, \lambda)$ can be written in the form

$$\begin{pmatrix} \frac{1}{\varepsilon} M_{22}^{(-1)} - \frac{\mu}{4\pi} + \ldots & \frac{1}{\varepsilon^2} \left(M_{23}^{(-1)} + R_2 \right) + \frac{1}{\varepsilon} \left(M_{23}^{(-1)} + R_1 \right) + \ldots \\ \frac{1}{\varepsilon^2} \left(M_{32}^{(-1)} + R_2 \right) + \frac{1}{\varepsilon} R_1 + \ldots & \frac{1}{\varepsilon^3} M_{33}^{(-1)} + \ldots \end{pmatrix}.$$

If

$$M_{33}^{(-1)}M_{22}^{(-1)} - \left(M_{23}^{(-1)} + R_2\right)\left(M_{32}^{(-1)} + R_2\right) \neq 0,$$

then $D^{-1}(\varepsilon, \lambda) \to 0$ when $\varepsilon \to 0$ as ε^3 and the limit of the family (4) in resolvent sense is the Laplace operator.

Set

$$\begin{aligned}
d_3 = {} & M_{22}^{(-1)}M_{33}^{(-1)} - \frac{\mu}{4\pi}M_{33}^{(-1)} - \left(M_{23}^{(-1)} + R_2\right)\left(M_{32}^{(-1)} + R_1\right) \\
& - \left(M_{23}^{(-1)} + R_1\right)\left(M_{32}^{(-1)} + R_2\right).
\end{aligned}$$

If the *resonance condition*

$$M_{33}^{(-3)}M_{22}^{(-1)} - \left(M_{23}^{(-2)} + R_2\right)\left(M_{32}^{(-2)} + R_2\right) = 0$$

is fulfilled and $d_3 \neq 0$, the matrix $D^{-1}(\varepsilon, \lambda)$ can be written in the form

$$\frac{1}{d_3}\left(\begin{array}{cc} M_{33}^{(-1)} + \dots & -\varepsilon\left(M_{32}^{(-1)} + R_2\right) - \varepsilon^2 R_1 + \dots \\ -\varepsilon\left(M_{23}^{(-1)} + R_2\right) - \varepsilon^2 R_1 + \dots & \varepsilon^2 M_{22}^{(-1)} - \varepsilon^3\frac{\mu}{4\pi} + \dots \end{array}\right)$$

and

$$D^{-1}(\varepsilon, \lambda) \to \left(\begin{array}{cc} \frac{M_{33}^{(-1)}}{d_3} & 0 \\ 0 & 0 \end{array}\right).$$

But in this case conditions, similar to the conditions (11), are not fulfilled. In particular the expression from (11) includes the term of the form

$$C\varepsilon^2 f_3(\varepsilon)E_3(\varepsilon),$$

which for some $f \in L_2(\mathbb{R}^3)$ can satisfy

$$\left\|\varepsilon^2 f_3(\varepsilon)E_3(\varepsilon)\right\|_{L_2} \to +\infty.$$

So the finite limit of the family of resolvents (6) does not exist.   □

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ON THE FORM OF INSTANTON-TYPE SOLUTIONS FOR EQUATIONS OF THE FIRST PAINLEVÉ HIERARCHY BY MULTIPLE-SCALE ANALYSIS

Abstract. We construct, using multiple-scale analysis, a formal solution containing sufficiently many free parameters for the first Painlevé hierarchy $(P_I)_m$ with a large parameter. This note is a short summary of our forthcoming paper [3].

1. Introduction

Aoki, Kawai and Takei, in 1990's, investigated the traditional Painlevé equations with a large parameter η from a viewpoint of the exact WKB analysis and local structure of formal solutions near turning points. In the papers [4, 8, 9, 10, 12], they constructed the formal solutions with 2-parameters called *instanton-type solutions* and established the connection formula among these solutions.

Several Painlevé hierarchies have recently been found in various areas of mathematics and it is also expected to establish the connection formula of instanton-type solutions for these hierarchies with a large parameter. For that purpose, we need to construct instanton-type solutions with sufficiently many free parameters so that Stokes phenomena are correctly caught.

In this note, we consider the first Painlevé hierarchy $(P_I)_m$ $(m = 1, 2, \dots)$ with a large parameter η and construct its instanton-type solutions. For the second member $(P_I)_2$ of the hierarchy, Y. Takei [13] had constructed instanton-type solutions by using singular perturbative reduction of a Hamiltonian system to its Birkhoff normal form. The first author [2] also constructed them by multiple-scale analysis. We follow the latter method and construct instanton-type solutions for a general member $(P_I)_m$. Detailed construction will be given in our forthcoming article [3].

2. Instanton-type solutions and multiple-scale analysis

2.1. The first Painlevé hierarchy with a large parameter

Let w_j $(j = 1, 2, \ldots)$ be the polynomial of variables u_k and v_l $(1 \leq k, l \leq j)$ defined by the recurrence relation

$$(1) \qquad w_j := \frac{1}{2} \sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2} \sum_{k=1}^{j-1} v_k v_{j-k} + c_j + \delta_{jm} t.$$

Here c_j is a constant and δ_{jm} stands for the Kronecker delta. Then the first Painlevé hierarchy $(P_{\mathrm{I}})_m$ with a large parameter η $(m = 1, 2, \ldots)$ is the system of non-linear equations

$$(2) \qquad \begin{cases} \eta^{-1} \dfrac{du_j}{dt} = 2v_j, & j = 1, 2, \ldots, m, \\[2mm] \eta^{-1} \dfrac{dv_j}{dt} = 2(u_{j+1} + u_1 u_j + w_j), & j = 1, 2, \ldots, m, \end{cases}$$

where u_j and v_j are unknown functions of t with the additional condition $u_{m+1} = 0$.

Note that the first member $(P_{\mathrm{I}})_1$ gives the traditional first Painlevé equation P_{I} with a large parameter η.

As the definition of the system is very complicated, we rewrite the system into the simpler form with the generating functions defined by

$$(3) \qquad \begin{aligned} & U(\theta) := \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) := \sum_{k=1}^{\infty} v_k \theta^k, \quad W(\theta) := \sum_{k=1}^{\infty} w_k \theta^{k+1} \\ & C(\theta) := \sum_{k=1}^{\infty} (c_k + \delta_{km} t) \theta^{k+1}. \end{aligned}$$

Here θ denotes an independent variable. Then the system (2) becomes

$$(4) \qquad \eta^{-1} \frac{d}{dt} \begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv \begin{pmatrix} 2V\theta \\ -(1+2u_1\theta)(1-U) + \dfrac{1+2C - \theta V^2}{1-U} \end{pmatrix}$$

with the condition that the coefficients of θ^{m+1} of U and V are zero. Here $A \equiv B$ implies that $A - B$ is equal to zero modulo θ^{m+2}.

2.2. 0-parameter solutions of $(P_{\mathrm{I}})_m$

For the construction of instanton-type solutions, we first construct a special kind of the solution of $(P_{\mathrm{I}})_m$ called a 0-*parameter solution*. We rewrite the result [7] on the

0-parameter solution of $(P_{\mathrm{I}})_m$ by using generating functions. Let us consider formal series in η^{-1} of the form

$$(5) \qquad \bar{u}_j(t) := \sum_{k=0}^{\infty} \eta^{-k} \hat{u}_{j,k}(t), \quad \bar{v}_j(t) := \sum_{k=0}^{\infty} \eta^{-k} \hat{v}_{j,k}(t), \quad j = 1, \ldots, m,$$

and let us define the generating functions with respect to the leading terms $\hat{u}_{j,0}$ and $\hat{v}_{j,0}$ of $\bar{u}_j$ and $\bar{v}_j$ by

$$(6) \qquad \hat{u}_0(\theta) := \sum_{j=1}^{\infty} \hat{u}_{j,0} \theta^j \quad \text{and} \quad \hat{v}_0(\theta) := \sum_{j=1}^{\infty} \hat{v}_{j,0} \theta^j,$$

respectively. Then, putting (5) into (2), we find the following equations for the generating functions:

$$(7) \qquad \hat{v}_0 = 0, \qquad (1 + 2\hat{u}_{1,0}\theta) = \frac{1 + 2C}{(1 - \hat{u}_0)^2}.$$

The equations can be easily solved and we have

$$(8) \qquad \hat{u}_0 = 1 - \sqrt{\frac{1 + 2C}{1 + 2\hat{u}_{1,0}\theta}}.$$

Note that the $\hat{u}_{1,0}$ in the right-hand side of (8) is taken so that the coefficient $\hat{u}_{m+1,0}$ of θ^{m+1} in $\hat{u}_0$ is zero.

2.3. Instanton-type solutions of $(P_{\mathrm{I}})_m$

Let $\alpha = -\frac{1}{2}$, and we fix it in what follows. We first introduce several notations to define instanton-type solutions.

Let $u_{k,j\alpha}$ and $v_{k,j\alpha}$ ($k = 1, 2, \ldots$, $j = 0, 1, 2, \ldots$) be unknown functions of the variable t. We define

$$(9) \qquad u := \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} u_{k,j\alpha}(t) \theta^k \eta^{j\alpha}, \qquad v := \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} v_{k,j\alpha}(t) \theta^k \eta^{j\alpha},$$

and denote by $\sigma_k^{\theta}(u)$ (resp. $\sigma_k^{\theta}(v)$) the coefficient of θ^k in u (resp. v).

Let Θ be the set of formal power series of θ without constant terms, and let $Q : (\Theta\theta)^2 \longrightarrow \Theta^2$ be the map defined by the relation

$$(10) \qquad Q\begin{pmatrix} x\theta \\ y\theta \end{pmatrix} := 2 \begin{pmatrix} y\theta \\ (1 + 2\hat{u}_{1,0}\theta)x - \sigma_1^{\theta}(x)\theta \end{pmatrix}$$

for $x = \sum_{j=1}^{\infty} x_j \theta^j, y = \sum_{j=1}^{\infty} y_j \theta^j \in \Theta$.

Then, by the change of unknown functions in (4),

(11) $U = \hat{u}_0 + \eta^\alpha(1 - \hat{u}_0)u, \qquad V = \hat{v}_0 + \eta^\alpha(1 - \hat{u}_0)v,$

we obtain the system of unknown functions (u, v) in the form

$$
\left(\eta^{-1}\frac{d}{dt} - Q\right)\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} \equiv \eta^\alpha\left(\begin{pmatrix} h\theta \\ S(u, v) \end{pmatrix} - uQ\begin{pmatrix} u\theta \\ v\theta \end{pmatrix}\right)
$$
(12)
$$
- \eta^{2\alpha}\left(u\begin{pmatrix} h \\ 2\sigma_1^\theta(u)u \end{pmatrix} + h\begin{pmatrix} u \\ v \end{pmatrix}\right)\theta
$$
$$
+ \eta^{3\alpha}u\left(h + \frac{d}{dt}\right)\begin{pmatrix} u \\ v \end{pmatrix}\theta,
$$

with

(13) $S(u, v) := \dfrac{1}{2}(-v, u)Q\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} + 3\sigma_1^\theta(u)u\theta \ \text{ and } \ h := \dfrac{d}{dt}(\log(1 - \hat{u}_0)).$

As the form of the above system suggests, the map Q plays an important role in the study of $(P_I)_m$ and its eigenvector $A(\lambda)$ corresponding to an eigenvalue λ in the sense of $Q(A(\lambda)\theta) = \lambda A(\lambda)\theta$ has the special form $\begin{pmatrix} a(\lambda) \\ \lambda a(\lambda)/2 \end{pmatrix}$ with

(14) $a(\lambda) := \dfrac{\theta}{1 - g(\lambda)\theta} = \displaystyle\sum_{k=0}^{\infty} g(\lambda)^k\theta^{k+1}, \quad g(\lambda) := \dfrac{\lambda^2 - 8\hat{u}_{1,0}}{4}.$

Since the coefficients of θ^{m+1} in U and V are zero, the coefficient $(1 - \hat{u}_0)a(\lambda)$ of θ^{m+1} must be zero. Hence the eigenvalue λ of Q is a root of the algebraic equation

(15) $\Lambda(\lambda, t) := g(\lambda)^m - \displaystyle\sum_{k=1}^{m} \hat{u}_{k,0}g(\lambda)^{m-k} = 0,$

where $\hat{u}_{k,0}$ is given by (5). Note that $\Lambda(\lambda, t)$ is an even function of λ.

Let $v_{\pm 1}(t), \ldots, v_{\pm m}(t)$ be the roots of the algebraic equation of λ where we set $v_k = -v_{-k}$, and let Ω be an open subset in $\mathbb{C}_t$. We always assume the following two conditions from now on.

(A1) The roots $v_i(t)$'s $(1 \leq |i| \leq m)$ are mutually distinct for each $t \in \Omega$.

(A2) The function $p_1v_1(t) + \cdots + p_mv_m(t)$ does not vanish identically on Ω for any $(p_1, \ldots, p_m) \in \mathbb{Z}^m \setminus \{0\}$.

Let $\tau := (\tau_1, \ldots, \tau_m)$ be m-independent variables, and let us define the rings

(16)
$$\mathcal{A}_\alpha(\Omega) := \mathcal{M}(\Omega)\left[\left[\eta^\alpha e^{\tau_1}, \ldots, \eta^\alpha e^{\tau_m}, \eta^\alpha e^{-\tau_1}, \ldots, \eta^\alpha e^{-\tau_m}\right]\right],$$
$$\mathcal{A}_\alpha^O(\Omega) := O(\Omega)\left[\left[\eta^\alpha e^{\tau_1}, \ldots, \eta^\alpha e^{\tau_m}, \eta^\alpha e^{-\tau_1}, \ldots, \eta^\alpha e^{-\tau_m}\right]\right],$$

where $\mathcal{M}(\Omega)$ (resp. $O(\Omega)$) denotes the set of formal power series in θ with coefficients in multi-valued holomorphic functions with a finite number of branching points and poles (resp. holomorphic functions) on Ω. We also denote by $\hat{\mathcal{A}}_\alpha(\Omega)$ (resp. $\hat{\mathcal{A}}_\alpha^O(\Omega)$) the subset in $\mathcal{A}_\alpha(\Omega)$ (resp. $\mathcal{A}_\alpha^O(\Omega)$) consisting of a formal power series of order less than or equal to α with respect to η. For $\varphi(\tau_1, \ldots, \tau_m, t, \theta, \eta) \in \mathcal{A}_\alpha(\Omega)$, we define the morphism ι by

(17)
$$\iota(\varphi) = \varphi\left(\eta \int^t v_1(s)ds, \ldots, \eta \int^t v_m(s)ds, t, \theta, \eta\right).$$

By replacing $\dfrac{d}{dt}$ in (12) with

(18)
$$\frac{\partial}{\partial t} + \eta v_1 \frac{\partial}{\partial \tau_1} + \eta v_2 \frac{\partial}{\partial \tau_2} + \cdots + \eta v_m \frac{\partial}{\partial \tau_m},$$

we obtain the partial differential equation associated with (12) of the form

(19)
$$\begin{aligned}
P\left(\begin{array}{c} u\theta \\ v\theta \end{array}\right) \equiv{}& \eta^\alpha\left(\left(\begin{array}{c} h\theta \\ S(u,v) \end{array}\right) + u\,P\left(\begin{array}{c} u\theta \\ v\theta \end{array}\right)\right) \\
& - \eta^{2\alpha}\left(u\left(\begin{array}{c} h \\ 2\sigma_1^\theta(u)u \end{array}\right) + \left(h + \frac{\partial}{\partial t}\right)\left(\begin{array}{c} u \\ v \end{array}\right)\right)\theta \\
& + \eta^{3\alpha} u\left(h + \frac{\partial}{\partial t}\right)\left(\begin{array}{c} u \\ v \end{array}\right)\theta.
\end{aligned}$$

Here the operator P is defined by

(20)
$$P := \chi_\tau - Q, \qquad \chi_\tau := v_1 \frac{\partial}{\partial \tau_1} + \cdots + v_m \frac{\partial}{\partial \tau_m}.$$

Then, for a solution $(u, v) \in \mathcal{A}_\alpha^2(\Omega) := (\mathcal{A}_\alpha(\Omega))^2$ of the system (19), the $(\iota(u), \iota(v))$ becomes a formal solution of the system (12).

DEFINITION 1. *We say that a formal solution (U, V) on Ω of the system* (4) *is of instanton-type if (U, V) has the form $(\hat{u}_0, \hat{v}_0) + \eta^\alpha(1 - \hat{u}_0)(\iota(u), \iota(v))$ for which $(u, v) \in \mathcal{A}_\alpha^2(\Omega)$ is a solution of* (19).

2.4. Existence of instanton-type solutions for $(P_I)_m$

Now we state our main theorem whose proof is given in [3].

THEOREM 1. *Let Ω be an open subset in $\mathbb{C}_t$ and we assume the conditions* (A1) *and* (A2). *Then we have instanton-type solutions of equations of* $(P_{\mathrm{I}})_m$ *with free $2m$-parameters* $(\beta_{-m}, \ldots, \beta_m) \in \mathbb{C}^{2m}[[\eta^{-1}]]$. *In particular, we can construct the solution* (u, v) *in* $\mathcal{A}_\alpha^2(\Omega)$ *for* (19) *of the form*

$$(21) \qquad \binom{u}{v} = \sum_{1 \le |k| \le m} f_k(\tau, t; \eta) A(\nu_k),$$

with

$$f_k(\tau, t; \eta) = \sum_{j=0, \ell=0}^{\infty} \eta^{(j+2\ell)\alpha} \left(\sum_{p \in \mathbb{Z}^m, |p|=j} f_{k,p,\ell}(t) e^{p \cdot \tau} \right),$$

where $|p| := |p_1| + \cdots + |p_m|$.

We can give the more precise form of f_k appearing in the above theorem. The leading term $f_{k,0}$ and the subleading term $f_{k,\alpha}$ of f_k, for example, are described by the following Lemmas 1 and 2.

LEMMA 1. *We have*

$$(22) \qquad f_{k,0} = \omega_k e^{\tau_k} \quad (1 \le |k| \le m),$$

where ω_k, ω_{-k} $(1 \le k \le m)$ are multi-valued holomorphic functions on Ω of the form

$$
\begin{aligned}
(23) \quad &\omega_k = \beta_k^{(0)} \exp\left(\int^t \left(\frac{1}{\nu_k} \sum_{j=1}^m \phi(k, j) \beta_j^{(0)} \beta_{-j}^{(0)} \exp\left(-2\int^t h_j dt\right) - h_k \right) dt \right), \\
&\omega_{-k} = \beta_{-k}^{(0)} \exp\left(\int^t \left(-\frac{1}{\nu_k} \sum_{j=1}^m \phi(k, j) \beta_j^{(0)} \beta_{-j}^{(0)} \exp\left(-2\int^t h_j dt\right) - h_k \right) dt \right)
\end{aligned}
$$

with free $2m$-parameters $(\beta_{-m}^{(0)}, \ldots, \beta_m^{(0)}) \in \mathbb{C}^{2m}$. Here $\phi(k, j)$ are rational functions of the variables ν_k's and h_k are holomorphic functions in Ω with the conditions

$$(24) \qquad \phi(k, j) = \phi(-k, j) \quad (1 \le j \le m), \quad h_k = h_{-k}.$$

For the explicit forms of $\phi(k, j)$ and h_k, see [3]. Furthermore the subleading term of the solution is given by the following.

LEMMA 2. *For any k $(1 \le |k| \le m)$, the $f_{k,\alpha}$ is given by*

$$
\begin{aligned}
(25) \quad f_{k,\alpha} &= \sum_{\substack{1 \le |j| \le m, \\ j \ne -k}} \frac{2}{(\nu_k + \nu_j)\nu_k \nu_j} \left((2\nu_k + \nu_j)\omega_k \omega_j e^{\tau_k + \tau_j} - \nu_j \omega_{-k} \omega_{-j} e^{-\tau_k - \tau_j} \right) \\
&\quad - \left(\sum_{j=1}^m \frac{\nu_j^2}{\nu_k} h_{j,k} \omega_j \omega_{-j} + \frac{6}{\nu_k} \omega_k \omega_{-k} + \frac{1}{2}\gamma_k \right) \times \frac{1}{\nu_k}.
\end{aligned}
$$

Here γ_k are holomorphic functions in Ω with $\gamma_k = \gamma_{-k}$ and $h_{k,j}$ are defined by

$$(26) \qquad h_{k,j} := \frac{4 \displaystyle\prod_{\substack{1 \le l \le m, \\ l \ne k, j}} \left(\nu_k^2 - \nu_l^2\right)}{\displaystyle\prod_{\substack{1 \le l \le m, \\ l \ne j}} \left(\nu_j^2 - \nu_l^2\right)} \quad (j \ne k), \qquad h_{k,k} := \sum_{\substack{l=1, \\ l \ne k}}^{m} \frac{4}{\nu_k^2 - \nu_l^2}$$

with the convention $h_{k,j} := h_{|k|,|j|}$.

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ANALYTIC FUNCTIONS, CAUCHY KERNEL, AND CAUCHY INTEGRAL IN TUBES

Abstract. Analytic functions in tubes in association with ultradistributional boundary values are analyzed. Conditions are stated on the analytic functions satisfying a certain norm growth which force the functions to be in the Hardy space H^2. Properties of the Cauchy kernel and Cauchy integral are obtained which extend results obtained previously by the author and collaborators.

1. Introduction

The definitions of regular cone $C \subset \mathbb{R}^n$ and the corresponding dual cone C^* of C are given in [2, Chapter 1] where the notation used in this paper is also contained. The Cauchy and Poisson kernels corresponding to the tube $T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$ with $t \in \mathbb{R}^n$ are defined by

$$K(z-t) = \int_{C^*} \exp(2\pi i \langle z-t, u \rangle) du, \quad z \in T^C = \mathbb{R}^n + iC, \quad t \in \mathbb{R}^n,$$

and

$$Q(z;t) = \frac{|K(z-t)|^2}{K(2iy)}, \quad z = x + iy \in T^C = \mathbb{R}^n + iC, \quad t \in \mathbb{R}^n,$$

respectively; see [2, Chapter 1]. The sequences $M_p, p = 0, 1, 2, ...$, of positive integers with conditions $(M.1)$ through $(M.3')$ and the subsequently defined spaces of functions and ultradistributions of Beurling and Roumieu type $\mathcal{D}(*, L^s)$ and $\mathcal{D}'(*, L^s)$, where $*$ is either (M_p) of Beurling type or $\{M_p\}$ of Roumieu type, are given in [2, Chapter 2]. For sequences M_p which satisfy the conditions $(M.1)$ and $(M.3')$, the Cauchy kernel $K(z-t) \in \mathcal{D}(*, L^s), 1 < s \le \infty$, [2, Theorem 4.1.1] as a function of $t \in \mathbb{R}^n$ for $z \in T^C$ where C is a regular cone in $\mathbb{R}^n$; and the Poisson kernel $Q(z;t) \in \mathcal{D}(*, L^s), 1 \le s \le \infty$, [2, Theorem 4.1.2] as a function of $t \in \mathbb{R}^n$ for $z \in T^C$. For $U \in \mathcal{D}'(*, L^s)$ the Cauchy and Poisson integrals are defined as $C(U;z) = \langle U_t, K(z-t) \rangle$ and $P(U;z) = \langle U_t, Q(z;t) \rangle$, respectively, for $z \in T^C$ and $t \in \mathbb{R}^n$ for appropriate values of s; see [2, Chapter 4].

In this paper we extend results in [2] concerning the norm growth of $C(U;z)$, $U \in \mathcal{D}'(*, L^s)$, to the values $1 < s < 2$. We obtain a new boundary value result for $C(U;z)$ and obtain a decomposition theorem for $U \in \mathcal{D}'(*.L^s), 1 < s < 2$. Considering functions analytic in the tube T^C which are known to have $\mathcal{D}'((M_p), L^2)$ boundary values, we impose conditions on the boundary value which force the analytic functions to be in the Hardy space $H^2(T^C)$.

2. Cauchy kernel and integral

Let the sequence M_p satisfy $(M.1)$ and $(M.3')$. For $U \in \mathcal{D}'(*,L^s), 1 < s < \infty, C(U;z)$ is an analytic function in $T^C = \mathbb{R}^n + iC$ [2, Theorem 4.2.1]; and we have a pointwise growth estimate on $C(U;z)$ ([1], [2, Theorem 4.2.2]). We have a norm growth estimate [2, Theorem 4.2.3] on $C(U;z)$ for $2 \leq s < \infty$; we extend this to $1 < s < 2$ by obtaining a norm growth on $C(U;z)$ for these cases. We recall the associated function $M^*(\rho)$ given in [2, p. 15].

THEOREM 1. *Let C be a regular cone in $\mathbb{R}^n$ and let the sequence M_p satisfy properties $(M.1)$ and $(M.3')$.*

Let $U \in \mathcal{D}'((M_p),L^s), 1 < s < 2$. For $1/r + 1/s = 1$

(1) $$\|C(U;z)\|_{L^r} \leq A|y|^{-n}e^{M^*(T/|y|)}, \quad |y| \leq 1.$$

If $n = 1$, (1) holds for $y \in C = (0,\infty)$ or $y \in C = (-\infty,0)$ where A depends on r and s and $T > 0$ is a fixed constant. If $n \geq 2$, (1) holds for $y \in C$ in which case A depends on y,r,s,n, and C; and $T > 0$ is a fixed constant which depends on y. If $n \geq 2$, (1) also holds for $y \in C' \subset C$, for any compact subcone C' of C, in which case A depends on C,C',r,s, and n; and $T > 0$ is a fixed constant which depends on C'.

Let $U \in \mathcal{D}'(\{M_p\},L^s), 1 < s < 2$, and $1/r + 1/s = 1$. If $n = 1$, (1) holds for $y \in C = (0,\infty)$ or $y \in C = (-\infty,0)$ where A depends on r and s and $T > 0$ is arbitrary. If $n \geq 2$, (1) holds for $y \in C$ in which case A depends on y,r,s,n, and C; and $T > 0$ is arbitrary. If $n \geq 2$, (1) also holds for $y \in C' \subset C$, for any compact subcone C' of C, in which case A depends on C,C',r,s, and n; and $T > 0$ is arbitrary.

Proof. Both cases for $* = (M_p)$ or $* = \{M_p\}$ when the dimension $n = 1$ are proved by analysis similar to that contained in the proof of [2, Theorem 5.4.2, pp. 126–128]. By this proof we in fact have for $n = 1$

$$\|C(U;z)\|_{L^r} \leq Ae^{M^*(T/|y|)}$$

for $y \in (0,\infty)$ or $y \in (-\infty,0)$; but the constant A depends on y in this case. By restricting $|y| \leq 1$, (2.1) is obtained in both cases where A is independent of y.

We now prove (1) for dimension $n \geq 2$. Using [2, Theorems 2.3.1 and 2.3.2]

(2) $$C(U;z) = \langle U_t, K(z-t) \rangle = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} F_\alpha(x,y)$$

where

$$F_\alpha(x,y) = \int_{\mathbb{R}^n} f_\alpha(t) D_t^\alpha K(z-t)dt$$

and the $f_\alpha \in L^r, 1/r + 1/s = 1$, satisfy the properties in [2, Theorems 2.3.1 and 2.3.2]. We note the estimate [3, (3.22)] on $D_t^\alpha K(z-t)$ which holds for $z = x + iy \in \mathbb{R}^n + iC$. In [3, (3.22)] the $\delta > 0$ depends on $y \in C$; whereas this δ depends on $C' \subset C$ if y is

restricted to compact subcones $C' \subset C$. From this estimate [3, (3.22)] and restricting $|y| \leq 1$ we have a constant Q_δ, depending on δ, such that

$$|D_t^\alpha K(z-t)| \leq S(C^*)\Gamma(n)\pi^{-n-|\alpha|}|\alpha|^{|\alpha|}Q_\delta^{1+|\alpha|}|y|^{-n-|\alpha|}(\delta+|x-t|^2)^{-n+1};$$

and recall the other constants in this estimate from [3, (3.22)]. Using this estimate with $|y| \leq 1$,

$$|F_\alpha(x,y)| \leq S(C^*)\Gamma(n)\pi^{-n-|\alpha|}|\alpha|^{|\alpha|}Q_\delta^{1+|\alpha|}|y|^{-n-|\alpha|}\widetilde{F}_\alpha(x,y)$$

where

$$\widetilde{F}_\alpha(x,y) = \int_{\mathbb{R}^n} |f_\alpha(t)|(\delta+|x-t|^2)^{-n+1}dt$$

from which

$$|F_\alpha(x,y)| \leq S(C^*)\Gamma(n)\pi^{-n-|\alpha|}|\alpha|^{|\alpha|}Q_\delta^{1+|\alpha|}|y|^{-n-|\alpha|}Q'_{\delta,s,r}$$

$$\times \left(\int_{\mathbb{R}^n} |f_\alpha(t)|^r(\delta+|x-t|^2)^{-1/2-r/4}dt \right)^{1/r}$$

follows using Hölder's inequality. Now using Fubini's theorem

$$\|F_\alpha(x,y)\|_{L^r} \leq S(C^*)\Gamma(n)\pi^{-n-|\alpha|}|\alpha|^{|\alpha|}Q_\delta^{1+|\alpha|}Q''_{\delta,s,r}|y|^{-n-|\alpha|}\|f_\alpha\|_{L^r}.$$

Using this estimate we return to (2) and obtain

$$\|C(U;z)\|_{L^r} \leq \sum_{|\alpha|=0}^{\infty} \|F_\alpha(x,y)\|_{L^r}$$

$$\leq S(C^*)\Gamma(n)\pi^{-n}Q'''_{\delta,s,r}|y|^{-n} \sum_{|\alpha|=0}^{\infty} \pi^{-|\alpha|}|\alpha|^{|\alpha|}(Q_\delta/|y|)^{|\alpha|}\|f_\alpha\|_{L^r}.$$

From the proof of Stirling's formula

$$|\alpha|^{|\alpha|} \leq e^{|\alpha|}|\alpha|!, \qquad |\alpha| = 1,2,3,\ldots,$$

and we have the convention that $|\alpha|^{|\alpha|} = 1$ if $|\alpha| = 0$. Using these facts, the norm properties of f_α from [2, Theorems 2.3.1 and 2.3.2] and proceeding as in [2, (4.73) and (4.60)] the growth (1) follows where $T = 2eQ_\delta/k\pi$ for some $k > 0$ if $* = (M_p)$ Beurling and for all $k > 0$ if $* = \{M_p\}$ Roumieu. Throughout the analysis the constant Q_δ depends on $y \in C$ if y is not restricted to compact subcones $C' \subset C$. If $y \in C' \subset C$, the constant Q_δ, and hence the constants A and T, is not dependent on y but is dependent on the compact subcone $C' \subset C$. The proof of Theorem 1 is complete. $\square$

In addition to completing the L^r norm growth properties for the considered Cauchy integral for all $s, 1 < s < \infty$, Theorem 1 shows that the Cauchy integral $C(U;z)$ studied there is an example of the type of analytic function with norm growth that we study in section 3 below in this paper.

We make a comment concerning the relation between Theorem 1 and [2, Theorem 5.4.2, p. 126]. For $y \in C$

$$|y|^{-n} e^{M^*(T/|y|)} \leq Q e^{M^*(T_1/|y|)}$$

where the constant Q does not depend on y for $T_1 > T$. The estimate obtained in the proof of [2, Theorem 5.4.2] is entirely correct, and the estimate obtained in Theorem 1 is a different one which is more precise.

The Fourier transform of a L^1 function ϕ will be symbolized by $\mathcal{F}[\phi(t);x]$ or by $\hat{\phi}(x)$ with $\mathcal{F}^{-1}[\phi(t);x]$ denoting the inverse Fourier transform. We have proved

$$\lim_{y \to 0, y \in C} \langle K(x+iy-t), \phi(x) \rangle = \mathcal{F}^{-1}[I_{C^*}(u)\hat{\phi}(u);t], \quad \phi \in \mathcal{D}(*, \mathbb{R}^n),$$

in $\mathcal{D}(*, L^s), 2 \leq s < \infty$, [2, Theorems 4.2.5 and 4.2.6]; here C is a regular cone, C^* is the dual cone, and $I_{C^*}(t)$ is the characteristic function of C^*. This result is used to obtain a boundary value result and a decomposition theorem for $U \in \mathcal{D}'(*, L^s), 2 \leq s < \infty$, [2, Corollary 4.2.1 and Theorem 4.2.7]. We extend the above limit property and subsequent results to $1 < s < 2$ for the cases that $C = (0, \infty)$ or $C = (-\infty, 0)$ in $\mathbb{R}^1$ or $C = C_\mu$ is a n-rant cone in $\mathbb{R}^n$ where

$$C_\mu = \{y \in \mathbb{R}^n : \mu_j y_j > 0, j = 1, ..., n\}, \quad \mu_j \in \{-1, 1\}, \quad j = l, ..., n.$$

THEOREM 2. *Let C_μ be any n-rant cone in $\mathbb{R}^n$, and let $I_{C_\mu^*}$ be the characteristic function of the dual cone $C_\mu^* = \overline{C}_\mu$. Let $\phi \in \mathcal{D}(*, \mathbb{R}^n)$ where the sequence M_p satisfies the properties $(M.1)$, $(M.2)$, and $(M.3')$. We have*

$$\lim_{y \to 0, y \in C_\mu} \langle K(x+iy-t), \phi(x) \rangle = \int_{\mathbb{R}^n} I_{C_\mu^*}(u)\hat{\phi}(u) e^{-2\pi i \langle t, u \rangle} du$$

in $\mathcal{D}(*, L^s), 1 < s < 2$.

Proof. Since the *n*-rant cone C_μ, its dual cone $C_\mu^* = \overline{C}_\mu$, and the corresponding Cauchy kernel function are products of one-dimensional half lines and the one-dimensional Cauchy kernel function, it is sufficient to prove the result in one dimension. We give an outline of the proof for the case that $C = (0, \infty)$. For $\phi \in \mathcal{D}(*, \mathbb{R}^n)$ we know

$$\mathcal{F}[D_x^\alpha \phi(x); u] = u^\alpha \mathcal{F}[\phi(x); u].$$

As noted in [2, p. 14], condition $(M.2)$ on the sequence M_p implies the existence of constants A and H larger than 1 such that

$$M_{p+q} \leq AH^{p+q} M_p M_q.$$

Using these facts and integration by parts techniques we prove the following for the cone $C = (0, \infty)$ with $1 < s < 2$:

$$\langle K(x+iy-t), \phi(x) \rangle \in \mathcal{D}(*, L^s), \quad t \in \mathbb{R}^1, \quad y \in C;$$

$$\int_0^\infty \hat{\phi}(u)e^{-2\pi itu}du \in \mathcal{D}(*,L^s), \quad t \in \mathbb{R}^1;$$

$$\left\| D_t^\alpha(\langle K(x+iy-t),\phi(x)\rangle - \int_0^\infty \hat{\phi}(u)e^{-2\pi itu}du) \right\|_{L^s} \le Nh^\alpha M_\alpha,$$

$\alpha = 0,1,2,\ldots$, for every $h > 0, (M_p)$ Beurling, or for some $h > 0, \{M_p\}$ Roumieu, with $N > 0$ independent of $y > 0$ and α; and

$$\lim_{y\to 0,\, y\in(0,\infty)} \|D_t^\alpha(\langle K(x+iy-t),\phi(x)\rangle - \int_0^\infty \hat{\phi}(u)e^{-2\pi itu}du)\|_{L^s} = 0,$$

$\alpha = 0,1,2,\ldots$, which proves the result. $\qquad\square$

As noted above Theorem 2 extends [2, Theorems 4.2.5 and 4.2.6] to the cases $1 < s < 2$ for half line cones $C = (0,\infty)$ and $C = (-\infty,0)$ and for n-rant cones $C = C_\mu$.

The following result extends [2, Corollary 4.2.1] to the cases $1 < s < 2$ for the n-rant cones $C = C_\mu$ considered in Theorem 2.

THEOREM 3. *Let* $U \in \mathcal{D}'(*,L^s), 1 < s < 2$, *and* $\phi \in \mathcal{D}(*,\mathbb{R}^n)$. *Let the sequence* M_p *satisfy* $(M.1),(M.2)$, *and* $(M.3')$. *We have*

$$\lim_{y\to 0,\, y\in C_\mu} \langle C(U;x+iy),\phi(x)\rangle = \left\langle U, \int_{\mathbb{R}^n} I_{C_\mu^*}(u)\hat{\phi}(u)e^{-2\pi i\langle t,u\rangle}du \right\rangle.$$

Proof. Using the change of order of integration formula [2, Theorem 4.2.4], Theorem 2, and the continuity of $U \in \mathcal{D}'(*,L^s)$ we have

$$\lim_{y\to 0,\, y\in C_\mu} \langle C(U;x+iy),\phi(x)\rangle = \lim_{y\to 0, y\in C_\mu} \langle U, \langle K(x+iy-t),\phi(x)\rangle\rangle$$

$$= \left\langle U, \int_{\mathbb{R}^n} I_{C_\mu^*}(u)\hat{\phi}(u)e^{-2\pi i\langle t,u\rangle}du \right\rangle.$$

$\qquad\square$

Now we may obtain a decomposition result for $U \in \mathcal{D}'(*,L^s), 1 < s < 2$, similar to that which we have obtained for $2 \le s < \infty$ in [2, Theorem 4.2.7]. For each C_μ we form

$$f_\mu(z) = \left\langle U, \int_{C_\mu^*} \exp(2\pi i\langle z-t,u\rangle)du \right\rangle, \quad z \in T^{C_\mu},$$

and note that there are 2^n n-tuples μ. As in the proof of [2, Theorem 4.2.7] we use Theorem 3 here and obtain

$$\langle U,\phi\rangle = \langle U, \sum_\mu \int_{C_\mu^*} \hat{\phi}(u)e^{-2\pi i\langle t,u\rangle}du \rangle = \sum_\mu \lim_{y\to 0, y\in C_\mu} \langle f_\mu(x+iy),\phi(x)\rangle$$

for $U \in \mathcal{D}'(*,L^s), 1 < s < 2$, and $\phi \in \mathcal{D}(*,\mathbb{R}^n)$. This extends [2, Theorem 4.2.7] to $1 < s < 2$ for n-rant cones $C = C_\mu$.

3. Analytic functions

Let B denote a proper open subset of $\mathbb{R}^n$, and let $d(y)$ denote the distance from $y \in B$ to the complement of B in $\mathbb{R}^n$. In [2, Chapter 5] we have considered analytic functions in tubes $T^B = \mathbb{R}^n + iB$ satisfying

$$(3) \qquad \|f(x+iy)\|_{L^r} \leq K(1+(d(y))^{-m})^q e^{M^*(T/|y|)}, \quad y \in B,$$

where $K > 0, T > 0, m \geq 0$, and $q \geq 0$ are all independent of $y \in B$ and $M^*(\rho)$ is the associated function of the sequence M_p defined in [2, p. 15].

For $B = C$, a regular cone in $\mathbb{R}^n$, we have shown in [2, section 5.2] that analytic functions $f(z), z \in T^C$, which satisfy (3) for $m = 0$ or $q = 0$ and $1 < r \leq 2$, obtain a boundary value $U \in \mathcal{D}'((M_p), L^1)$ as $y \to 0, y \in C$, [2, Theorem 5.2.1]. A converse result is proved in [2, Theorem 5.2.2]. In this converse result we can now easily prove as an additional conclusion that

$$f(z) = \langle U_t, K(z-t) \rangle, \quad z \in T^C,$$

using the proof of [2, Theorem 5.2.2]; that is, in [2, Theorem 5.2.2] we can add as a conclusion that the analytic function $f(z)$ constructed there can be recovered as the Cauchy integral of its boundary value.

Additionally we note that the result [2, Theorem 5.3.1], and hence the results [2, Theorems 5.3.2 and 5.3.3], can be stated and proved under the more general hypothesis that the set C is any open connected subset of $\mathbb{R}^n$ which is contained in or is any of the 2^n n-rants C_μ in $\mathbb{R}^n$. The only sacrifice in the conclusion is that the support of the constructed function $g(t)$ can not be determined under this more general hypothesis.

Let us recall the Hardy H^r functions in tubes $T^C = \mathbb{R}^n + iC$, for C being a regular cone, which have been studied extensively by Stein and Weiss [5]. An analytic function $f(z), z \in T^C$, is in the Hardy space $H^r = H^r(T^C), r > 0$, if

$$\|f(x+iy)\|_{L^r} \leq A, \quad y \in C,$$

where the constant $A > 0$ is independent of $y \in C$. In [4] we showed that if an analytic function $f(z), z \in T^C$, has a distributional boundary value in $\mathcal{S}'$ which is a $L^r, 1 \leq r \leq \infty$, function, the analytic function must be in H^r. Results of this type have applications in quantum field theory.

The Hardy spaces H^r are subspaces of the analytic functions in T^C which satisfy (3) for $m = 0$ or $q = 0$, which are the analytic functions we considered in [2, section 5.2] with respect to the existence of boundary values in $\mathcal{D}'((M_p), L^r)$. Thus for the values of r that we have considered in [2, section 5.2], $f(z) \in H^r$ will have an ultradistributional boundary value. We now obtain a result, like those in [4], in which we show for $r = 2$ that any analytic function $f(z), z \in T^C$, which satisfies (3) with $m = 0$ or $q = 0$ and with $r = 2$ and whose boundary value in $\mathcal{D}'((M_p), L^2)$, which exists by [2, Corollary 5.2.3], is a bounded L^2 function in $\mathcal{D}'((M_p), L^2)$ must be a H^2 function.

THEOREM 4. *Let $f(z)$ be analytic in T^C, C being a regular cone, and satisfy*

(4) $$\|f(x+iy)\|_{L^2} \leq Ke^{M^*(T/|y|)}, \, y \in C.$$

Let the $\mathcal{D}'((M_p),L^2)$ boundary value of $f(z)$ be a bounded function $h \in \mathcal{D}'((M_p),L^2)$. We have $f(z) \in H^2(T^C)$ and

$$f(z) = \int_{\mathbb{R}^n} h(t)K(z-t)dt = \int_{\mathbb{R}^n} h(t)Q(z;t)dt, \quad z \in T^C.$$

Proof. From [2, Corollary 5.2.3] and its proof we have

(5) $$f(z) = \int_{\mathbb{R}^n} h(t)K(z-t)dt = \int_{\mathbb{R}^n} g(t)e^{2\pi i\langle z,t\rangle}dt, \quad z \in T^C,$$

where $\mathrm{supp}(g) \subseteq C^*$ almost everywhere and $h = \mathcal{F}^{-1}[\tilde{g}]$ with this inverse Fourier transform being an element in $\mathcal{D}'((M_p),L^2)$ [2, (2.52), p. 27]. Now let $w = u + iv \in T^C$ be arbitrary but fixed and consider $K(z+w)f(z), z \in T^C$, where

$$K(z+w) = \int_{C^*} \exp(2\pi i\langle z+w,u\rangle)du.$$

Using [4, Lemma 3.2] we have that $K(z+w)$ is analytic in $z \in T^C$ and

$$|K(z+w)| \leq M_v < \infty, \quad z \in T^C,$$

where $M_v > 0$ is a constant that depends only on $v = \mathrm{Im}(w)$. Thus $K(z+w)f(z)$ is analytic in $z \in T^C$ and satisfies

$$\|K(x+iy+w)f(x+iy)\|_{L^2} \leq KM_v e^{M^*(T/|y|)}, \quad y \in C,$$

with M_v being independent of $z \in T^C$. We have $K(x+iy+w)f(x+iy) \to K(x+w)h(x)$ in $\mathcal{D}'((M_p).L^2)$ as $y \to 0, y \in C$; and $K(x+w)h(x) \in \mathcal{D}'((M_p),L^2)$ since $K(x+w)$ is bounded in $x \in \mathbb{R}^n$. By the proof of [2, Corollary 5.2.3] applied to $K(z+w)f(z), z \in T^C$, we have

(6) $$K(z+w)f(z) = \int_{\mathbb{R}^n} K(t+w)h(t)K(z-t)dt, z \in T^C,$$

for any fixed $w \in T^C$. Now corresponding to $z = x + iy \in T^C$ choose $w = -x + iy \in T^C$ and obtain

$$K(t+w)K(z-t) = |K(z-t)|^2$$

and

$$K(z+w) = K(2iy).$$

With this choice of $w = -x + iy \in T^C$, (6) becomes

(7) $$f(z) = \int_{\mathbb{R}^n} h(t)Q(z;t)dt, \quad z \in T^C,$$

where $Q(z;t)$ is the Poisson kernel for $z \in T^C$ and $t \in \mathbb{R}^n$. From (7) and the proof of [4, Lemma 3.5] we have

$$\|f(x+iy)\|_{L^2} \leq \|h\|_{L^2} < \infty, \quad y \in C;$$

and $f(z) \in H^2(T^C)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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A COHOMOLOGY VANISHING THEOREM AND LAPLACE HYPERFUNCTIONS WITH HOLOMORPHIC PARAMETERS

Abstract. From 1987 onwards, the theory of Laplace hyperfunctions has been developed by H. Komatsu. Laplace hyperfunctions are represented as a class of holomorphic functions of exponential type. The aim of this paper is to give the vanishing theorem of cohomology groups on a pseudoconvex open subset for holomorphic functions with exponential growth at infinity. As an application of the theorem, we construct the sheaf of Laplace hyperfunctions and that with holomorphic parameters, and we also study several properties of these sheaves. This is a short summary of our paper [1].

1. Introduction

The theory of Laplace hyperfunctions has been developed by H. Komatsu (in [3]–[8]) to give a rigid framework of operational calculus for functions without growth conditions at infinity.

Let us briefly recall the definition of Laplace hyperfunctions with support in $[a, \infty]$ $(a \in \mathbb{R} \sqcup \{+\infty\})$ and that of their Laplace transforms (see [3]–[8]). Let $\mathbb{D}^2$ be the radial compactification $\mathbb{C} \sqcup S^1$ of the complex plane whose topology is defined in the usual way (see the next section). Let $O_{\mathbb{C}}^{\exp}$ be the sheaf of holomorphic functions of exponential type, that is, if V is an open set in $\mathbb{D}^2$, then $O_{\mathbb{C}}^{\exp}(V)$ denotes the space of all holomorphic functions $F(z)$ on $V \cap \mathbb{C}$ such that for any compact set K in V there are positive constants H and C for which we have

$$|F(z)| \leqq Ce^{H|z|}, \qquad z \in K \cap \mathbb{C}.$$

Then the space $\mathcal{B}_{[a,\infty]}^{\exp}$ of Laplace hyperfunctions with support in $[a, \infty]$ is defined as the quotient space

$$(1) \qquad \mathcal{B}_{[a,\infty]}^{\exp} := \frac{O_{\mathbb{C}}^{\exp}(\mathbb{D}^2 \setminus [a, \infty])}{O_{\mathbb{C}}^{\exp}(\mathbb{D}^2)}.$$

Let $f(x)$ be a hyperfunction with support in $[a, \infty]$ with its defining function $F(z) \in O_{\mathbb{C}}^{\exp}(\mathbb{D}^2 \setminus [a,\infty])$. Then the Laplace transform $\widehat{f}(\lambda)$ of $f(x)$ is defined by the integral

$$\widehat{f}(\lambda) := \int_C e^{-\lambda z} F(z) dz,$$

where the path C of integration is composed of a ray from $e^{i\alpha}\infty$ $(-\pi/2 < \alpha < 0)$ to a point $c < a$ and a ray from c to $e^{i\beta}\infty$ $(0 < \beta < \pi/2)$.

As we have seen, the Laplace hyperfunctions are defined by global sections of holomorphic functions of exponential type. Therefore it is an important problem to

localize the notion of Laplace hyperfunctions and to construct the sheaf of Laplace hyperfunctions whose global sections with support in $[a, \infty]$ give ones introduced by H. Komatsu. In this paper, we construct the sheaf of Laplace hyperfunctions and that with holomorphic parameters by establishing the vanishing theorem of cohomology groups on a pseudoconvex open subset for holomorphic functions with exponential growth at infinity. The vanishing theorem established here not only plays an important role in the construction of the sheaf of Laplace hyperfunctions but also has independent interest as Example 2 shows. For the details and the proof of the theorems in this paper, see N. Honda and K. Umeta [1]

Acknowledgement. To conclude the introduction, the authors would like to express their sincere gratitude to Professor Hikosaburo Komatsu for the valuable lectures and advice in Hokkaido University.

2. The vanishing theorem for holomorphic functions of exponential type

We need to introduce several notions before stating our vanishing theorem. Let $n \in \mathbb{N}$, let m be a non-negative integer and let $\mathbb{D}^{2n}$ be the *radial compactification* of $\mathbb{C}^n$, that is, the set $\mathbb{D}^{2n}$ is the disjoint union of $\mathbb{C}^n$ and the real $(2n-1)$-dimensional unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$.

Let $X := \mathbb{C}^{n+m}$ and $\hat{X}$ be the partial radial compactification $\mathbb{D}^{2n} \times \mathbb{C}^m$ of $\mathbb{C}^{n+m}$. We denote by X_∞ the closed subset $\hat{X} \setminus X$ in $\hat{X}$, and we denote by

$$p_1 : \hat{X} = \mathbb{D}^{2n} \times \mathbb{C}^m \to \mathbb{D}^{2n} \quad (\text{resp. } p_2 : \hat{X} = \mathbb{D}^{2n} \times \mathbb{C}^m \to \mathbb{C}^m)$$

the canonical projection to the first (resp. second) space. A family of fundamental neighborhoods of $(z_0, w_0) \in X \subset \hat{X}$ consists of

$$(2) \qquad\qquad B_\varepsilon(z_0, w_0) := \{(z, w) \in X; |z - z_0| < \varepsilon, |w - w_0| < \varepsilon\}$$

for $\varepsilon > 0$, and that of $(z_0, w_0) \in X_\infty$ consists of a product of an open cone and an open ball

$$(3) \quad G_r(\Gamma, w_0) := \left(\left\{ z \in \mathbb{C}^n; |z| > r, \frac{z}{|z|} \in \Gamma \right\} \cup \Gamma \right) \times \left\{ w \in \mathbb{C}^m; |w - w_0| < \frac{1}{r} \right\},$$

where $r > 0$ and Γ runs through open neighborhoods of z_0 in S^{2n-1}.

We denote by O_X the sheaf of holomorphic functions on X.

DEFINITION 1. *Let Ω be an open subset in $\hat{X}$. The set $O_X^{\exp}(\Omega)$ of holomorphic functions of exponential type on Ω consists of holomorphic functions $f(z, w)$ on $\Omega \cap X$ which satisfy, for any compact set K in Ω,*

$$(4) \qquad\qquad |f(z, w)| \leq C_K e^{H_K |z|}, \qquad ((z, w) \in K \cap X),$$

with some positive constants C_K and H_K. We denote by $O_X^{\exp}$ the associated sheaf on $\hat{X}$ of the presheaf $\{O_X^{\exp}(\Omega)\}_\Omega$.

Let A be a subset in $\hat{X}$. We define the set $\mathrm{clos}^1_\infty(A) \subset X_\infty$ as follows. A point $(z,w) \in X_\infty$ belongs to $\mathrm{clos}^1_\infty(A)$ if and only if there exist points $\{(z_k, w_k)\}_{k \in \mathbb{N}}$ in $A \cap X$ that satisfy

$$(z_k, w_k) \to (z, w) \text{ in } \hat{X} \text{ and } \frac{|z_{k+1}|}{|z_k|} \to 1 \quad (k \to \infty).$$

Set

$$N^1_\infty(A) := X_\infty \setminus \mathrm{clos}^1_\infty(X \setminus A).$$

DEFINITION 2. *Let U be an open subset in $\hat{X}$. We say that U is regular at ∞ if $N^1_\infty(U) = U \cap X_\infty$ is satisfied.*

EXAMPLE 1. We give some examples of open subsets which are regular at ∞.

- Let U be the open set $G_r(\Gamma, 0) \cup \tilde{U}$, where $\tilde{U}$ is a bounded open subset in X and the cone $G_r(\Gamma, 0)$ is defined by (3) with $r > 0$ and Γ an open subset in S^{2n-1}. Then U is regular at ∞. In particular, $\mathbb{D}^2$ and $\mathbb{D}^2 \setminus [a, +\infty]$ $(a \in [-\infty, \infty))$ are regular at ∞.

- For the set $U := \mathbb{D}^2 \setminus \{1, 2, 3, 4, \ldots, +\infty\}$ we have $N^1_\infty(U) = S^1 \setminus \{+\infty\}$, and hence U is regular at ∞. However, $U := \mathbb{D}^2 \setminus \{1, 2, 4, 8, 16, \ldots, +\infty\}$ is not regular because of $N^1_\infty(U) = S^1$.

For a subset A in X, we denote by $\mathrm{dist}(p, A)$ the distance between a point p and A, i.e.,

$$\mathrm{dist}(p, A) := \inf_{q \in A} |p - q|.$$

For convenience, set $\mathrm{dist}(p, A) = +\infty$ if A is empty. We also define, for $q = (z, w) \in X$,

$$\mathrm{dist}_{\mathbb{D}^{2n}}(q, A) := \mathrm{dist}\left(q, A \cap p_2^{-1}(p_2(q))\right) = \inf_{(\zeta, w) \in A} |z - \zeta|.$$

Let Ω be an open subset in $\hat{X}$. We set

(5)
$$\psi(p) := \min\left\{\frac{1}{2}, \frac{\mathrm{dist}_{\mathbb{D}^{2n}}(p, X \setminus \Omega)}{1 + |z|}\right\}, \qquad (p = (z, w) \in X),$$

$$\Omega_\varepsilon := \left\{p = (z, w) \in \Omega \cap X; \mathrm{dist}(p, X \setminus \Omega) > \varepsilon, |w| < \frac{1}{\varepsilon}\right\}, \quad (\varepsilon > 0).$$

Now we give the main theorem.

THEOREM 1. *Assume the following two conditions:*

1. *$\Omega \cap X$ is pseudoconvex in X and Ω is regular at ∞.*

2. *At a point in $\Omega \cap X$ sufficiently close to $z = \infty$ the function $\psi(z, w)$ is continuous and uniformly continuous with respect to the variables w, that is, for any $\varepsilon > 0$,*

there exist $\delta_\varepsilon > 0$ and $R_\varepsilon > 0$ for which $\psi(z,w)$ is continuous on the open set $\Omega_{\varepsilon,R_\varepsilon} := \Omega_\varepsilon \cap \{|z| > R_\varepsilon\}$ and satisfies

$$\left|\psi(z,w) - \psi(z,w')\right| < \varepsilon, \qquad \left((z,w),(z,w') \in \Omega_{\varepsilon,R_\varepsilon},\ |w-w'| < \delta_\varepsilon\right).$$

Then we have

(6) $$H^k(\Omega, O_X^{\exp}) = 0, \qquad (k \neq 0).$$

As condition 2. in the theorem is automatically satisfied for a product of open sets, we have the following corollary.

COROLLARY 1. *Let U (resp. W) be an open subset in $\mathbb{D}^{2n}$ (resp. $\mathbb{C}^n$). If $U \cap \mathbb{C}^n$ and W are pseudoconvex in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively and if U is regular at ∞ in $\mathbb{D}^{2n}$, then* (6) *holds for $\Omega := U \times W$.*

Note that, in the later section, we will see that, if $n = 1$, the vanishing theorem still holds for an open subset of product type without the regularity condition at ∞. However, if n is greater than one, one cannot expect the vanishing theorem anymore without the regularity condition at ∞ as the following example shows.

EXAMPLE 2. Assume $n = 2$ and $m = 0$, i.e., $X = \mathbb{C}^2_{(z_1,z_2)}$ and $\hat{X} = \mathbb{D}^4$. Set

$$U := \left\{(z_1, z_2) \in X; |\arg(z_1)| < \frac{\pi}{4}, |z_2| < |z_1|\right\},$$
$$\Omega := \left(\overline{U}\right)^{\circ} \setminus \{p_\infty\} \subset \hat{X},$$

where p_∞ denotes the point $(1,0,0,0)$ in $S^3 \subset \mathbb{D}^4$. The open subset $\Omega \cap X = U$ is pseudoconvex in X, while Ω is not regular at ∞. Then we have $H^1(\Omega, O_X^{\exp}) \neq 0$.

3. Laplace hyperfunctions with holomorphic parameters

By Theorem 1, we can construct cohomologically the sheaf $\mathcal{B}_\mathbb{R}^{\exp}$ of Laplace hyperfunctions and the sheaf $\mathcal{B}O_N^{\exp}$ of Laplace hyperfunctions with holomorphic parameters.

Let $N = \mathbb{R} \times \mathbb{C}^m (m \geq 0)$, and let $\overline{N} = \overline{\mathbb{R}} \times \mathbb{C}^m$ be the closure of N inside $\hat{X} = \mathbb{D}^2 \times \mathbb{C}^m$. Then we have the following theorem.

THEOREM 2. *The closed set $\overline{N}$ is purely 1-codimensional with respect to the sheaf $O_X^{\exp}$, i.e.,*

$$\mathcal{H}^k_{\overline{N}}(O_X^{\exp}) = 0, \quad (k \neq 1).$$

Here $\mathcal{H}^k_{\overline{N}}(O_X^{\exp})$ is the k-th derived sheaf of $O_X^{\exp}$ with support in $\overline{N}$.

As a particular case, we have the following corollary.

COROLLARY 2. $\overline{\mathbb{R}}$ *is purely 1-codimensional with respect to the sheaf* $O_{\mathbb{C}}^{\exp}$, *that is,*

$$\mathscr{H}_{\overline{\mathbb{R}}}^{k}(O_{\mathbb{C}}^{\exp}) = 0 \quad (k \neq 1).$$

DEFINITION 3. *The sheaf* $\mathscr{B}O_{N}^{\exp}$ *of Laplace hyperfunctions of one variable with holomorphic parameters is defined by*

$$\mathscr{B}O_{N}^{\exp} := \mathscr{H}_{\overline{N}}^{1}(O_{X}^{\exp}) \underset{\mathbb{Z}_{\overline{N}}}{\otimes} \omega_{\overline{N}},$$

where $\mathbb{Z}_{\overline{N}}$ *denotes the constant sheaf on* $\overline{N}$ *having stalk* $\mathbb{Z}$ *and* $\omega_{\overline{N}}$ *denotes the orientation sheaf* $\mathscr{H}_{\overline{N}}^{1}(\mathbb{Z}_{\hat{X}})$ *on* $\overline{N}$.

The global sections of the sheaf $\mathscr{B}O_{N}^{\exp}$ can be written in terms of cohomology groups by Theorem 2. For an open set $\Omega \subset \overline{\mathbb{R}}$ and a pseudoconvex open subset $T \subset \mathbb{C}^{m}$, by taking a complex neighborhood V of Ω in $\mathbb{D}^{2}$, we have

$$\mathscr{B}O_{N}^{\exp}(\Omega \times T) = H_{\Omega \times T}^{1}(V \times T, O_{X}^{\exp}) = \frac{O_{X}^{\exp}((V \setminus \Omega) \times T)}{O_{X}^{\exp}(V \times T)}.$$

Note that the above representation does not depend on a choice of the complex neighborhood V.

DEFINITION 4. *We define the sheaf* $\mathscr{B}_{\mathbb{R}}^{\exp}$ *of Laplace hyperfunctions of one variable on* $\overline{\mathbb{R}}$ *by*

$$\mathscr{B}_{\mathbb{R}}^{\exp} := \mathscr{H}_{\overline{\mathbb{R}}}^{1}(O_{\mathbb{C}}^{\exp}) \underset{\mathbb{Z}_{\overline{\mathbb{R}}}}{\otimes} \omega_{\overline{\mathbb{R}}},$$

where $\mathbb{Z}_{\overline{\mathbb{R}}}$ *denotes the constant sheaf on* $\overline{\mathbb{R}}$ *having stalk* $\mathbb{Z}$ *and* $\omega_{\overline{\mathbb{R}}}$ *denotes the orientation sheaf* $\mathscr{H}_{\overline{\mathbb{R}}}^{1}(\mathbb{Z}_{\hat{X}})$ *on* $\overline{\mathbb{R}}$.

The restriction of $\mathscr{B}_{\mathbb{R}}^{\exp}$ to $\mathbb{R}$ is isomorphic to the sheaf $\mathscr{B}_{\mathbb{R}}$ of ordinary hyperfunctions because of $O_{\mathbb{C}}^{\exp}|_{\mathbb{C}} = O_{\mathbb{C}}$. By Corollary 2 we have

$$\Gamma_{[a,\infty]}(\overline{\mathbb{R}}, \mathscr{B}_{\mathbb{R}}^{\exp}) = \frac{O_{\mathbb{C}}^{\exp}(\mathbb{D}^{2} \setminus [a,\infty])}{O_{\mathbb{C}}^{\exp}(\mathbb{D}^{2})}.$$

Hence the set $\mathscr{B}_{[a,\infty]}^{\exp}$ defined by H. Komatsu coincides with $\Gamma_{[a,\infty]}(\overline{\mathbb{R}}, \mathscr{B}_{\mathbb{R}}^{\exp})$ in our framework.

4. Several properties of $\mathscr{B}O_{N}^{\exp}$

We can also show the vanishing theorem on an open subset which is not necessarily regular at ∞ if $n = 1$. This fact is deeply related to the flabbiness of $\mathscr{B}O_{N}^{\exp}$.

THEOREM 3. *Let U be an open subset in $\mathbb{D}^2$, and W a pseudoconvex open subset in $\mathbb{C}^m$. Then we have*

$$H^k(U \times W, O_X^{\mathrm{exp}}) = 0, \qquad (k \neq 0).$$

The sets $N, \overline{N}$ and $\hat{X}$ are the same as those in the previous section. Now we state the theorems for the flabbiness and the unique continuation property of $\mathcal{B}O_N^{\mathrm{exp}}$.

THEOREM 4. *Let Ω_1 and Ω_2 be open subsets in $\overline{\mathbb{R}}$ with $\Omega_1 \subset \Omega_2$, and let W be a pseudoconvex open subset in $\mathbb{C}^m$. Then the restriction*

$$\mathcal{B}O_N^{\mathrm{exp}}(\Omega_2 \times W) \to \mathcal{B}O_N^{\mathrm{exp}}(\Omega_1 \times W)$$

is surjective.

COROLLARY 3 ([3]). *The sheaf $\mathcal{B}_{\mathbb{R}}^{\mathrm{exp}}$ of Laplace hyperfunctions is flabby.*

The following theorem shows that the sheaf $\mathcal{B}O_N^{\mathrm{exp}}$ has a unique continuation property with respect to holomorphic parameters.

THEOREM 5. *Let W_1 and W_2 be non-empty connected open subsets in $\mathbb{C}^m$ with $W_1 \subset W_2$ and Ω an open subset in $\overline{\mathbb{R}}$. Then the restriction*

$$\mathcal{B}O_N^{\mathrm{exp}}(\Omega \times W_2) \longrightarrow \mathcal{B}O_N^{\mathrm{exp}}(\Omega \times W_1)$$

is injective.

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ABSTRACT STOCHASTIC PROBLEMS
IN SPACES OF DISTRIBUTIONS

Abstract. The Cauchy problem for the equation $u'(t) = Au(t) + B\mathbb{W}(t)$, $t \geq 0$, with white noise $\mathbb{W}$ and A being the generator of regularized semigroups is studied in different spaces of distributions. Solutions of the problem in spaces of distributions with respect to time variable, random variable and both time and random variables are studied.

1. Introduction

The Cauchy problem for operator-differential equations with white noise as an inhomogeneity often arises as a model of different evolution processes subject to random perturbations. The basic one among them is the Cauchy problem

$$(1) \qquad X'(t) = AX(t) + B\mathbb{W}(t), \quad t \in [0, \tau),\ \tau \leq \infty, \quad X(0) = \zeta,$$

where A is the generator of a C_0-semigroup. Because of irregularity of the white noise $\mathbb{W}$ it is usually reduced to an integral equation with the "primitive" of $\mathbb{W}$, i.e. with some Wiener process (e.g. [14, 12]).

Our work is devoted to generalized solutions of the stochastic Cauchy problem (1) with A not necessarily being the generator of a C_0 semigroup, but being the generator of a regularized, namely integrated semigroup $V = \{V(t),\ t \in [0, \tau)\}$ in a Hilbert space H. We suppose $\{\mathbb{W}(t), t \geq 0\}$ to be an $\mathbb{H}$-valued white noise which we define in our work rigorously in different spaces of distributions, $B \in \mathcal{L}(\mathbb{H}, H)$.

The fact that the operator A is generating only an integrated semigroup means that the solution operators $U(t)$, $t \in [0, \tau)$, of the corresponding homogeneous Cauchy problem are not bounded. Therefore one has to introduce some regularized family V instead of $\{U(t)\}$ or consider the solution operators of problem (1) in certain spaces of distributions. At the same time due to irregularity, particularly to discontinuity of the white noise (it is informally defined as a process with independent identically distributed random values with infinite variation) one has to reduce the Cauchy problem to the above-mentioned integral equation with a Wiener process which is defined axiomatically as the infinite dimensional generalization of Brownian motion, or to consider the problem (1) in certain spaces of distributions. The choice of a proper space of distributions depends on the conditions imposed on A and initial value ζ on one hand and on the properties of the noise $\mathbb{W}$ on the other hand.

In the next section (section 2) we give necessary definitions from the theory of regularized semigroups, Hilbert space-valued (abstract) Wiener processes, abstract Schwartz distributions and stochastic distributions.

In section 3 we consider the problem (1) in spaces of Hilbert space valued distributions with respect to one variable. If it is the time variable t (subsection 3.1) we obtain existence of a unique solution for A generating an n-times integrated semigroup, but $\mathbb{W}$ must be a Q-white noise $\mathbb{W}_Q$, where Q is a nuclear operator in $\mathbb{H}$. If we consider the problem in the space of distributions with respect to the random variable ω (subsection 3.2), we obtain the result for the equation with the singular white noise ($Q = I$), but A must be the generator of a C_0-semigroup. In section 4 we introduce the space of distributions with respect to both t and ω and obtain the result for A generating an n-times integrated semigroup and $\mathbb{W}$ being the singular white noise.

The beginning of research in this direction was made in [14, 11, 10]. In [13, 15] different approaches to defining of distributions in t and ω were studied.

2. Definitions: regularized semigroups, abstract Wiener processes and abstract distributions

2.1. Regularized semigroups

Let A be a closed linear operator and $R(t)$, $t \geq 0$, be bounded linear operators in a Banach space H.

DEFINITION 1. *A strongly continuous family $V = \big\{V(t),\ t \in [0, \tau)\big\}$, $\tau \leq \infty$, of bounded operators in H is called an R-regularized semigroup with the generator A if*

$$V(t)A\zeta = AV(t)\zeta, \quad \zeta \in \mathrm{dom}\, A, \qquad V(t)\zeta = A\int_0^t V(s)\zeta ds + R(t)\zeta, \quad \zeta \in H.$$

The semigroup V is called exponentially bounded if $\|V(t)\| \leq Me^{\varpi t}$, $t \geq 0$, for some $M > 0$, $\varpi \in \mathbb{R}$, and local if $\tau < \infty$.

If $R(t) = \big(t^n/n!\big)I$, then V is also an *n-times integrated semigroup*. If $\overline{\mathrm{dom}\, A} = H$ and $R(t) \equiv R$ is invertible, bounded and densely defined, then V is an *R-semigroup*. If $R = I$, then an R-semigroup is a *C_0-semigroup*.

Note that an R-semigroup in [3] is defined as a strongly continuous family of bounded operators satisfying the R-semigroup property

$$V(t+s)R = V(t)V(s), \quad s, t, s+t \in [0, \tau), \ V(0) = R$$

with the infinitesimal generator

$$\mathcal{G}f := (\lambda - L_\lambda^{-1})f, \lambda > \varpi, \ \mathrm{dom}\, \mathcal{G} = \big\{f \in H :\ Rf \in \mathrm{ran}\, L_\lambda\big\}, \ L_\lambda f := \int_0^\infty e^{\lambda t}V(t)f dt.$$

It is called there a *C*-semigroup. We prefer the term "*R*-semigroup" that reflects its regularizing property and makes it differ from C_0-semigroups, where *C* comes from "continuity".

As to integrated semigroups, they are also defined via corresponding "semigroup property" in [2] with the infinitesimal generator, but we will use the equivalent general Definition 1. We refer to [9, 8] for examples of integrated, convolution, *R*-semigroups and their generators, including important differential operators.

2.2. Wiener processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathbb{H}$ be a Hilbert space and Q be a linear symmetric positive trace class operator with a system of eigenvectors $\{e_i\}$, forming a basis of $\mathbb{H}$, such that $Qe_i = \sigma_i^2 e_i$, $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$.

DEFINITION 2. *A stochastic process* $W_Q = \{W_Q(t), t \geq 0\}$ *with values in* $\mathbb{H}$ *is called a Q-Wiener process, if*

$(W1)$ $W_Q(0) = 0$ *a.s.;*

$(W2)$ W_Q *has independent increments;*

$(W3)$ *the increments* $W_Q(t) - W_Q(s)$ *are normally distributed with mean zero and covariance operator equal to* $(t-s)Q$;

$(W4)$ *the trajectories of* W_Q *are continuous a.s.*

Thus defined *Q*-Wiener process is a generalization of Brownian motion. It is well known that Brownian motion $\{\beta(t), t \geq 0\}$, where $\beta(t) = \beta(\omega, t)$, $\omega \in \Omega$, is defined via conditions (W1)–(W4) in the case $\mathbb{H} = \mathbb{R}$ and $Q = I$. A finite-dimensional Brownian motion has form $\sum_{i=1}^{n} \beta_i(t)e_i$, where $\{e_i\}$ is an orthonormal basis in $\mathbb{R}^n$ and β_i are independent Brownian motions. When passing to infinite dimensions, to avoid divergency in $\mathbb{H}$, one has to consider a regularized sum

$$W_Q(t) := \sum_{i=1}^{\infty} \sigma_i \beta_i(t)e_i, \, t \geq 0, \qquad W_Q(t) \in L_2(\Omega; \mathbb{H}),$$

which happens to be an $\mathbb{H}$-valued *Q*-Wiener process.

The formal series $\sum_{i=1}^{\infty} \beta_i(t)e_i =: W(t)$ is called a *cylindrical Wiener process*.

2.3. Spaces of abstract distributions. White noise in spaces of abstract distributions

For any Banach space X by $\mathcal{D}'(X)$ we denote the space of all X-valued distributions over the space of test function $\mathcal{D}$. In contrary to the $\mathbb{R}$-valued Schwartz distributions they are called *abstract distributions*. By $\mathcal{D}'_0(X)$ we denote the subspace of distributions having supports in $[0, \infty)$.

Let $\mathbb{H}$ now be a Hilbert space and W_Q be an $\mathbb{H}$-valued Q-Wiener process. Since W_Q has continuous in $t \geq 0$ trajectories for almost all $\omega \in \Omega$, define *Q-white noise* $\mathbb{W}_Q$ (with trajectories) in $\mathcal{D}'_0(\mathbb{H})$ as generalized derivative of W_Q set to be zero at $t < 0$, i.e. by the following equality:

(2) $$\langle \mathbb{W}_Q, \theta \rangle := -\int_0^\infty W_Q(t)\theta'(t)\,dt = \int_0^\infty \theta(t)\,dW_Q(t), \quad \theta \in \mathcal{D}.$$

The first integral in (2) is understood as Bochner integral of an $L_2(\Omega; \mathbb{H})$-valued function, the second one — as an abstract Ito integral with respect to the Wiener process. The equality of the integrals follows from the Ito formula.

We will further use convolution of distributions defined as follows (see. [4]).

DEFINITION 3. *Let* X, $\mathcal{Y}$ *and* $\mathcal{Z}$ *be Banach spaces, such that there exists a continuous bilinear operation* $(u, v) \mapsto uv \in \mathcal{Z}$ *defined on* $X \times \mathcal{Y}$. *For any* $G \in \mathcal{D}'_0(X)$ *and* $F \in \mathcal{D}'_0(\mathcal{Y})$ *the convolution* $G * F \in \mathcal{D}'_0(\mathcal{Z})$ *is defined by the equality*

$$\langle G * F, \theta \rangle := \langle (g * f)^{(n+m)}, \theta \rangle = (-1)^{n+m} \int_0^\infty (g * f)(t)\theta^{n+m}(t)\,dt, \quad \theta \in \mathcal{D},$$

where $g : \mathbb{R} \to X$, $f : \mathbb{R} \to \mathcal{Y}$ *are continuous functions such that*

$$\langle G, \theta \rangle = (-1)^n \int_0^\infty g(t)\theta^{(n)}(t)\,dt, \quad \langle F, \theta \rangle = (-1)^m \int_0^\infty f(t)\theta^{(m)}(t)\,dt,$$

$(g * f)(t) := \int_0^t g(t - s)f(s)\,ds$.

Note that in the particular case when G is a regular distribution, i.e. $\langle G, \theta \rangle = \int_0^\infty G(t)\theta(t)\,dt$, the equality $\langle G * F, \theta \rangle = \int_0^\infty G(t)\langle F(\cdot), \theta(t + \cdot)\rangle\,dt$ holds.

2.4. Spaces of abstract stochastic distributions. Singular white noise

The theory of stochastic distributions uses the white noise probability space. It is the triple $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$, where $\mathcal{B}(\mathcal{S}')$ is the Borel σ-field of $\mathcal{S}'$ (the Schwartz space of tempered distributions), μ is the centered Gaussian or white noise measure on $\mathcal{B}(\mathcal{S}')$ satisfying the equality

$$\int_{\mathcal{S}'} e^{i\langle \omega, \theta \rangle}\,d\mu(\omega) = e^{-\frac{1}{2}|\theta|_0^2}, \quad \theta \in \mathcal{S},$$

where $|\cdot|_0$ is the norm of $L_2(\mathbb{R})$. Existence of such measure is stated by the Bochner–Minlos theorem (see, e.g. [6]).

The construction of spaces of *abstract stochastic distributions* [6] is analogous to the construction of the Gelfand triple $\mathcal{S} \subset L_2(\mathbb{R}) \subset \mathcal{S}'$. Its central element is the space (L^2) of all functions of $\omega \in \mathcal{S}'$ which are square integrable with respect to the measure μ. Hermite functions $\xi_k(x) = \pi^{-\frac{1}{4}}\left((k-1)!\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{k-1}(x)$ (where

$h_k(x) = (-1)^k e^{\frac{x^2}{2}} (d/dx)^k e^{-\frac{x^2}{2}}$, are Hermite polynomials) are the eigenfunctions of the differential operator $\hat{D} = -\frac{d^2}{dt^2} + t^2 + 1$ with $\hat{D}\xi_k = (2k)\xi_k$, $k \in \mathbb{N}$, and form an orthonormal basis of $L_2(\mathbb{R})$. Stochastic Hermite polynomials $\mathbf{h}_\alpha(\omega) := \prod_k h_{\alpha_k}(\langle \omega, \xi_k \rangle)$, $\omega \in \mathcal{S}'$, where $\alpha \in \mathcal{T}$ (the set of all finite multi-indices) form an orthogonal basis of (L^2) with

$$(\mathbf{h}_\alpha, \mathbf{h}_\beta)_{(L^2)} = \alpha! \delta_{\alpha,\beta}, \qquad \alpha! := \prod_k \alpha_k!.$$

They are the eigenfunctions of the second quantization operator $\Gamma(\hat{D})$. We have

$$\Gamma(\hat{D})\mathbf{h}_\alpha = \prod_k (2k)^{\alpha_k} \mathbf{h}_\alpha =: (2\mathbb{N})^\alpha \mathbf{h}_\alpha.$$

The space of test functions $(\mathcal{S})$ is a countably-Hilbert space $(\mathcal{S}) = \bigcap_{p \in \mathbb{N}} (\mathcal{S}_p)$ with the projective limit topology, where

$$(\mathcal{S}_p) = \left\{ \varphi = \sum_{\alpha \in \mathcal{T}} \varphi_\alpha \mathbf{h}_\alpha \in (L^2) : \sum_{\alpha \in \mathcal{T}} \alpha! |\varphi_\alpha|^2 (2\mathbb{N})^{2p\alpha} < \infty \right\}$$

with the norm $|\cdot|_p$, generated by the scalar product

$$(\varphi, \psi)_p = (\Gamma(\hat{D})^p \varphi, \Gamma(\hat{D})^p \psi)_{(L^2)} = \sum_{\alpha \in \mathcal{T}} \alpha! \varphi_\alpha \overline{\psi}_\alpha (2\mathbb{N})^{2p\alpha}.$$

Its adjoint space $(\mathcal{S})'$ is called the space of stochastic (Hida) distributions (random variables). We have $(\mathcal{S})' = \cup_{p \in \mathbb{N}} (\mathcal{S}_{-p})$ with the inductive limit topology, where $(\mathcal{S}_{-p})$ is the adjoint of $(\mathcal{S}_p)$. The space $(\mathcal{S}_{-p})$ can be identified with the space of all formal expansions $\Phi = \sum_{\alpha \in \mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha$, satisfying $\sum_{\alpha \in \mathcal{T}} \alpha! |\Phi_\alpha|^2 (2\mathbb{N})^{-2p\alpha} < \infty$, with scalar product

$$(\Phi, \Psi)_{-p} = (\Gamma(\hat{D})^{-p} \varphi, \Gamma(\hat{D})^{-p} \psi)_{(L^2)} = \sum_{\alpha \in \mathcal{T}} \alpha! \Phi_\alpha \overline{\Psi}_\alpha (2\mathbb{N})^{-2p\alpha}.$$

Denote the corresponding norm by $|\cdot|_{-p}$. We have:

$$\langle \Phi, \varphi \rangle = \sum_{\alpha \in \mathcal{T}} \alpha! \Phi_\alpha \varphi_\alpha \text{ for } \Phi = \sum_{\alpha \in \mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})', \quad \varphi = \sum_{\alpha \in \mathcal{T}} \varphi_\alpha \mathbf{h}_\alpha \in (\mathcal{S}).$$

Thus we have the following Gelfand triple: $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})'$.

Define $(\mathcal{S})'(\mathbb{H})$, the space of $\mathbb{H}$-valued generalized random variables over $(\mathcal{S})$ as the space of linear continuous operators $\Phi : (\mathcal{S}) \to \mathbb{H}$ with the topology of uniform convergence on bounded subsets of $(\mathcal{S})$. Denote the action of $\Phi \in (\mathcal{S})'(\mathbb{H})$ on $\varphi \in (\mathcal{S})$ by $\Phi[\varphi]$. The structure of $(\mathcal{S})'(\mathbb{H})$ is due to the next proposition (see the proof in [7]).

PROPOSITION 1. *Any $\Phi \in (\mathcal{S})'(\mathbb{H})$ can be extended to a bounded operator from* $(\mathcal{S}_p)$ *to $\mathbb{H}$ for some $p \in \mathbb{N}$.*

The space $(\mathcal{S})$ is a nuclear countably Hilbert space since for any $p \in \mathbb{N}$ the embedding $I_{p,p+1} : (\mathcal{S}_{p+1}) \hookrightarrow (\mathcal{S}_p)$ is a Hilbert–Schmidt operator. From this fact and proposition 1 one deduces

COROLLARY 1. *Any* $\Phi \in (\mathcal{S})'(\mathbb{H})$ *is a Hilbert–Schmidt operator from* $(\mathcal{S}_p)$ *to* $\mathbb{H}$ *for some* $p \in \mathbb{N}$.

For any $\Phi \in (\mathcal{S})'(\mathbb{H})$ denote by Φ_j the linear functional defined on $\varphi \in (\mathcal{S})$ by $\langle \Phi_j, \varphi \rangle := (\Phi[\varphi], e_j)$. Let p be such that Φ is Hilbert–Schmidt from $(\mathcal{S}_p)$ to $\mathbb{H}$. Then all Φ_j, $j \in \mathbb{N}$, belong to the corresponding space $(\mathcal{S}_{-p})$, thus we have

$$\Phi_j = \sum_{\alpha \in \mathcal{T}} \Phi_{\alpha,j} \mathbf{h}_\alpha, \quad \sum_{\alpha \in \mathcal{T}} \alpha! |\Phi_{\alpha,j}|^2 (2\mathbb{N})^{-2p\alpha} < \infty.$$

For the Hilbert–Schmidt norm of $\Phi : (\mathcal{S}_p) \to \mathbb{H}$ we obtain:

$$\|\Phi\|^2_{\mathrm{HS},p} = \sum_{\alpha \in \mathcal{T}} \left\| \Phi \left[\frac{\mathbf{h}_\alpha}{(\alpha!)^{\frac{1}{2}}(2\mathbb{N})^{p\alpha}} \right] \right\|^2 = \sum_{\alpha \in \mathcal{T}} \sum_{j=1}^{\infty} \left| \left\langle \Phi_j, \frac{\mathbf{h}_\alpha}{(\alpha!)^{\frac{1}{2}}(2\mathbb{N})^{p\alpha}} \right\rangle \right|^2$$

$$= \sum_{\alpha \in \mathcal{T}, j \in \mathbb{N}} \alpha! |\Phi_{\alpha,j}|^2 (2\mathbb{N})^{-2p\alpha}.$$

Denote by $(\mathcal{S}_{-p})(\mathbb{H})$ the space of all Hilbert–Schmidt operators acting form $(\mathcal{S}_p)$ to $\mathbb{H}$. It is a separable Hilbert space with an orthogonal basis consisting of operators $\mathbf{h}_\alpha \otimes e_j$, $\alpha \in \mathcal{T}$, $j \in \mathbb{N}$, defined by

$$(\mathbf{h}_\alpha \otimes e_j)\varphi := (\mathbf{h}_\alpha, \varphi)_{(L^2)} e_j, \quad \varphi \in (\mathcal{S}_p).$$

It follows from corollary 1 that $(\mathcal{S})'(\mathbb{H}) = \bigcup_{p \in \mathbb{N}} (\mathcal{S}_{-p})(\mathbb{H})$ and any $\Phi \in (\mathcal{S})'(\mathbb{H})$ has the decomposition

$$\Phi = \sum_{j \in \mathbb{N}} \Phi_j e_j = \sum_{\alpha \in \mathcal{T}, j \in \mathbb{N}} \Phi_{\alpha,j} (\mathbf{h}_\alpha \otimes e_j) = \sum_{\alpha \in \mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha,$$

where $\Phi_j = (\Phi[\cdot], e_j) \in (\mathcal{S}_{-p})$ for some $p \in \mathbb{N}$, $\Phi_\alpha = \sum_{j \in \mathbb{N}} \Phi_{\alpha,j} e_j \in \mathbb{H}$. For the norm $\| \cdot \|^2_{-p} := \| \cdot \|^2_{\mathrm{HS},p}$ we have

$$\|\Phi\|^2_{-p} = \sum_{j \in \mathbb{N}} |\Phi_j|^2_{-p} = \sum_{\alpha \in \mathcal{T}, j \in \mathbb{N}} \alpha! |\Phi_{\alpha,j}|^2 (2\mathbb{N})^{-2p\alpha} = \sum_{\alpha \in \mathcal{T}} \alpha! \|\Phi_\alpha\|^2 (2\mathbb{N})^{-2p\alpha} < \infty.$$

We evidently have

$$(\mathcal{S}_{-p_1})(\mathbb{H}) \subseteq (\mathcal{S}_{-p_2})(\mathbb{H}) \text{ for } p_1 < p_2,$$

and

$$\|\Phi\|_{-p_1} \geq \|\Phi\|_{-p_2} \quad \text{for all } \Phi \in (\mathcal{S}_{-p_1})(\mathbb{H}).$$

To define singular white noise in these spaces first define a sequence of independent Brownian motions $\{\beta_j(t)\}$. Let $n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection with the property $n = n(i, j) \geq ij$. As it was done in [11, 5], we use the Fourier coefficients of the decomposition of Brownian motion $\beta(t)$ in $(L_2)(\mathbb{R})$:

$$\beta(t, \omega) = \langle \omega, \mathbf{1}_{[0,t]} \rangle = \Big\langle \omega, \sum_{i=1}^{\infty} \int_0^t \xi_i(s) ds \xi_i \Big\rangle = \sum_{i=1}^{\infty} \int_0^t \xi_i(s) \, ds \, \mathbf{h}_{\varepsilon_i},$$

where $\varepsilon_i := (0, 0, ..., \underset{i}{1}, 0, ...)$. Defining $\beta_j(t) = \sum_{i=1}^{\infty} \int_0^t \xi_i(s) ds \mathbf{h}_{\varepsilon_{n(i,j)}}$, we obtain the next decomposition for the Wiener process $W(t), t \geq 0$:

$$W(t) = \sum_{j=1}^{\infty} \beta_j(t) e_j = \sum_{i,j \in \mathbb{N}} \int_0^t \xi_i(s) ds (\mathbf{h}_{\varepsilon_{n(i,j)}} \otimes e_j) = \sum_{n=1}^{\infty} \int_0^t \xi_{i(n)}(s) ds (\mathbf{h}_{\varepsilon_n} \otimes e_{j(n)}).$$

Its derivative with respect to t is called singular $\mathbb{H}$-valued white noise. It has the following decomposition:

$$\mathbb{W}(t) = \sum_{i,j \in \mathbb{N}} \xi_i(t)(\mathbf{h}_{\varepsilon_{n(i,j)}} \otimes e_j) = \sum_{i,j \in \mathbb{N}} \mathbb{W}_{\varepsilon_{n(i,j)}}(t) \mathbf{h}_{\varepsilon_{n(i,j)}}, \quad \mathbb{W}_{\varepsilon_{n(i,j)}}(t) = \xi_i(t) e_j.$$

By the well known estimates $\left| \int_0^t \xi_i(s) \, ds \right|^2 = O(i^{-\frac{3}{2}})$ and $|\xi_i(t)| = O(i^{-1/4})$ of the Hermite functions, we obtain

$$\|W(t)\|_{-1}^2 = \sum_{i,j \in \mathbb{N}} \left| \int_0^t \xi_i(s) \, ds \right|^2 \left(2n(i,j)\right)^{-2} \leq C \sum_{i,j \in \mathbb{N}} i^{-7/2} j^{-2} < \infty,$$

$$\|\mathbb{W}(t)\|_{-1}^2 = \sum_{i,j \in \mathbb{N}} |\xi_i(t)|^2 \left(2n(i,j)\right)^{-2} \leq C \sum_{i,j \in \mathbb{N}} i^{-5/2} j^{-2} < \infty.$$

Thus $W(t) \in (\mathcal{S}_{-1})(\mathbb{H}) \subset (\mathcal{S})'(\mathbb{H})$ and $\mathbb{W}(t) \in (\mathcal{S}_{-1})(\mathbb{H}) \subset (\mathcal{S})'(\mathbb{H})$ for all $t \geq 0$.

Convergence in the space $(\mathcal{S})'(\mathbb{H})$ is characterized by the next proposition [7].

PROPOSITION 2. *Let* $\Phi_n = \sum_\alpha \Phi_\alpha^{(n)} \mathbf{h}_\alpha, \Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})'(\mathbb{H})$. *The following assertions are equivalent:*

  (i) $\Phi_n \to \Phi$ *in* $(\mathcal{S})'(\mathbb{H})$;

  (ii) *all elements of the sequence* $\{\Phi_n\}$ *and* Φ *belong to* $(\mathcal{S}_{-p})(\mathbb{H})$ *for some* $p \in \mathbb{N}$ *and* $\lim_{n \to \infty} \|\Phi_n - \Phi\|_{-p} = 0$,

It follows from this propositions that differentiation with respect to t of an $(\mathcal{S})'(\mathbb{H})$-valued function $\Phi(t)$ is equivalent to its differentiation as of a function with values in $(\mathcal{S}_{-p})(\mathbb{H})$ for some $p \in \mathbb{N}_0$. It is easy to see that $\dfrac{d}{dt} W(t) = \mathbb{W}(t)$ for all $t \in \mathbb{R}$. We will call an $(\mathcal{S})'(\mathbb{H})$-valued function $\Phi(t)$ integrable on $[a; b] \subset \mathbb{R}$ if it is Bochner integrable as an $(\mathcal{S}_{-p})(\mathbb{H})$-valued function for some p.

3. Solutions of stochastic Cauchy problem generalized with respect to one of the variables

3.1. Generalized solutions with respect to *t*

Let A be a closed linear operator acting from H to $[\mathrm{dom}A]$ (the domain of A endowed with the graph-norm), $B \in \mathcal{L}(\mathbb{H};H)$, $\zeta \in H$ and let $\mathbb{W}_Q$ be an $\mathbb{H}$-valued Q-white noise, defined by (2).

We define the *generalized solution* of the Cauchy problem (1) with $\mathbb{W} = \mathbb{W}_Q$ to be a distribution $X \in \mathcal{D}'_0(L_2(\Omega;[\mathrm{dom}A]))$ satisfying the equation

$$(3) \qquad\qquad P * X = \delta \otimes \zeta + B \mathbb{W}_Q, \,{}^{1}$$

where $P := \delta' \otimes I - \delta \otimes A \in \mathcal{D}'_0\big(\mathcal{L}([\mathrm{dom}A],H)\big)$

A distribution $G \in \mathcal{D}'_0\big(\mathcal{L}(H,[\mathrm{dom}A])\big)$ is called the *convolution inverse* for $P \in \mathcal{D}'_0\big(\mathcal{L}([\mathrm{dom}A],H)\big)$ if $G * P = \delta \otimes I_{[\mathrm{dom}A]}$, $P * G = \delta \otimes I_H$.

By the properties of the convolution inverse it is proved in [1] that the generalized random process X defined by

$$(4) \qquad\qquad \langle X, \theta \rangle := \langle G\zeta, \theta \rangle + \langle G * B\mathbb{W}_Q, \theta \rangle, \quad \theta \in \mathcal{D},$$

is the unique solution of (3) in the space $\mathcal{D}'_0(L_2(\Omega,[\mathrm{dom}A]))$. As a consequence we obtain the next result.

THEOREM 1. *Suppose that A is the generator of an n-times integrated semigroup $V = \{V(t),\, t \ge 0\}$. The Cauchy problem (1) with $\mathbb{W} = \mathbb{W}_Q$ has a unique solution $X \in \mathcal{D}'_0(L_2(\Omega;[\mathrm{dom}A]))$ given by the formula*

$$\langle X, \theta \rangle = (-1)^n \left[\int_0^\infty \theta^{(n)}(t)V(t)\zeta dt - \int_0^\infty \theta^{(n+1)}(t)dt \int_0^t V(t-s)B W_Q(s)\,ds \right].$$

This follows from the fact that the convolution inverse G in this case is the generalized derivative of V of order n; therefore, by (4) and (2) we obtain the result.

3.2. Generalized solutions with respect to ω

To consider the Cauchy problem (1) in the space $(\mathcal{S})'(H)$ we define the action of a linear closed operator $A : H \to H$ and linear bounded operator $B : \mathbb{H} \to H$ in the next way.

For any $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})'(\mathbb{H})$ define $B\Phi := \sum_\alpha B\Phi_\alpha \mathbf{h}_\alpha$. Thus B evidently becomes a linear continuous mapping of $(\mathcal{S})'(\mathbb{H})$ into $(\mathcal{S})'(H)$.

---

[1] For $u \in \mathcal{D}'$, $h \in H$ by $u \otimes h$ we denote the distribution from $\mathcal{D}'(H)$ defined by the equality $\langle u \otimes h, \theta \rangle := \langle u, \theta \rangle h$.

Define $(\mathrm{dom}A)$ to be the set of all $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})'(H)$ such that $\Phi_\alpha \in \mathrm{dom}A$ for all $\alpha \in \mathcal{T}$ and $\sum_\alpha \|A\Phi_\alpha\|_H^2 (2\mathbb{N})^{-2p\alpha} < \infty$ for some p. For any $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathrm{dom}A)$ define $A\Phi := \sum_\alpha A\Phi_\alpha \mathbf{h}_\alpha$.

The following theorem is proved in [1].

THEOREM 2. *Let A be the generator of a C_0-semigroup $\{U(t),\ t \geq 0\}$ in a Hilbert space H. Then for any $\zeta \in (\mathrm{dom}A) \subset (\mathcal{S})'(H)$ the Cauchy problem (1) with the singular white noise $\mathbb{W}$ has the unique solution*

$$X(t) = U(t)\zeta + \int_0^t U(t-s)B\mathbb{W}(s)ds \in (\mathcal{S})'(H), t \geq 0.$$

The solution is constructed as the series $X(t) = \sum_\alpha X_\alpha(t)\mathbf{h}_\alpha, t \geq 0$, where

$$X_\alpha(t) = \begin{cases} U(t)\zeta_{\varepsilon_n} + \int_0^t U(t-s)B\mathbb{W}_{\varepsilon_n}(s)\,ds, & \alpha = \varepsilon_n, \\ U(t)\zeta_\alpha, & \alpha \neq \varepsilon_n \end{cases}$$

are the solutions of the well-posed Cauchy problems

$$X'_{\varepsilon_n}(t) = AX_{\varepsilon_n}(t) + B\mathbb{W}_{\varepsilon_n}(t), \quad X_{\varepsilon_n}(0) = \zeta_{\varepsilon_n},$$
$$X'_\alpha(t) = AX_\alpha(t), \quad X_\alpha(0) = \zeta_\alpha \text{ for } \alpha \neq \varepsilon_n.$$

4. Generalized solutions with respect to t and ω

We see from the results of the previous section that in order to solve the Cauchy problem (1) with weaker conditions imposed on A, namely with A generating an n-times integrated semigroup, one has to consider it in the space $\mathcal{D}'_0(L_2(\Omega;H))$ of distributions in variable t. At the same time this forces one to take a Q-white noise $\mathbb{W}_Q$ with a nuclear operator Q as $\mathbb{W}$. In order to introduce the white noise with $Q = I$ into the equation one has to state the problem (1) in the space $(\mathcal{S})'(H)$ of distributions with respect to the random variable ω, but under this approach one has to impose more restrictive conditions on A, namely it must be the generator of a C_0-semigroup. This suggests the idea of combining the two approaches and considering the problem (1) in a suitable space of distributions in both t and ω.

Recall that the singular white noise $\mathbb{W}(t)$ belongs to the Hilbert space $(\mathcal{S}_{-1})(\mathbb{H}) \subset (\mathcal{S})'(\mathbb{H})$ of all Hilbert–Schmidt operators acting from $(\mathcal{S}_1)$ to $\mathbb{H}$ for each $t \in \mathbb{R}$. Consider the space $\mathcal{D}'_0((\mathcal{S}_{-1})(\mathbb{H}))$ of abstract $(\mathcal{S}_{-1})(\mathbb{H})$-valued distributions with supports in $[0;\infty)$ over the space $\mathcal{D}$ of test functions. Denote by $\mathbb{W}$ the distribution defined by

$$\langle \mathbb{W}, \theta \rangle = \int_0^\infty \mathbb{W}(t)\theta(t)dt, \quad \theta \in \mathcal{D}.$$

It is easy to see that $\mathbb{W}(t)$ is a continuous $(\mathcal{S}_{-1})(\mathbb{H})$-valued function of t, therefore $\mathbb{W} \in \mathcal{D}'_0((\mathcal{S}_{-1})(\mathbb{H}))$.

In section 3.2 we defined the action of $B \in \mathcal{L}(\mathbb{H}, H)$ as a linear continuous mapping of $(\mathcal{S})'(\mathbb{H})$ into $(\mathcal{S})'(H)$. Denote by the same symbol its restriction to $(\mathcal{S}_{-1})(\mathbb{H})$. It is easy to see that it is a linear bounded operator from $(\mathcal{S}_{-1})(\mathbb{H})$ to $(\mathcal{S}_{-1})(H)$.

Now we define the action of A in $(\mathcal{S}_{-1})(H)$. By $(\mathrm{dom}A)_{-1}$ we denote the set of all $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S}_{-1})(H)$ such that

$$\Phi_\alpha \in \mathrm{dom}A \ \text{ for all } \ \alpha \in \mathcal{T} \ \text{ and } \ \sum_\alpha \|A\Phi_\alpha\|_H^2 (2\mathbb{N})^{-2\alpha} < \infty.$$

For any $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathrm{dom}A)_{-1}$ define $A\Phi := \sum_\alpha A\Phi_\alpha \mathbf{h}_\alpha$. Since A is linear and closed as an operator in H, this defines A as a closed linear operator in $(\mathcal{S}_{-1})(H)$. Denote by $[(\mathrm{dom}A)_{-1}]$ the space $(\mathrm{dom}A)_{-1}$ with the graph norm. With such defined A we can consider the operator $P = \delta' \otimes I - \delta \otimes A$ as a distribution belonging to the space $\mathcal{D}_0'\big(\mathcal{L}([(\mathrm{dom}A)_{-1}]; (\mathcal{S}_{-1})(H))\big)$.

We will call $X \in \mathcal{D}_0'\big([(\mathrm{dom}A)_{-1}]\big)$ a solution of problem (1) with $\zeta \in (\mathcal{S}_{-1})(H)$ if it satisfies the equation $P * X = \delta \otimes \zeta + B\mathbb{W}$.

As a straightforward generalization of Theorem 1 to the case of $(\mathcal{S}_{-1})(H)$-valued functions we obtain the following result.

THEOREM 3. *Let A be the generator of an n-times integrated semigroup $V = \{V(t), t \geq 0\}$. Then for any $\zeta \in (\mathcal{S}_{-1})(H)$ the Cauchy problem (1) has a unique solution $X \in \mathcal{D}_0'\big([(\mathrm{dom}A)_{-1}]\big)$ given by the formula*

$$\langle X, \theta \rangle = (-1)^n \left[\int_0^\infty \theta^{(n)}(t) V(t) \zeta \, dt + \int_0^\infty \theta^{(n)}(t) dt \int_0^t V(t-s) B\mathbb{W}(s) ds \right].$$

where the integrals are understood as the Bochner integrals of $(\mathcal{S}_{-1})(H)$-valued functions.

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EQUIVALENCE OF SEQUENTIAL DEFINITIONS
OF THE CONVOLUTION OF DISTRIBUTIONS

Abstract. The equivalence of various sequential definitions of the convolution of distributions is proved. The list of known equivalent definitions is extended by adding definitions in terms of upper unit-sequences.

1. Introduction

The convolution of distributions (tempered distributions) is closely connected with the space $\mathcal{D}'_{L^1}$ of integrable distributions, the dual of the space $\mathcal{B}_0$. General definitions of the convolution in $\mathcal{D}'$ (in $\mathcal{S}'$), given in different ways in terms of integrability of certain distributions by various authors (C. Chevalley [1], L. Schwartz [10], R. Shiraishi [11]), appeared to be equivalent (see [11]). Later the list of equivalent definitions was gradually extended, for example by adding various sequential definitions (see V.S. Vladimirov [12, pp. 102–105], P. Dierolf–J. Voigt [2], A. Kamiński [4]). Sequential approaches are interesting, because they lead to natural generalizations connected with suitable restrictions of the considered classes of sequences (see [4]; for another type of generalizations see [13]). A similar situation concerns ultradistributions and tempered ultradistributions: various equivalent definitions of the convolution, including sequential ones, are related to integrability of certain ultradistributions (see [9, 5, 6]).

In sequential definitions of convolvable and integrable distributions and ultradistributions, an essential role is played by specific classes of sequences (called *unit-sequences*) of smooth functions of bounded support approximating the constant function 1. In this paper (see also [7]), we study another type (inspired by papers of B. Fisher, see e.g. [3]) of approximation of the function 1 by specific classes of sequences (called *upper unit-sequences*) of smooth functions with supports bounded only from below. In the next section, we give definitions of the classes Π of unit-sequences and Γ of upper unit-sequences as well as the classes $\overline{\Pi}$ and $\overline{\Gamma}$, narrower than Π and Γ.

Using these classes, we give in section 4 several sequential definitions of the convolution in $\mathcal{D}'$ and prove Theorem 3, the main result of the paper, that all of them are equivalent to the classical definitions mentioned above.

In the proof of Theorem 3 we apply the results and methods from [11, 2, 4, 8] (see section 3) as well as Lemma 1 proved in [7]. Note that a counterpart of Theorem 3 for tempered distributions is also true.

2. Preliminaries

The sets of all positive integers, non-negative integers, reals are denoted by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and their Cartesian powers for a fixed $d \in \mathbb{N}$ by $\mathbb{N}^d$, $\mathbb{N}_0^d$, $\mathbb{R}^d$, respectively. Elements of $\mathbb{R}^d$ and $\mathbb{N}_0^d$ are denoted by Latin and their coordinates by the corresponding Greek letters. Our multi-dimensional notation is mostly standard. In particular, for $x = (\xi_1, \ldots, \xi_d), y = (\eta_1, \ldots, \eta_d) \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$, the symbols $x \leq y$, $x \leq \alpha$ and $\alpha \leq x$ mean that the respective inequalities $\xi_i \leq \eta_i$, $\xi_i \leq \alpha$ and $\alpha \leq \xi_i$ hold for all $i = 1, \ldots, d$. A similar notation concerns strict inequalities. For $a = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ we set $[a, \infty) := [\alpha_1, \infty) \times \ldots \times [\alpha_d, \infty)$. If $a_n = (\alpha_{n,1}, \ldots, \alpha_{n,d}) \in \mathbb{R}^d$ for $n \in \mathbb{N}$, we write $a_n \to -\infty$ ($a_n \to \infty$) as $n \to \infty$, whenever $\alpha_{n,i} \to -\infty$ ($\alpha_{n,i} \to \infty$) as $n \to \infty$ for every $i = 1, \ldots, d$. Moreover, let $\alpha^k := \alpha^{\kappa_1 + \ldots + \kappa_d}$ for $\alpha \in \mathbb{R}$ and $k = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}_0^d$.

We will consider, beside the usual support, supp φ, also the *unitary support*, $s^1(\varphi) := \{x \in \mathbb{R}^d : \varphi(x) = 1\}$, of a function φ on $\mathbb{R}^d$.

To mark that a set $K \subset \mathbb{R}^d$ is compact we will write $K \sqsubset \mathbb{R}^d$. We use the standard notation: L^∞, C^∞, $\mathcal{E}$, $\mathcal{B}_0$, $\mathcal{B}$, $\mathcal{D}_K$ ($K \sqsubset \mathbb{R}^d$), $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}'_{L^1}$ for known spaces of functions and distributions on $\mathbb{R}^d$ and $\langle f, \varphi \rangle$ for the value of $f \in \mathcal{D}'$ on $\varphi \in \mathcal{D}$, or we use the more precise notation: $L^\infty(\mathbb{R}^d), \ldots, \mathcal{D}(\mathbb{R}^d)$, $\mathcal{D}'(\mathbb{R}^d)$, $\mathcal{D}'_{L^1}(\mathbb{R}^d)$ and $\langle f, \varphi \rangle_d$ to indicate the dimension d. For $k \in \mathbb{N}_0$, $K \sqsubset \mathbb{R}^d$ and a smooth (i.e. C^∞) function φ on $\mathbb{R}^d$, we define

$$q_{k,K}(\varphi) := \max_{0 \leq i \leq k} \max_{x \in K} |\varphi^{(i)}(x)|, \quad q_k(\varphi) := \max_{0 \leq i \leq k} \|\varphi^{(i)}\|_\infty,$$

where $\| \cdot \|_\infty$ denotes the supremum norm; evidently, $q_{k,K}(\varphi) \leq q_k(\varphi)$.

Recall that the sets $\mathcal{B}_0$; $\mathcal{B}$; and $\mathcal{D}_K$ ($K \sqsubset \mathbb{R}^d$) consist of all smooth functions φ such that $|\varphi^{(i)}(x)| \to 0$ as $|x| \to \infty$ for $i \in \mathbb{N}_0^d$; $q_k(\varphi) < \infty$ for $k \in \mathbb{N}_0$; and supp $\varphi \subset K$, respectively. Moreover, we have $\mathcal{E} = C^\infty$ and $\mathcal{D} = \cup_{K \sqsubset \mathbb{R}^d} \mathcal{D}_K$ in the sense of equalities of sets. The sets under consideration are endowed with the topologies defined by the respective families of seminorms: $\mathcal{B}_0$ and $\mathcal{B}$ by the family $\{q_k : k \in \mathbb{N}_0\}$; $\mathcal{E}$ by the family $\{q_{k,K} : k \in \mathbb{N}_0, K \sqsubset \mathbb{R}^d\}$; and $\mathcal{D}_K$ by the family $\{q_{k,K} : k \in \mathbb{N}_0\}$ (for $K \sqsubset \mathbb{R}^d$). The space $\mathcal{D}$ is endowed with the inductive limit topology of the spaces $\mathcal{D}_K$. Clearly,

$$(1) \qquad\qquad q_k(\varphi\psi) \leq 2^k q_k(\varphi) q_k(\psi), \qquad \varphi, \psi \in \mathcal{B}, \ k \in \mathbb{N}_0.$$

DEFINITION 1. *By unit-sequence we mean a sequence of functions $\pi_n \in \mathcal{D}$, convergent to 1 in $\mathcal{E}$, such that $\sup_{n \in \mathbb{N}} \|\pi_n^{(k)}\|_\infty < \infty$ for $k \in \mathbb{N}_0^d$, i.e.*

$$(2) \qquad\qquad \sup_{n \in \mathbb{N}} q_k(\pi_n) =: M_k < \infty, \qquad k \in \mathbb{N}_0^d.$$

By special unit-sequence we mean such a unit-sequence $\{\pi_n\}$ that for every bounded $K \subset \mathbb{R}^d$ there is an $n_0 \in \mathbb{N}$ such that $\pi_n(x) = 1$ for $x \in K$, $n \geq n_0$.

DEFINITION 2. *A set $E \subset \mathbb{R}^d$ is called bounded from below if $E \subset [a, \infty)$ for some $a \in \mathbb{R}^d$. By upper unit-sequence we mean a sequence $\{\Gamma_n\}$ of smooth functions, with supports bounded from below, convergent to 1 in $\mathcal{E}$ (i.e., there are $a_n \in \mathbb{R}^d$ with $a_n \to -\infty$ so that $\operatorname{supp} \Gamma_n \subset [a_n, \infty)$ for $n \in \mathbb{N}$) such that $\sup_{n \in \mathbb{N}} \| \Gamma_n^{(k)} \|_\infty < \infty$ for every $k \in \mathbb{N}_0^d$, i.e.*

$$(3) \qquad \sup_{n \in \mathbb{N}} q_k(\Gamma_n) =: N_k < \infty, \qquad k \in \mathbb{N}_0^d.$$

By special upper unit-sequence we mean an upper unit-sequence $\{\Gamma_n\}$ such that if $E \subset \mathbb{R}^d$ is bounded from below, then there is an $n_0 \in \mathbb{N}$ so that $s^1(\Gamma_n) \supset E$ for $n \geq n_0$ (i.e. there are $a_n, b_n \in \mathbb{R}^d$, $a_n < b_n$ $(n \in \mathbb{N})$, with $a_n \to -\infty$, and an index n_1 so that $[a_n, \infty) \supset \operatorname{supp} \Gamma_n \supset s^1(\Gamma_n) \supset [b_n, \infty)$ for $n > n_1$).

The classes of all *unit-sequences*, *special unit-sequences*, *upper unit-sequences*, and *special upper unit-sequences* of functions defined on $\mathbb{R}^d$ will be denoted, respectively, by Π, $\overline{\Pi}$, Γ and $\overline{\Gamma}$ or by Π_d, $\overline{\Pi}_d$, Γ_d and $\overline{\Gamma}_d$ to mark the dimension of $\mathbb{R}^d$.

For arbitrary $\{\pi_n\} \in \Pi$, $\{\Gamma_n\} \in \Gamma$, $\psi \in \mathcal{B}$ and $k \in \mathbb{N}_0^d$, we have

$$(4) \qquad \sup_{n \in \mathbb{N}} q_k(\pi_n \psi) \leq 2^k M_k q_k(\psi); \qquad \sup_{n \in \mathbb{N}} q_k(\Gamma_n \psi) \leq 2^k N_k q_k(\psi).$$

Given a class $\mathcal{Y}$ of sequences of functions consider the property:

$(\ast)$ *Class $\mathcal{Y}$ satisfies the implication: $\{\rho_n\}, \{\sigma_n\} \in \mathcal{Y} \Rightarrow \{\tau_n\} \in \mathcal{Y}$, where the sequence is defined by $\tau_{2n-1} := \rho_n$ and $\tau_{2n} := \sigma_n$ for $n \in \mathbb{N}$.*

Clearly, the above defined classes Π, $\overline{\Pi}$, Γ and $\overline{\Gamma}$ satisfy condition $(\ast)$.

DEFINITION 3. *A distribution f is called extendible for a function $\psi \in \mathcal{B}$ if $\{\langle f, \pi_n \psi \rangle\}$ is a Cauchy sequence for every $\{\pi_n\} \in \Pi$. The mapping $f_\psi \colon \mathcal{D} \cup (\psi) \to \mathbb{C}$ (where (ψ) denotes the singleton set), uniquely defined for such a distribution by*

$$(5) \qquad \langle f_\psi, \omega \rangle := \lim_{j \to \infty} \langle f, \pi_j \omega \rangle, \qquad \omega \in \mathcal{D} \cup (\psi),$$

for an arbitrary $\{\pi_n\} \in \Pi$, is called the extension of f for the function ψ.

If f is extendible for a $\psi \in \mathcal{B}$, then the limit in (5) does not depend on the choice of the sequence $\{\pi_n\}$ from Π, because the class Π satisfies (I). Consequently, the left side of (5) is well defined for $\omega = \psi$. Moreover, $\langle f_\psi, \varphi \rangle = \lim_{j \to \infty} \langle f, \pi_j \varphi \rangle = \langle f, \varphi \rangle$ for all $\varphi \in \mathcal{D}$ and $\{\pi_n\} \in \Pi$, due to the continuity of f on $\mathcal{D}$, i.e. $f_\psi|_\mathcal{D} = f$.

DEFINITION 4. *A distribution f is called extendible for a sequence $\{\Gamma_n\} \in \Gamma$ if is extendible for all Γ_n. The mapping $f^\diamond \colon \mathcal{D}^\diamond \to \mathbb{C}$, where $\mathcal{D}^\diamond := \mathcal{D} \cup \{\Gamma_n : n \in \mathbb{N}\}$, uniquely defined for such a distribution by formula (5) for $\psi \in \mathcal{D}^\diamond$, i.e. $\langle f^\diamond, \Gamma_n \rangle := \lim_{j \to \infty} \langle f, \pi_j \Gamma_n \rangle$ for $n \in \mathbb{N}$, is called the extension of f for the sequence $\{\Gamma_n\}$.*

DEFINITION 5. *A distribution f is called extendible to the space $\mathcal{B}$ if it is extendible for every $\psi \in \mathcal{B}$. The mapping $\tilde{f} \colon \mathcal{B} \to \mathbb{C}$, uniquely defined for such a distribution by means of formula (5) for every $\psi \in \mathcal{B}$ and $\{\pi_n\} \in \Pi$, i.e. by*

$$(6) \qquad \langle \tilde{f}, \psi \rangle := \lim_{j \to \infty} \langle f, \pi_j \psi \rangle, \qquad \psi \in \mathcal{B},$$

is called the extension of f to the space $\mathcal{B}$.

3. Integrable distributions

Integrable distributions, elements of the topological dual $\mathcal{B}'_0$ of $\mathcal{B}_0$, were described by P. Dierolf and J. Voigt in [2] by several equivalent conditions. To formulate below the extension of their result, proved in [7] and used in the proof of Theorem 3 in section 4, we apply for $\mathcal{A} \subseteq \mathcal{B}$ and $K \sqsubset \mathbb{R}^d$ the notation: $\mathcal{A}^K := \{\varphi \in \mathcal{A} \colon \operatorname{supp} \varphi \cap K = \emptyset\}$.

THEOREM 1. *Let $f \in \mathcal{D}'$. The following conditions are equivalent:*

(a_1)   *there are an $l \in \mathbb{N}_0$ and a $C > 0$ such that*

$$(7) \qquad |\langle f, \varphi \rangle| \leq C q_l(\varphi), \qquad \varphi \in \mathcal{D};$$

(A_1)   *f is extendible to $\mathcal{B}$ and its extension $\tilde{f}$ given on $\mathcal{B}$ by (6) is an element of $\mathcal{B}'$, i.e. there are an $l \in \mathbb{N}_0$ and a $C > 0$ such that*

$$(8) \qquad |\langle \tilde{f}, \psi \rangle| \leq C q_l(\psi), \qquad \psi \in \mathcal{B};$$

(a_2)   *there exists such an $l \in \mathbb{N}_0$ that for every $\varepsilon > 0$ there is a $K \sqsubset \mathbb{R}^d$ such that*

$$(9) \qquad |\langle f, \varphi \rangle| \leq \varepsilon q_l(\varphi), \qquad \varphi \in \mathcal{D}^K;$$

(A_2)   *f is extendible to $\mathcal{B}$ and its extension $\tilde{f}$ given on $\mathcal{B}$ by (6) is an element of $\mathcal{B}'$ with the property: there exists such an $l \in \mathbb{N}_0$ that for every $\varepsilon > 0$ there is a $K \sqsubset \mathbb{R}^d$ for which the inequality holds:*

$$(10) \qquad |\langle \tilde{f}, \psi \rangle| \leq \varepsilon q_l(\psi), \qquad \psi \in \mathcal{B}^K;$$

(a_3)   *there are an $l \in \mathbb{N}_0$, a $C > 0$ and a $K \sqsubset \mathbb{R}^d$ so that (7) holds for all $\varphi \in \mathcal{D}^K$;*

(A_3)   *f is extendible to $\mathcal{B}$ and its extension $\tilde{f}$ given on $\mathcal{B}$ by (6) is an element of $\mathcal{B}'$ with the property: there are an $l \in \mathbb{N}_0$, a $C > 0$ and a $K \sqsubset \mathbb{R}^d$ so that (8) holds for all $\varphi \in \mathcal{B}^K$;*

(b)   *$\{\langle f, \pi_n \rangle\}$ is a Cauchy sequence for every $\{\pi_n\} \in \Pi$;*

(B)   *f is extendible for every sequence $\{\Gamma_n\} \in \Gamma$ and $\{\langle f^\diamond, \Gamma_n \rangle\}$ is a Cauchy sequence, where $f^\diamond$ is the extension of f for the sequence $\{\Gamma_n\}$;*

$(\bar{b})$ $\{\langle f, \pi_n \rangle\}$ *is a Cauchy sequence for every* $\{\pi_n\} \in \overline{\Pi}$;

$(\overline{B})$ *f is extendible for every sequence $\{\Gamma_n\} \in \overline{\Gamma}$ and $\{\langle f^\diamond, \Gamma_n \rangle\}$ is a Cauchy sequence, where $f^\diamond$ is the extension of f for the sequence $\{\Gamma_n\}$.*

If any of the above conditions holds, then

$$(11) \qquad \langle \tilde{f}, 1 \rangle = \lim_{n \to \infty} \langle f, \pi_n \rangle = \lim_{n \to \infty} \langle f^\diamond, \Gamma_n \rangle,$$

for all $\{\pi_n\} \in \Pi$ and $\{\Gamma_n\} \in \Gamma$.

4. Convolution of distributions

Let $f, g \in \mathcal{D}'(\mathbb{R}^d)$. If the following condition introduced by C. Chevalley in [1]:

$$(C) \qquad (f * \varphi)(\check{g} * \psi) \in L^1(\mathbb{R}^d) \qquad \text{for all } \varphi, \psi \in \mathcal{D}(\mathbb{R}^d),$$

is assumed, there is a unique $f \overset{C}{*} g \in \mathcal{D}'$, the Chevalley convolution of f, g such that

$$\langle (f \overset{C}{*} g) * \varphi, \psi \rangle_d := \int_{\mathbb{R}^d} (f * \varphi)(x)(\check{g} * \psi)(x) \, dx, \qquad \varphi, \psi \in \mathcal{D}(\mathbb{R}^d).$$

L. Schwartz considered in [10] the condition:

$$(S) \qquad (f \otimes g) \varphi^\triangle \in \mathcal{D}'_{L^1}(\mathbb{R}^{2d}) \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d),$$

where $\varphi^\triangle(x, y) := \varphi(x + y)$ for $x, y \in \mathbb{R}^d$, and R. Shiraishi in [11] the conditions:

$$(S_1) \qquad f(\check{g} * \varphi) \in \mathcal{D}'_{L^1}(\mathbb{R}^d) \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d);$$
$$(S_2) \qquad (\check{f} * \varphi)g \in \mathcal{D}'_{L^1}(\mathbb{R}^d) \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d),$$

where the symbol $\check{h}$ for a given $h \in \mathcal{D}'(\mathbb{R}^d)$ means the distribution on $\mathbb{R}^d$ defined by $\langle \check{h}, \psi \rangle := \langle h, \check{\psi} \rangle$ and $\check{\psi}(x) := \psi(-x)$ for all $\psi \in \mathcal{D}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

Assuming (S), (S_1), (S_2), they defined the convolutions $f \overset{S}{*} g, f \overset{S_1}{*} g, f \overset{S_2}{*} g$:

$$\langle f \overset{S}{*} g, \varphi \rangle_d := \langle (f \otimes g) \varphi^\triangle, 1_{2d} \rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d);$$
$$\langle f \overset{S_1}{*} g, \varphi \rangle_d := \langle f(\check{g} * \varphi), 1_d \rangle_d, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d);$$
$$\langle f \overset{S_2}{*} g, \varphi \rangle_d := \langle (\check{g} * \varphi)g, 1_d \rangle_d, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d),$$

respectively, where 1_d and 1_{2d} are the constant functions equal to 1 on $\mathbb{R}^d$ and $\mathbb{R}^{2d}$.

R. Shiraishi proved in [11] the following theorem:

THEOREM 2. *Let $f, g \in \mathcal{D}'(\mathbb{R}^d)$. Conditions (C), (S), (S_1) and (S_2) are equivalent. If any of the conditions holds, then $f \overset{C}{*} g = f \overset{S}{*} g = f \overset{S_1}{*} g = f \overset{S_2}{*} g$.*

Due to Theorem 2, we may use for $f, g \in \mathcal{D}'(\mathbb{R}^d)$ the common notation

$$(12) \qquad f * g := f \overset{C}{*} g = f \overset{S}{*} g = f \overset{S_1}{*} g = f \overset{S_2}{*} g,$$

whenever one of conditions (C), (S), (S_1) and (S_2) is satisfied.

V.S. Vladimirov gave in [12] for $f, g \in \mathcal{D}'(\mathbb{R}^d)$ the following sequential definition of the convolution, denoted here by $f \overset{V}{*} g$:

$$(13) \qquad \langle f \overset{V}{*} g, \varphi \rangle_d = \lim_{n \to \infty} \langle f \otimes g, \pi_n \varphi^{\triangle} \rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d),$$

whenever the limit in (13) exists for all $\{\pi_n\} \in \overline{\Pi}_{2d}$ or, in other words, whenever

$$(\overline{V}) \qquad \{\langle f \otimes g, \pi_n \varphi^{\triangle} \rangle_{2d}\} \in \mathfrak{C} \qquad \text{for all } \{\pi_n\} \in \overline{\Pi}_{2d} \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d),$$

where $\mathfrak{C}$ denotes the class of all numerical Cauchy sequences. Clearly, condition $(\overline{V})$ implies that the limit in (13) does not depend on $\{\pi_n\} \in \overline{\Pi}_{2d}$.

P. Dierolf and J. Voigt proved in [2] for $f, g \in \mathcal{D}'(\mathbb{R}^d)$ that Vladimirov's condition $(\overline{V})$ and its extension in the following form:

$$(V) \qquad \{\langle f \otimes g, \pi_n \varphi^{\triangle} \rangle_{2d}\} \in \mathfrak{C} \qquad \text{for all } \{\pi_n\} \in \Pi_{2d} \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d)$$

are equivalent to conditions (C), (S), (S_1), (S_2) and the convolution $f \overset{V}{*} g$ defined for any $\{\pi_n\}$ from the classes $\overline{\Pi}_{2d}$ and Π_{2d} by common formula (13), coincides with the convolution $f * g$ given by (12).

A. Kamiński considered in [4], in connection with J. Mikusiński's irregular operations, the following conditions for $f, g \in \mathcal{D}'(\mathbb{R}^d)$:

$$(K) \qquad \{\langle (\pi_n f) * (\widetilde{\pi}_n g), \varphi \rangle_d\} \in \mathfrak{C} \quad \text{for all } \{\pi_n\}, \{\widetilde{\pi}_n\} \in \Pi_d \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$(K_1) \qquad \{\langle (\pi_n f) * g, \varphi \rangle_d\} \in \mathfrak{C} \qquad \text{for all } \{\pi_n\} \in \Pi_d \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$(K_2) \qquad \{\langle f * (\pi_n g), \varphi \rangle_d\} \in \mathfrak{C} \qquad \text{for all } \{\pi_n\} \in \Pi_d \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d)$$

as well as their variants: $(\overline{K})$, $(\overline{K}_1)$, $(\overline{K}_2)$, in which the class Π_d is replaced by $\overline{\Pi}_d$. He defined the convolutions $f \overset{K}{*} g$, $f \overset{K_1}{*} g$ and $f \overset{K_2}{*} g$ by the following formulas:

$$\langle f \overset{K}{*} g, \varphi \rangle_d := \lim_{n \to \infty} \langle (\pi_n f) * (\widetilde{\pi}_n g), \varphi \rangle_d, \quad \{\pi_n\}, \{\widetilde{\pi}_n\} \in \Pi_d, \ \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$\langle f \overset{K_1}{*} g, \varphi \rangle_d := \lim_{n \to \infty} \langle (\pi_n f) * g, \varphi \rangle_d, \qquad \{\pi_n\} \in \Pi_d, \ \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$\langle f \overset{K_2}{*} g, \varphi \rangle_d := \lim_{n \to \infty} \langle f * (\pi_n g), \varphi \rangle_d, \qquad \{\pi_n\} \in \Pi_d, \ \varphi \in \mathcal{D}(\mathbb{R}^d)$$

under conditions (K), (K_1), (K_2), respectively, and by the above formulas restricted to the class $\overline{\Pi}_d$ under conditions $(\overline{K})$, $(\overline{K}_1)$, $(\overline{K}_2)$, respectively. It was shown in [4], due to the results from [11] and [2], that conditions (K), (K_1), (K_2), $(\overline{K})$, $(\overline{K}_1)$, $(\overline{K}_2)$ are equivalent to conditions (V) and $(\overline{V})$ (and to those mentioned previously) and the corresponding convolutions coincide.

Consider now for $f, g \in \mathcal{D}'(\mathbb{R}^d)$, in connection with the classes Γ and $\overline{\Gamma}$ of upper unit-sequences and special upper unit-sequences, the following conditions:

(F_V) $\quad \{\langle f \otimes g, \Gamma_n \varphi^\triangle \rangle_{2d}\} \in \mathfrak{C}$ $\quad$ for all $\{\Gamma_n\} \in \Gamma_{2d}$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$;

(F_K) $\quad \{\langle (\Gamma_n f) * (\widetilde{\Gamma}_n g), \varphi \rangle_d\} \in \mathfrak{C}$ $\quad$ for all $\{\Gamma_n\}, \{\widetilde{\Gamma}_n\} \in \Gamma_d$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$

together with their variants $(\overline{F}_V)$ and $(\overline{F}_K)$, in which the classes Γ_{2d} and Γ_d are replaced by $\overline{\Gamma}_{2d}$ and $\overline{\Gamma}_d$, respectively. Define the convolutions $f \overset{F_V}{*} g$ and $f \overset{F_K}{*} g$ by the formulas

$$\langle f \overset{F_V}{*} g, \varphi \rangle_{2d} := \lim_{n \to \infty} \langle f \otimes g, \Gamma_n \varphi^\triangle \rangle_{2d}, \qquad \{\Gamma_n\} \in \Gamma_{2d}, \ \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$\langle f \overset{F_K}{*} g, \varphi \rangle_d := \lim_{n \to \infty} \langle (\Gamma_n f) * (\widetilde{\Gamma}_n g), \varphi \rangle_d, \qquad \{\Gamma_n\}, \{\widetilde{\Gamma}_n\} \in \Gamma_d, \ \varphi \in \mathcal{D}(\mathbb{R}^d),$$

under conditions (F_V) and (F_K), respectively, and by the same formulas but restricted to the classes $\overline{\Gamma}_{2d}$ and $\overline{\Gamma}_d$, under conditions $(\overline{F}_V)$ and $(\overline{F}_K)$, respectively.

The following theorem is true:

THEOREM 3. *Let $f, g \in \mathcal{D}'(\mathbb{R}^d)$. Each of the conditions (V), $(\overline{V})$, (K), $(\overline{K})$, (K_1), $(\overline{K}_1)$, (K_2), $(\overline{K}_2)$, (F_V), $(\overline{F}_V)$, (F_K) and $(\overline{F}_K)$ is equivalent to any of the conditions listed in Theorem 2. If any of the conditions is satisfied, then*

$$(14) \qquad f * g = f \overset{V}{*} g = f \overset{K}{*} g = f \overset{K_1}{*} g = f \overset{K_2}{*} g = f \overset{F_V}{*} g = f \overset{F_K}{*} g.$$

In the proof of Theorem 3, the following lemma plays an important role:

LEMMA 1. *Let $h \in \mathcal{D}'(\mathbb{R}^{2d})$. Assume that*

$$\operatorname{supp} h \subset K^\triangle := \{(x,y) \in \mathbb{R}^{2d} : x + y \in K\}$$

for some $K \sqsubset \mathbb{R}^d$ and there is a scalar α such that

$$(15) \qquad \lim_{n \to \infty} \langle h, \Gamma_n^1 \otimes \Gamma_n^2 \rangle_{2d} = \alpha$$

for all $\{\Gamma_n^1\}, \{\Gamma_n^2\} \in \overline{\Gamma}_d$. Then for arbitrary special unit-sequences $\{\pi_n^1\}, \{\pi_n^2\} \in \overline{\Pi}_d$ there exists an increasing sequence $\{q_n\}$ of positive integers such that

$$(16) \qquad \lim_{n \to \infty} \langle h, \pi_{q_n}^1 \otimes \pi_{q_n}^2 \rangle_{2d} = \alpha.$$

The proof of Lemma 1 is not trivial and requires an induction construction. Its full presentation is beyond the scope of this article. We show a complete proof of Lemma 1 with all its nuances in a separate publication (see [7]).

Proof of Theorem 3. We present the proof of all implications necessary to conclude the formulated equivalence.

$(S) \Rightarrow (F_V)$; $(S) \Rightarrow (V)$.   Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$, denote $h := (f \otimes g)\varphi^{\triangle}$ and assume condition (S), which means that $h \in \mathcal{D}'_{L^1}(\mathbb{R}^{2d})$. By Theorem 1, it follows that

$$\lim_{n \to \infty} \langle h, \Gamma_n \rangle_{2d} = \lim_{n \to \infty} \langle h, \pi_n \rangle_{2d} = \langle h, 1_{2d} \rangle_{2d}$$

for $\{\pi_n\} \in \Pi_{2d}$ and $\{\Gamma_n\} \in \Gamma_{2d}$, i.e. (F_V) and (V) hold. Also $f \overset{F_V}{*} g = f \overset{V}{*} g = f \overset{S}{*} g$.

$(F_V) \Rightarrow (\overline{F}_V)$; $(F_K) \Rightarrow (\overline{F}_K)$; $(V) \Rightarrow (\overline{V})$; $(K_i) \Rightarrow (\overline{K}_i)$ $(i = 1, 2)$.   The implications are obvious, because of the inclusions $\overline{\Gamma} \subset \Gamma$ and $\overline{\Pi} \subset \Pi$ between the considered classes of (upper) unit-sequences.

$(F_V) \Rightarrow (F_K)$; $(\overline{F}_V) \Rightarrow (\overline{F}_K)$; $(V) \Rightarrow (K)$; $(\overline{V}) \Rightarrow (\overline{K})$.   Clearly, the equalities

(17) $\langle (\Gamma_n^1 f) * (\Gamma_n^2 g), \varphi \rangle_d = \langle (\Gamma_n^1 f) \otimes (\Gamma_n^2 g), \varphi^{\triangle} \rangle_{2d} = \langle (f \otimes g)\varphi^{\triangle}, \Gamma_n^1 \otimes \Gamma_n^2 \rangle_{2d}$

hold for all $n \in \mathbb{N}$ and $\{\Gamma_n^1\}, \{\Gamma_n^2\} \in \Gamma_d$. Similarly, the equalities

(18) $\langle (\pi_n^1 f) * (\pi_n^2 g), \varphi \rangle_d = \langle (\pi_n^1 f) \otimes (\pi_n^2 g), \varphi^{\triangle} \rangle_{2d} = \langle (f \otimes g)\varphi^{\triangle}, \pi_n^1 \otimes \pi_n^2 \rangle_{2d}$

hold for all $n \in \mathbb{N}$ and for all $\{\pi_n^1\}, \{\pi_n^2\} \in \Pi_d$.

If $\{\Gamma_n^1\}$ and $\{\Gamma_n^2\}$ are arbitrary sequences in Γ_d (respectively, in $\overline{\Gamma}_d$), then the sequence $\{\Gamma_n^1 \otimes \Gamma_n^2\}$ is in Γ_{2d} (respectively, in $\overline{\Gamma}_{2d}$). Hence, by (17), condition (F_V) (respectively, $(\overline{F}_V)$) implies condition (F_K) (respectively, $(\overline{F}_K)$). Moreover, $f \overset{F_V}{*} g = f \overset{F_K}{*} g$. Similarly, if $\{\pi_n^1\}$ and $\{\pi_n^2\}$ are in Π_d (respectively, in $\overline{\Pi}_d$), then the sequence $\{\pi_n^1 \otimes \pi_n^2\}$ is in Π_{2d} (respectively, in $\overline{\Pi}_{2d}$) and, due to (18), condition (V) (respectively, $(\overline{V})$) implies condition (K) (respectively, $(\overline{K})$). Moreover, $f \overset{V}{*} g = f \overset{K}{*} g$.

$(\overline{F}_K) \Rightarrow (\overline{K})$.   Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and denote $h := (f \otimes g)\varphi^{\triangle}$. Assume that condition $(\overline{F}_K)$ holds, i.e. there is a number α such that (15) holds for all $\{\Gamma_n^1\}, \{\Gamma_n^2\} \in \overline{\Gamma}_d$. Fix $\{\pi_n^1\}, \{\pi_n^2\} \in \overline{\Pi}_d$ and let $\{\widetilde{\pi}_n^1\}$ and $\{\widetilde{\pi}_n^2\}$ be arbitrary subsequences of $\{\pi_n^1\}$ and $\{\pi_n^2\}$. By Lemma 1, there exist subsequences $\{\widetilde{\pi}_{q_n}^1\}$ and $\{\widetilde{\pi}_{q_n}^2\}$ of $\{\widetilde{\pi}_n^1\}$ and $\{\widetilde{\pi}_n^2\}$, respectively, such that (16) holds. Then

(19) $$\lim_{n \to \infty} \langle h, \pi_n^1 \otimes \pi_n^2 \rangle_{2d} = \alpha$$

for our arbitrarily fixed sequences $\{\pi_n^1\}, \{\pi_n^2\} \in \overline{\Pi}_d$, i.e. condition $(\overline{K})$ is satisfied. Moreover, $f \overset{F_K}{*} g = f \overset{K}{*} g$ (for the classes $\overline{\Gamma}$ and $\overline{\Pi}$).

$(K) \Rightarrow (K_i)$ $(i = 1, 2)$; $(\overline{K}) \Rightarrow (\overline{K}_i)$ $(i = 1, 2)$.   Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and let $h := (f \otimes g)\varphi^{\triangle}$. Assume that condition (K) (respectively, $(\overline{K})$) is satisfied, i.e. equality (19) is true for some α and for all $\{\pi_n^1\}$ and $\{\pi_n^2\}$ in Π_d (respectively, in $\overline{\Pi}_d$). For arbitrarily fixed sequences $\{\pi_n^1\}$ and $\{\pi_n^2\}$ in Π_d (respectively, in $\overline{\Pi}_d$), the assumption implies

(20) $$\lim_{n, m \to \infty} \langle h, \pi_n^1 \otimes \pi_m^2 \rangle_{2d} = \alpha$$

with the double limit on the left side. If not, there would exist an $\varepsilon_0 > 0$ and increasing sequences $\{p_n\}$, $\{q_n\}$ of indices such that $|\langle h, \pi^1_{p_n} \otimes \pi^2_{q_n} \rangle_{2d} - \alpha| > \varepsilon_0$ for $n \in \mathbb{N}$. But the sequences $\{\pi^1_{p_n}\}$ and $\{\pi^2_{q_n}\}$ are also in Π_d (respectively, in $\overline{\Pi}_d$), so the last inequality contradicts our assumption concerning (19) and proves (20).

Since $\varphi^\triangle(\pi^1_n \otimes 1_d)$ and $\varphi^\triangle(1_d \otimes \pi^2_m)$ are in $\mathcal{D}(\mathbb{R}^{2d})$ for $n, m \in \mathbb{N}$, we have

$$\lim_{m \to \infty} \langle h, \pi^1_n \otimes \pi^2_m \rangle_{2d} = \langle f \otimes g, \varphi^\triangle(\pi^1_n \otimes 1_d) \rangle_{2d} = \langle (\pi^1_n f) * g, \varphi \rangle_d, \qquad n \in \mathbb{N}$$

and

$$\lim_{n \to \infty} \langle h, \pi^1_n \otimes \pi^2_m \rangle_{2d} = \langle f \otimes g, \varphi^\triangle(1_d \otimes \pi^2_m) \rangle_{2d} = \langle f * (\pi^2_m g), \varphi \rangle_d, \qquad m \in \mathbb{N}.$$

Hence, by (20),

$$\lim_{n \to \infty} \langle (\pi^1_n f) * g, \varphi \rangle_d = \lim_{n \to \infty} \lim_{m \to \infty} \langle h, \pi^1_n \otimes \pi^2_m \rangle_{2d} = \alpha = \lim_{m \to \infty} \langle f * (\pi^2_m g), \varphi \rangle_d.$$

The implications and the equalities $f \overset{K}{*} g = f \overset{K_1}{*} g = f \overset{K_2}{*} g$ are proved.

$(\overline{K}_i) \Rightarrow (S_i)$ $(i = 1, 2)$.   For all $n \in \mathbb{N}$, we have

$$\langle (\pi^1_n f) * g, \varphi \rangle_d = \langle f(\check{g} * \varphi), \pi^1_n \rangle_d \quad \text{and} \quad \langle f * (\pi^2_n g, \varphi \rangle_d = \langle (\check{f} * \varphi)g, \pi^2_n \rangle_d,$$

so conditions $(\overline{K}_1)$ and $(\overline{K}_2)$ imply (S_1) and (S_2), respectively, by Theorem 1.

Since, by Theorem 2, conditions (S), (S_1) and (S_2) are equivalent, the proof of the equivalence of all conditions and of all equalities in (14) is thus completed. $\square$

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ABOUT THE STABILITY PROBLEM FOR STRICTLY HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

Abstract. We consider jump-type initial data for a strictly hyperbolic quasilinear system of conservation laws in one space dimension. Suppose that the initial jump is associated formally with a shock wave. Our aim is the consideration of sufficient stability conditions for this problem in the case of arbitrary jump amplitude.

1. Introduction

We consider a strictly hyperbolic system of conservation laws

$$
(1) \qquad \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad t > 0, \, x \in \mathbb{R}^1,
$$

$u = (u_1, \ldots, u_n)$, $f = (f_1, \ldots, f_n) \in C^\infty$, supplemented with jump-type initial data

$$
(2) \qquad u\big|_{t=0} = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases}
$$

We assume that the states u_- and u_+ are connected by the Hugoniot locus (see overleaf) so that the problem (1), (2) has formally the shock wave solution. Our aim is to investigate the stability criterion for shocks with arbitrary amplitudes $u_- - u_+$.

Let us recall that solutions to the Cauchy problem for quasilinear hyperbolic equations are not unique in the D' sense (see, e.g. [1, 2, 13]). The standard example is the Hopf (Burgers) equation associated with the initial data of the form of a jump function with a positive amplitude,

$$
\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = 0, \quad t > 0, \, x \in \mathbb{R}^1,
$$

$$
u\big|_{t=0} = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \quad u_- < u_+, \end{cases}
$$

when both the family of shock waves with an arbitrary number N of jumps and the centered rarefaction are the weak solutions.

For the scalar conservation law with a flux function $f(u)$ it is known that only one of the possible weak solutions is stable and it should be an entropy solution. In particular, the shock wave is stable if and only if the Oleinik E-condition,

$$
(3) \qquad \frac{f(u) - f(u_-)}{u - u_-} \geq \frac{f(u_+) - f(u_-)}{u_+ - u_-} \geq \frac{f(u_+) - f(u)}{u_+ - u}
$$

is verified for every u between u_+ and u_- [1, 2, 13].

The same problem for systems remains unsolved up to now in the general statement and only the case of weak shocks (that is for sufficiently small shock amplitudes) has been studied in detail so far.[1, 2].

Theoretically, there is a possibility to investigate strictly hyperbolic systems passing to a parabolic regularization. In the framework of this approach we obtain the uniqueness of the prelimiting solution automatically [1, 2]. However, it is not clear how to identify the resulting solution profile with the specific initial data. In particular, the shock wave stability criterion remains unknown.

We guess that progress in this problem can be achieved at present in the framework of the weak asymptotics method, see [3]–[12] and references therein. The main idea here is to satisfy the equation with a remainder which is small in the weak sense. The advantage is such that we should investigate some ODEs instead of PDEs. This approach allows to recognize the principal structure of the solution and to calculate the main characteristics of the limiting solution. For the case of shock waves it means the uniform-in-time description of the interaction processes for weak solutions of hyperbolic problems.

2. Assumptions and definitions

As it has been noted in Introduction, we do not consider the Riemann problem but investigate the stability of shock-wave solutions. Thus, we assume that:

A. The states u_- and u_+ are joined by the Hugoniot locus in the sense that there is a C^2-curve $u = u(\xi)$ in the state space and C^2-function $s = s(\xi)$ such that $u(0) = u_-$, $u(\xi_N) = u_+$, and for any $\xi \in [0, \xi_N]$ the Rankine–Hugoniot condition

$$(4) \qquad\qquad s(\xi)\big(u(\xi) - u_-\big) = f\big(u(\xi)\big) - f(u_-)$$

is satisfied.

In what follows, we will use the alternative form of the condition (4). Let

$$(5) \qquad\qquad \mathcal{H}(u_-, u) = \int_0^1 Df\big(u_- + (u - u_-)\omega\big)d\omega,$$

then (4) is equivalent to the following:

$$(6) \qquad\qquad \big(sI - \mathcal{H}(u_-, u(\xi))\big)\big(u_- - u(\xi)\big) = 0.$$

Obviously, the last equality means that the shock speed s should be the eigenvalue $\Lambda(u_-, u)$ of the matrix $\mathcal{H}(u_-, u)$,

$$(7) \qquad\qquad s(\xi) = \Lambda\big(u_-, u(\xi)\big),$$

whereas $u_- - u(\xi)$ can be interpreted as the corresponding eigenvector.

To present our next assumption let us define a sequence $\{u(\xi_i)\}$, $i = 0,\ldots,N$, $\xi_0 = 0$, $\xi_i < \xi_{i+1}$ such that $u(\xi_i)$ belongs to the Hugoniot locus for all i and

$$(8) \qquad |u(\xi_{i+1}) - u(\xi_i)| \ll 1.$$

B. Assume that the Liu E-condition

$$(9) \qquad \Lambda\big(u(\xi), u(\xi_{i+1})\big) < \Lambda\big(u(\xi_i), u(\xi_{i+1})\big) < \Lambda\big(u(\xi_i), u(\xi)\big)$$

is verified for any sequence $\{u(\xi_i)\}$ under condition (8), any $i = 0,\ldots,N-1$ and all $\xi \in (\xi_i, \xi_{i+1})$.

In fact, this assumption is very restrictive, in particular, it implies the convexity condition in the scalar case.

Now let us regularize the problem (1), (2). To this aim we replace firstly the initial jump by a chain of equally spaced elementary jumps

$$(10) \qquad u_\Delta\big|_{t=0} = \begin{cases} u_0^0, & x < \Delta_1, \\ u_1^0, & \Delta_1 < x < \Delta_2, \\ \ldots & \ldots \\ u_{N-1}^0, & \Delta_{N-1} < x < \Delta_N, \\ u_N^0, & x > \Delta_N, \end{cases}$$

where $u_i^0 = u(\xi_i)$, $i = 0,\ldots,N$ and $\Delta_1 < \Delta_2 \cdots < \Delta_N$ are small parameters. In view of the condition (9) the solution of any Cauchy problem

$$\frac{\partial u_{\Delta_i}}{\partial t} + \frac{\partial f(u_{\Delta_i})}{\partial x} = 0, \quad t > 0,\, x \in \mathbb{R}^1,$$

$$u_{\Delta_i}\big|_{t=0} = \begin{cases} u_{i-1}^0, & x < \Delta_i, \\ u_i^0, & x > \Delta_i, \end{cases}$$

will be a stable shock wave.

Next, fixing the parameters Δ_i, we pass to the parabolic regularization of the problem (1), (2) with smoothed initial data (10):

$$(11) \qquad \frac{\partial u_{\Delta,\varepsilon}}{\partial t} + \frac{\partial f(u_{\Delta,\varepsilon})}{\partial x} = \varepsilon \frac{\partial^2 u_{\Delta,\varepsilon}}{\partial x^2}, \quad t > 0,\, x \in \mathbb{R}^1,$$

$$(12) \qquad u_{\Delta,\varepsilon}\big|_{t=0} = u_{\Delta,\varepsilon}^0(x),$$

where $\varepsilon \ll |\Delta_i|$ is a small parameter, $u_{\Delta,\varepsilon}^0(x) \in C^\infty(\mathbb{R}_x^1)$ for $\varepsilon = \text{const} > 0$, and $u_{\Delta,\varepsilon}^0(x) \to u_\Delta\big|_{t=0}$ as $\varepsilon \to 0$ in $\mathcal{D}'$ sense.

To describe the problem (11), (12) solution uniformly in time we use the weak asymptotics method [5].

DEFINITION 1. *Let $u_{\Delta,\varepsilon} = u_{\Delta,\varepsilon}(t,x)$ be a function belonging to $C^{\infty}([0,T] \times \mathbb{R}_x^1)$ for each $\varepsilon = \mathrm{const} > 0$ and let $u_{\Delta,\varepsilon}$, $f(u_{\Delta,\varepsilon})$ belong to $C([0,T]; \mathcal{D}'(\mathbb{R}_x^1))$ for all $\varepsilon \in [0,\mathrm{const}]$. We say that $u_{\Delta,\varepsilon}(t,x)$ is a weak asymptotic* mod $O_{\mathcal{D}'}(\varepsilon)$ *solution of equation (11) if the relation*

$$(13) \qquad \frac{d}{dt} \int_{-\infty}^{\infty} u_{\Delta,\varepsilon} \cdot \psi dx - \int_{-\infty}^{\infty} f(u_{\Delta,\varepsilon}) \cdot \frac{d\psi}{dx} dx = O(\varepsilon)$$

holds uniformly in $t \in (0,T]$ for any test vector function $\psi = \psi(x) \in \mathcal{D}(\mathbb{R}_x^1)$.

Here and below $a \cdot b$ denotes the inner product of vectors a, b, and the estimate $O(\varepsilon^k)$ is understood in the $C([0,T])$ sense: $\left|O(\varepsilon^k)\right| \leq C_T \varepsilon^k$ for $t \in [0,T]$.

DEFINITION 2. *A function $g(t,x,\varepsilon)$ is said to be of the order $O_{\mathcal{D}'}(\varepsilon^k)$ if the relation*

$$(g,\psi) = \int_{-\infty}^{\infty} g(t,x,\varepsilon) \cdot \psi(x) dx = O(\varepsilon^k)$$

holds uniformly in $t \in (0,T]$ for any test function $\psi = \psi(x) \in \mathcal{D}(\mathbb{R}_x^1)$.

Obviously, the jump (2) generates the stable shock wave if and only if

$$(14) \qquad u_{\Delta,\varepsilon}(t,x) \to u_- + (u_+ - u_-)H(x - st) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}_x^1) \quad \text{for} \quad t > 0$$

as $\varepsilon, \Delta_i \to 0$, $i = 1,\ldots,N$.

It has been proved for the scalar equations that the weak asymptotics with the property (14) exists if and only if the Oleinik E-condition (4) is verified (see [6], [11], and [12] for convex, concave-convex, and arbitrary flux function respectively). This approach has been used also to describe uniformly in time the interaction processes in gas dynamics [8, 9, 10]. Now we will construct the weak asymptotic solution for the system (1).

3. Asymptotic construction

We need to construct the weak asymptotics for some time interval $[0,T_\Delta]$ with T_Δ as big as the time instant of the last elementary solution interaction. Obviously, we should know how to construct the stable solution for any elementary interaction of the waves with arbitrary amplitudes without application of any additional conditions. At the same time it is clear that it is enough to describe only one elementary interaction. So we will consider the equation (11) supplemented with the initial condition

$$u_{\Delta,\varepsilon}(x,0) = u(\xi_\alpha) + \left(u(\xi) - u(\xi_\alpha)\right)\omega\left(\frac{x - x_\alpha}{\varepsilon}\right)$$

$$(15) \qquad\qquad + \left(u(\xi_\beta) - u(\xi)\right)\omega\left(\frac{x - x_\beta}{\varepsilon}\right).$$

Here $\xi_\alpha < \xi < \xi_\beta$,

$$(16) \qquad\qquad\qquad\qquad x_\alpha < x_\beta,$$

and $\omega(z/\varepsilon)$ is a Heaviside function regularization, such that $\omega(\eta) \in C^\infty$, $\omega(\eta)$ tends to its limiting values with an exponential rate, and

$$(17) \qquad \omega(\eta) + \omega(-\eta) = 1.$$

Let us denote $\varphi_{\alpha,\beta}^0(t)$ the trajectories of the non-interacting shock waves generated by the initial condition (15), namely,

$$(18) \qquad \varphi_\alpha^0(t) = \Lambda\big(u(\xi_\alpha), u(\xi)\big)t + x_\alpha, \quad \varphi_\beta^0(t) = \Lambda\big(u(\xi_\beta), u(\xi)\big)t + x_\beta.$$

Next we define the "fast time"

$$(19) \qquad \tau = \frac{\rho(t)}{\varepsilon}, \quad \rho(t) = \varphi_\beta^0(t) - \varphi_\alpha^0(t)$$

to measure the distance between the trajectories $\varphi_\alpha(t)$ and $\varphi_\beta(t)$. For the first stage of interactions, when $\xi_\alpha = \xi_{i-1}$, $\xi = \xi_i$, and $\xi_\beta = \xi_{i+1}$, so, the condition (8) is verified, we use the Liu E-condition (9) and obtain that the trajectories $x = \varphi_\alpha^0(t)$ and $x = \varphi_\beta^0(t)$ intersect at one point (x^*, t^*). For the next stages, when (8) may be violated, we obtain the trajectories intersection applying the sharpened version of the condition **B**:

B'. Let

$$(20) \qquad \Lambda\big(u(\xi), u(\xi_\beta)\big) < \Lambda\big(u(\xi_\alpha), u(\xi_\beta)\big) < \Lambda\big(u(\xi_\alpha), u(\xi)\big)$$

for any ξ_α, ξ_β such that $0 \leq \xi_\alpha < \xi_\beta \leq \xi_N$ and for all $\xi \in (\xi_\alpha, \xi_\beta)$.

In view of the choice (16), $\rho(0) = x_\beta - x_\alpha > 0$ and $\tau|_{t\approx 0} \to \infty$ as $\varepsilon \to 0$ if $x_\beta - x_\alpha \approx \varepsilon^{1-\kappa}$, $\kappa > 0$. Thus, outside of a small neighborhood of the point (x^*, t^*), we have that $\tau \to \infty$ before the time instant of interaction and $\tau \to -\infty$ after the interaction.

Let us write the asymptotic ansatz in the self-similar form:

$$u_{\Delta,\varepsilon}(x,t) = u(\xi_\alpha) + A_\alpha(\tau)\omega\Big(\frac{x - \varphi_\alpha}{\varepsilon}\Big)$$

$$(21) \qquad\qquad + A_\beta(\tau)\omega\Big(\frac{x - \varphi_\beta}{\varepsilon}\Big) + B(\tau)\omega'\Big(\frac{x - x^*}{\varepsilon}\Big).$$

Here $\omega'(z) = \partial\omega(z)/\partial z$, the phases $\varphi_{\alpha,\beta} = \varphi_{\alpha,\beta}(\tau,t)$ are smooth functions such that

$$(22) \qquad \varphi_{\alpha,\beta}(\tau,t) = \varphi_{\alpha,\beta}^0(t) + \rho(t)\varphi_{\alpha,\beta}^1(\tau),$$

where

$$(23) \qquad \varphi_{\alpha,\beta}^1(\tau) \to 0 \text{ as } \tau \to \infty, \quad \varphi_{\alpha,\beta}^1(\tau) \to \bar{\varphi}_{\alpha,\beta}^1 \quad \text{as} \quad \tau \to -\infty,$$

and $\bar{\varphi}_{\alpha,\beta}^1$ are constants. Next $A_{\alpha,\beta}(\tau)$, $B(\tau)$ are smooth functions such that

$$(24) \qquad A_\alpha(\tau) \to u(\xi) - u(\xi_\alpha), \quad A_\beta(\tau) \to u(\xi_\beta) - u(\xi) \quad \text{as} \quad \tau \to \infty,$$

(25) $$A_{\alpha,\beta}(\tau) \to \bar{A}_{\alpha,\beta} \quad \text{as} \quad \tau \to -\infty,$$

(26) $$B(\tau) \to 0 \quad \text{as} \quad \tau \to \infty, \quad B(\tau) \to \bar{B} \quad \text{as} \quad \tau \to -\infty,$$

where $\bar{A}_{\alpha,\beta}$ and $\bar{B}$ are some constants. We assume that $\varphi_{\alpha,\beta}(\tau,t)$, $A_{\alpha,\beta}(\tau)$, and $B(\tau)$ tend to the limiting values as $\tau \to \pm\infty$ with an exponential rate.

To continue our construction we should calculate the weak expansions for the functions $u_{\Delta,\varepsilon}(x,t)$ and $f\big(u_{\Delta,\varepsilon}(x,t)\big)$.

LEMMA 1. *For the function $u_{\Delta,\varepsilon}(x,t)$ of the form (21) the following relations hold:*

$$u_{\Delta,\varepsilon}(x,t) = u(\xi_\alpha) + A_\alpha(\tau)H(x-\varphi_\alpha) + A_\beta(\tau)H(x-\varphi_\beta)$$
(27)
$$+ \varepsilon B(\tau)\delta(x-x^*) + O_{\mathcal{D}'}(\varepsilon^2),$$

$$f\big(u_{\Delta,\varepsilon}(x,t)\big) = f\big(u(\xi_\alpha)\big) + \mathcal{H}_\alpha\big(u(\xi_\alpha),\sigma,\tau\big)A_\alpha(\tau)H(x-\varphi_\alpha)$$
(28)
$$+ \mathcal{H}_\beta\big(u(\xi_\alpha),\sigma,\tau\big)A_\beta(\tau)H(x-\varphi_\beta) + O_{\mathcal{D}'}(\varepsilon),$$

where the matrices $\mathcal{H}_{\alpha,\beta}$ are the convolutions

$$\mathcal{H}_\alpha\big(u(\xi_\alpha),\sigma,\tau\big) = \int_{-\infty}^{\infty} Df\Big(u(\xi_\alpha) + A_\alpha\omega(\eta) + A_\beta\omega(\eta-\sigma)\Big)\omega'(\eta)d\eta,$$
(29)
$$\mathcal{H}_\beta\big(u(\xi_\alpha),\sigma,\tau\big) = \int_{-\infty}^{\infty} Df\Big(u(\xi_\alpha) + A_\beta\omega(\eta) + A_\alpha\omega(\eta+\sigma)\Big)\omega'(\eta)d\eta$$

with the properties

$$\lim_{\sigma \to +\infty} \mathcal{H}_\alpha\big(u(\xi_\alpha),\sigma,\tau\big) = \mathcal{H}\big(u(\xi_\alpha),u(\xi_\alpha)+A_\alpha\big),$$
(30)
$$\lim_{\sigma \to +\infty} \mathcal{H}_\beta\big(u(\xi_\alpha),\sigma,\tau\big) = \mathcal{H}\big(u(\xi_\alpha)+A_\alpha,u(\xi_\alpha)+A_\alpha+A_\beta\big),$$

$$\lim_{\sigma \to -\infty} \mathcal{H}_\alpha\big(u(\xi_\alpha),\sigma,\tau\big) = \mathcal{H}\big(u(\xi_\alpha)+A_\beta,u(\xi_\alpha)+A_\alpha+A_\beta\big),$$
(31)
$$\lim_{\sigma \to -\infty} \mathcal{H}_\beta\big(u(\xi_\alpha),\sigma,\tau\big) = \mathcal{H}\big(u(\xi_\alpha),u(\xi_\alpha)+A_\beta\big),$$

and

$$\mathcal{H}_\alpha\big(u(\xi_\alpha),0,\tau\big) = \mathcal{H}_\beta\big(u(\xi_\alpha),0,\tau\big)$$
(32)
$$= \int_{-\infty}^{\infty} Df\Big(u(\xi_\alpha) + (A_\alpha+A_\beta)\omega(\eta)\Big)\omega'(\eta)d\eta.$$

Here $\mathcal{H}\big(u(\xi_\alpha),u(\xi)\big)$ is the matrix defined in (5) and $\sigma = \sigma(\tau,t,\varepsilon)$ characterizes the distance between the trajectories φ_α and φ_β, namely

(33) $$\sigma = \frac{\varphi_\beta - \varphi_\alpha}{\varepsilon}.$$

Proof. To prove the first assertion of the lemma it is enough to note that for any test-function $\psi(x) \in \mathcal{D}(\mathbb{R}^1)$

$$\left(\omega\left(\frac{x-\varphi}{\varepsilon}\right), \psi(x)\right) = \left(H(x-\varphi), \psi(x)\right) + \int_{-\infty}^{\varphi} \omega\left(\frac{x-\varphi}{\varepsilon}\right)\psi(x)dx$$
$$- \int_{\varphi}^{\infty}\left(1 - \omega\left(\frac{x-\varphi}{\varepsilon}\right)\right)\psi(x)dx.$$

In view of (17) we have:

$$\int_{-\infty}^{\varphi} \omega\left(\frac{x-\varphi}{\varepsilon}\right)\psi(x)dx - \int_{\varphi}^{\infty}\left(1 - \omega\left(\frac{x-\varphi}{\varepsilon}\right)\right)\psi(x)dx$$
$$= \varepsilon \int_{-\infty}^{0} \omega(\eta)\left(\psi(\varphi+\varepsilon\eta) - \psi(\varphi-\varepsilon\eta)\right)dx = O(\varepsilon^2).$$

Furthermore,

$$\left(\omega'\left(\frac{x-x^*}{\varepsilon}\right), \psi(x)\right) = \varepsilon \int_{-\infty}^{\infty} \omega'(\eta)\psi(x^* + \varepsilon\eta)d\eta = \varepsilon\psi(x^*) + O(\varepsilon^2).$$

To prove (28) let us write firstly:

$$\left(f\left(u_{\Delta,\varepsilon}(x,t)\right), \psi\right) = \left(f\left(\tilde{u}_{\Delta,\varepsilon}(x,t)\right), \psi\right) + R,$$

where

$$\tilde{u}_{\Delta,\varepsilon}(x,t) = u_{\Delta,\varepsilon}(x,t) - B(\tau)\omega'\left(\frac{x-x^*}{\varepsilon}\right),$$

and

$$|R| = \left|\int_{-\infty}^{\infty}\left(f\left(u_{\Delta,\varepsilon}(x,t)\right) - f\left(\tilde{u}_{\Delta,\varepsilon}(x,t)\right)\right) \cdot \psi(x)dx\right|$$

$$(34) \qquad\qquad\qquad \leq \varepsilon C \int_{-\infty}^{\infty} |B(\tau) \cdot \psi(x^* + \varepsilon\eta)| \, \omega'(\eta)d\eta \leq \varepsilon c.$$

Furthermore,

$$\left(f\left(\tilde{u}_{\Delta,\varepsilon}(x,t)\right), \psi\right) = -\int_{-\infty}^{\infty} f\left(\tilde{u}_{\Delta,\varepsilon}(x,t)\right) \cdot \frac{d\phi(x)}{dx}dx$$

$$(35) \qquad = f\left(u(\xi_\alpha)\right) \cdot \int_{-\infty}^{\infty} \psi(x)dx + \int_{-\infty}^{\infty} Df\left(\tilde{u}_{\Delta,\varepsilon}(x,t)\right)\frac{\partial \tilde{u}_{\Delta,\varepsilon}(x,t)}{\partial x} \cdot \phi(x)dx,$$

where $\phi(x) = \int_x^{\infty} \psi(x')dx'$.

Calculating the derivative $\partial \tilde{u}_{\Delta,\varepsilon}/\partial x$ and changing the variable $\eta = (x - \varphi_{\alpha,\beta})/\varepsilon$, we transform the right-hand side of the equality (35) to the following form:

$$(36) \qquad f\left(u(\xi_\alpha)\right) \cdot \int_{-\infty}^{\infty} \psi(x)dx +$$

$$+ \int_{-\infty}^{\infty} Df\Big(u(\xi_\alpha) + A_\alpha\omega(\eta) + A_\beta\omega(\eta - \sigma)\Big)\omega'(\eta) \cdot \phi(\phi_\alpha + \epsilon\eta)d\eta$$

$$+ \int_{-\infty}^{\infty} Df\Big(u(\xi_\alpha) + A_\beta\omega(\eta) + A_\alpha\omega(\eta + \sigma)\Big)\omega'(\eta) \cdot \phi(\phi_\beta + \epsilon\eta)d\eta.$$

Now applying the Taylor formula and taking into account the identity

$$\phi(\phi_{\alpha,\beta}) = \int_{\phi_{\alpha,\beta}}^{\infty} \psi(x)dx = \big(H(x - \phi_{\alpha,\beta}), \psi(x)\big),$$

we obtain the representation (28) for the flux-function.

To calculate the limiting values (30), (31) of the convolutions $\mathcal{H}_{\alpha,\beta}$ it is enough to use the stabilization properties of the function $\omega(\eta)$. Equality (32) is obvious. □

Substituting the expressions (27) and (28) into the left-hand side of (13), we derive the relation for obtaining the parameters of the asymptotic solution (21):

$$\frac{\rho'}{\epsilon}\Big\{A'_\alpha(\tau)H(x - \phi_\alpha) + A'_\beta(\tau)H(x - \phi_\beta)\Big\} + \rho'B'(\tau)\delta(x - x^*)$$

(37)
$$+ \Big\{ -\frac{d\phi_\alpha}{dt}A_\alpha(\tau) + \mathcal{H}_\alpha\big(u(\xi_\alpha), \sigma, \tau\big)A_\alpha(\tau)\Big\}\delta(x - \phi_\alpha)$$

$$+ \Big\{ -\frac{d\phi_\beta}{dt}A_\beta(\tau) + \mathcal{H}_\beta\big(u(\xi_\alpha), \sigma, \tau\big)A_\beta(\tau)\Big\}\delta(x - \phi_\beta) = O_{\mathcal{D}'}(\epsilon).$$

We now apply the following almost obvious statement [10]:

LEMMA 2. *Let $S = S(\tau)$ be a function from the Schwartz space and let a function $\phi_k = \phi_k(\tau, \epsilon) \in C^\infty$ have the representation*

(38)
$$\phi_k = x^* + \epsilon\chi_k,$$

where $x^* = \text{const}$ and $\chi_k = \chi_k(\tau, \epsilon)$ is a slowly increasing function. Then

(39)
$$S(\tau)H(x - \phi_k) = S(\tau)\big\{H(x - x^*) + \epsilon\chi_k\delta(x - x^*)\big\} + O_{\mathcal{D}'}(\epsilon^2).$$

According to (24), (25), $A'_{\alpha,\beta}(\tau)$ belong to the Schwartz space. Furthermore, one can rewrite the functions $\phi^0_{\alpha,\beta}(t)$ as follows:

$$\phi^0_\alpha(t) = \Lambda\big(u(\xi_\alpha), u(\xi)\big)(t - t^*) + x^*, \quad \phi^0_\beta(t) = \Lambda\big(u(\xi_\beta), u(\xi)\big)(t - t^*) + x^*.$$

Moreover, in view of the representations (19), (22)

(40)
$$\phi_{\alpha,\beta}(\tau, t) = x^* + \epsilon\chi_{\alpha,\beta},$$

where

(41)
$$\chi_{\alpha,\beta} = \tau\left(\frac{\Lambda\big(u(\xi_{\alpha,\beta}), u(\xi)\big)}{\Lambda\big(u(\xi_\beta), u(\xi)\big) - \Lambda\big(u(\xi_\alpha), u(\xi)\big)} + \phi^1_{\alpha,\beta}\right)$$

since

$$\tau = \left\{\Lambda\big(u(\xi_\beta),u(\xi)\big) - \Lambda\big(u(\xi_\alpha),u(\xi)\big)\right\}\frac{t-t^*}{\varepsilon}.$$

So that the relation (37) can be transformed to the following form:

$$\frac{\rho'}{\varepsilon}\left\{A'_\alpha(\tau)+A'_\beta(\tau)\right\}H(x-x^*)+\rho'\left\{\chi_\alpha A'_\alpha(\tau)+\chi_\beta A'_\beta(\tau)+B'(\tau)\right\}\delta(x-x^*)$$

$$+\left\{-\frac{d\varphi_\alpha}{dt}A_\alpha+\mathcal{H}_\alpha\big(u(\xi_\alpha),\sigma,\tau\big)A_\alpha(\tau)\right\}\delta(x-\varphi_\alpha)$$

$$(42)\qquad +\left\{-\frac{d\varphi_\beta}{dt}A_\beta(\tau)+\mathcal{H}_\beta\big(u(\xi_\alpha),\sigma,\tau\big)A_\beta(\tau)\right\}\delta(x-\varphi_\beta)=O_{\mathcal{D}'}(\varepsilon).$$

Setting equal to zero the terms of (42) of the value $O_{\mathcal{D}'}(\varepsilon^{-1})$ and taking into account the assumption (24) we obtain the identity

$$(43)\qquad\qquad A_\alpha(\tau)+A_\beta(\tau)=u(\xi_\beta)-u(\xi_\alpha).$$

Thus, setting equal to zero the coefficients of the δ-functions in the relation (42), we obtain the equations:

$$(44)\qquad\qquad \frac{d\varphi_{\alpha,\beta}}{dt}A_{\alpha,\beta}=\mathcal{H}_{\alpha,\beta}\big(u(\xi_\alpha),\sigma,\tau\big)A_{\alpha,\beta}(\tau),$$

$$(45)\qquad\qquad \frac{dB}{d\tau}=\sigma A'_\alpha(\tau).$$

Therefore, (44) imply that $A_{\alpha,\beta}(\tau)$ should be associated with the "eigenvector" of the matrix $\mathcal{H}_{\alpha,\beta}\big(u(\xi_\alpha),\sigma(\tau),\tau\big)$, whereas $d\varphi_{\alpha,\beta}/dt$ should be the corresponding eigenvalue, which we denote by $\Lambda_{\alpha,\beta}\big(u(\xi_\alpha),\sigma,\tau\big)$, that is

$$(46)\qquad\qquad \Lambda_{\alpha,\beta}\big(u(\xi_\alpha),\sigma,\tau\big)A_{\alpha,\beta}=\mathcal{H}_{\alpha,\beta}\big(u(\xi_\alpha),\sigma,\tau\big)A_{\alpha,\beta}(\tau).$$

These equations and (43) form the system to define $A_{\alpha,\beta}$. Thus we should suppose:

A'. Let the system (43), (46) be solvable with respect to the branch $A_{\alpha,\beta}$ with the property (24) for any $u(\xi_\alpha)$, $u(\xi_\beta)$, $0\le\xi_\alpha<\xi_\beta\le\xi_N$, for all $\xi\in(\xi_\alpha,\xi_\beta)$, and uniformly in $\sigma\ge 0$.

REMARK 1. The equations (44) describe uniformly in time the passage from the Rankine–Hugoniot conditions for the states $\big(u(\xi_\alpha),u(\xi)\big)$, $\big(u(\xi),u(\xi_\beta)\big)$ to some new states (in fact, to the state $\big(u(\xi_\alpha),u(\xi_\beta)\big)$). At the same time, in view of (24), (30),

$$\mathcal{H}_\alpha\big(u(\xi_\alpha),\sigma(\tau),\tau\big)\rightarrow\mathcal{H}\big(u(\xi_\alpha),u(\xi)\big),$$
$$(47)\qquad \mathcal{H}_\beta\big(u(\xi_\alpha),\sigma(\tau),\tau\big)\rightarrow\mathcal{H}\big(u(\xi),u(\xi_\beta)\big),$$
$$A_\alpha\rightarrow u(\xi)-u(\xi_\alpha),\quad A_\beta\rightarrow u(\xi_\beta)-u(\xi)\quad\text{as}\quad\tau\rightarrow\infty.$$

Since **A'** implies the validity of assumption **A**, one can treat **A'** as a sharpened version of **A**.

Furthermore, the system (43), (46) contains $3n - 2$ scalar equations for $2n$ scalar functions. So for $n = 2$ the assumption $\mathbf{A}'$ implies some natural conditions for the flux function, whereas for $n \geq 3$ (43), (46) is an overdetermined system. This is a little similar to the problem of the existence of Riemann invariants ($n(n-1)$ equations for n unknowns).

Under the assumption $\mathbf{A}'$ we obtain:

$$(48) \qquad \frac{d\varphi_\alpha}{dt} = \Lambda_\alpha\big(u(\xi_\alpha), \sigma, \tau\big), \quad \frac{d\varphi_\beta}{dt} = \Lambda_\beta\big(u(\xi_\alpha), \sigma, \tau\big).$$

Before the interaction ($\tau \to +\infty$) the first *a priori* assumptions (23) imply

$$(49) \qquad \sigma \to \tau \quad \text{as} \quad \tau \to +\infty.$$

This, the *a priori* assumptions (24), and the property (30) imply that

$$(50) \qquad \begin{aligned} &\mathcal{H}_\alpha\big(u(\xi_\alpha), \sigma(\tau), \tau\big) \to \mathcal{H}\big(u(\xi_\alpha), u(\xi)\big), \\ &\mathcal{H}_\beta\big(u(\xi_\alpha), \sigma(\tau), \tau\big) \to \mathcal{H}\big(u(\xi), u(\xi_\beta)\big) \quad \text{as} \quad \tau \to +\infty. \end{aligned}$$

Respectively,

$$(51) \qquad \begin{aligned} &\Lambda_\alpha\big(u(\xi_\alpha), \sigma(\tau), \tau\big) \to \Lambda\big(u(\xi_\alpha), u(\xi)\big), \\ &\Lambda_\beta\big(u(\xi_\alpha), \sigma(\tau), \tau\big) \to \Lambda\big(u(\xi), u(\xi_\beta)\big) \quad \text{as} \quad \tau \to +\infty. \end{aligned}$$

Therefore, the limiting relations (23) and (24) verify the concordance of the equations (48) with our definition (22) of $\varphi_{\alpha,\beta}^0(t)$.

To find the limiting behavior of $\varphi_{\alpha,\beta}(\tau, t)$ after the interaction ($\tau \to -\infty$) let us reduce the system (48) to a scalar equation. In view of (19), (33),

$$\frac{d(\varphi_\beta - \varphi_\alpha)}{dt} = \rho' \frac{d\sigma}{d\tau}.$$

Hence, by subtracting one equation in (48) from the other we obtain the equation

$$(52) \qquad \frac{d\sigma}{d\tau} = \frac{1}{\rho'} \Big\{ \Lambda_\beta\big(u(\xi_\alpha), \sigma, \tau\big) - \Lambda_\alpha\big(u(\xi_\alpha), \sigma, \tau\big) \Big\} \overset{\text{def}}{=} \mathcal{F}(\sigma, t),$$

which is completed by the condition (49).

LEMMA 3. *The value $\sigma = 0$ is the unique critical point for the problem (52), (49). Moreover, $\sigma \to 0$ with an exponential rate as $\tau \to -\infty$.*

Proof. First of all we note that

$$(53) \qquad \mathcal{F}|_{\sigma \to \infty} = 1, \quad \text{whereas} \quad \mathcal{F}|_{\sigma=0} = 0,$$

since

(54) $$\mathcal{H}_{\alpha,\beta}\big(u(\xi_\alpha),\sigma,\tau\big)\big|_{\sigma=0} = \mathcal{H}\big(u(\xi_\alpha),u(\xi_\beta)\big).$$

Let us prove that $\sigma = 0$ is the first zero-point of $\mathcal{F}$. We will use the identity

(55) $$\mathcal{H}_\alpha\big(u(\xi_\alpha),\sigma,\tau\big)A_\alpha(\tau) + \mathcal{H}_\beta\big(u(\xi_\alpha),\sigma,\tau\big)A_\beta(\tau) = f\big(u(\xi_\beta)\big) - f\big(u(\xi_\alpha)\big),$$

which follows directly from the definition (29) and the property (43). Thus,

(56) $$\Lambda_\alpha\big(u(\xi_\alpha),\sigma,\tau\big)A_\alpha(\tau) + \Lambda_\beta\big(u(\xi_\alpha),\sigma,\tau\big)A_\beta(\tau) = f\big(u(\xi_\beta)\big) - f\big(u(\xi_\alpha)\big).$$

Therefore, from (43) and (56) we derive the next identity

(57) $$\begin{aligned}\Big(\Lambda_\beta\big(u(\xi_\alpha),\sigma,\tau\big) &- \Lambda_\alpha\big(u(\xi_\alpha),\sigma,\tau\big)\Big)A_\beta(\tau) \\ &= \Big\{\Lambda\big(u(\xi_\alpha),u(\xi_\beta)\big) - \Lambda_\alpha\big(u(\xi_\alpha),\sigma,\tau\big)\Big\}\big(u(\xi_\beta) - u(\xi_\alpha)\big).\end{aligned}$$

So the supposition $\{\Lambda_\beta - \Lambda_\alpha\}|_{\sigma=\sigma_0>0} = 0$ implies that

(58) $$\Lambda\big(u(\xi_\alpha),u(\xi_\beta)\big) = \Lambda_\alpha\big(u(\xi_\alpha),\sigma_0,\tau\big),$$

which is impossible since

(59) $$\begin{aligned}u(\xi_\alpha) + A_\alpha\omega(\eta) &+ A_\beta\omega(\eta-\sigma) \\ &= u(\xi_\alpha) + \big(u(\xi_\beta) - u(\xi_\alpha) - \kappa(\sigma,\eta)A_\beta\big)\omega(\eta),\end{aligned}$$

where $0 < \kappa_1(\sigma) \le \kappa(\sigma,\eta) \le \kappa_2(\sigma) < 1$ for $\sigma > 0$. Next using (59) and the similar equality

(60) $$\begin{aligned}\int_{-\infty}^{\infty} &Df\Big(u(\xi_\alpha) + A_\beta\omega(\eta) + A_\alpha\omega(\eta+\sigma)\Big)\omega'(\eta)d\eta \\ &= \int_{-\infty}^{\infty} Df\Big(u(\xi_\beta) - \big(u(\xi_\beta) - u(\xi_\alpha) - \kappa(\sigma,\eta)A_\alpha\big)\omega(\eta)\Big)\omega'(\eta)d\eta,\end{aligned}$$

and applying the assumption $\mathbf{B}'$, we observe that

(61) $$\begin{aligned}\Lambda_\alpha\big(u(\xi_\alpha),\sigma,\tau\big) &\to \Lambda\big(u(\xi_\alpha),u(\xi_\beta)\big) + 0, \\ \Lambda_\beta\big(u(\xi_\alpha),\sigma,\tau\big) &\to \Lambda\big(u(\xi_\alpha),u(\xi_\beta)\big) - 0 \quad \text{as} \quad \sigma \to 0.\end{aligned}$$

Therefore, $d\mathcal{F}/d\sigma|_{\sigma=0} > 0$ and the limit value $\sigma = 0$ can be achieved only for $\tau \to -\infty$.

It remains to prove that $\sigma = 0$ is the stationary point for the equation (52). Let us note firstly that in accordance with (43)

$$\frac{\partial}{\partial\tau}\mathcal{H}_{\alpha,\beta}\big(u(\xi_\alpha),z,\tau\big)\big|_{z=0} = 0.$$

Thus, differentiating (46) with respect to τ, we obtain the relation

$$\left\{\frac{\partial}{\partial\tau}\Lambda_{\alpha,\beta}\big(u(\xi_\alpha),z,\tau\big)\right\}A_{\alpha,\beta}\Big|_{z=0}$$

$$=\left\{\mathcal{H}_{\alpha,\beta}\big(u(\xi_\alpha),z,\tau\big)-\Lambda_{\alpha,\beta}\big(u(\xi_\alpha),z,\tau\big)E\right\}\frac{\partial}{\partial\tau}A_{\alpha,\beta}(\tau)\Big|_{z=0}\to 0$$

as $\tau\to-\infty$. Therefore,

$$\frac{d\mathcal{F}}{d\tau}\Big|_{\sigma=0}=\frac{\partial\mathcal{F}}{\partial\tau}\Big|_{\sigma=0}\to 0\quad\text{as}\quad\tau\to-\infty.$$

Obviously, the derivative $d^k\mathcal{F}/d\tau^k|_{\sigma=0}$ for each k contains either the terms $d^i\sigma/d\tau^i|_{\sigma=0}$, $i<k$, or derivatives of $A_{\alpha,\beta}$. That is why $\sigma\to 0$ as $\tau\to-\infty$. $\qquad\square$

To complete the investigation of equations (48) we should prove the *a priori* assumptions (23). We write firstly:

$$\frac{d\varphi_{\alpha,\beta}}{dt}=\frac{d\varphi_{\alpha,\beta}^0}{dt}+\rho'\frac{d}{d\tau}(\tau\varphi_{\alpha,\beta}^1).$$

This and the equations (18), (48) imply that

$$\frac{d}{d\tau}(\tau\varphi_{\alpha,\beta}^1)=\frac{1}{\rho'}\left(\Lambda_{\alpha,\beta}\big(u(\xi_\alpha),\sigma,\tau\big)-\Lambda\big(u(\xi_{\alpha,\beta}),u(\xi)\big)\right).$$

Since $\sigma(t)$ is known now, we conclude that

$$\varphi_{\alpha,\beta}^1(\tau)=\frac{1}{\rho'\tau}\int_{-\infty}^{\tau}\left(\Lambda_{\alpha,\beta}\big(u(\xi_\alpha),\sigma,\tau\big)-\Lambda\big(u(\xi_{\alpha,\beta}),u(\xi)\big)\right)d\tau.$$

Accordingly (50), (61)

$$\varphi_{\alpha,\beta}^1\to 0\quad\text{as}\quad\tau\to\infty,$$

$$\varphi_{\alpha,\beta}^1\to\frac{1}{\rho'}\left(\Lambda\big(u(\xi_\alpha),u(\xi_\beta)\big)-\Lambda\big(u(\xi_{\alpha,\beta}),u(\xi)\big)\right)\quad\text{as}\quad\tau\to-\infty.$$

Therefore, for $\tau\to-\infty$

$$\varphi_{\alpha,\beta}\to\Lambda\big(u(\xi_{\alpha,\beta}),u(\xi)\big)(t-t^*)$$

$$+\left(\Lambda\big(u(\xi_\alpha),u(\xi_\beta)\big)-\Lambda\big(u(\xi_{\alpha,\beta}),u(\xi)\big)\right)(t-t^*)=\Lambda\big(u(\xi_\alpha),u(\xi_\beta)\big)(t-t^*).$$

We now note that according to the definition (29) the stabilization rates of the matrix $\mathcal{H}_{\alpha,\beta}$ and of the function ω coincide. The same is true for $A_{\alpha,\beta}$ and $\Lambda_{\alpha,\beta}$. Next coming back to the equation (45) and completing it by the condition $B|_{\tau\to\infty}\to 0$, we verify the assumption (26) and obtain the function B with the exponential rate of the stabilization.

Consequently, we conclude that the neighboring shock waves merge and this process implies the shock wave with the amplitude $u_- - u_+$ formation. This completes the proof of our main result:

THEOREM 1. *Let the assumptions* **A′** *and* **B′** *be satisfied. Then there exists a weak asymptotic* mod $O_{\mathcal{D}'}(\varepsilon)$ *solution of the problem (11), (12) with property (14).*

4. Examples

EXAMPLE 1. We consider firstly the system of isentropic elasticity

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0,$$

(62)

$$\frac{\partial v}{\partial t} - \frac{\partial g(u)}{\partial x} = 0,$$

which is strictly hyperbolic under the condition $g'(u) > 0$.

The non-zero elements of the matrix $\mathcal{H}(\bar{u}_-, \bar{u})$ (see (5), $\bar{u} = (u, v)$) are equal to -1 and

$$h(\bar{u}_-, \bar{u}) = -\int_0^1 g'(u_- + (u - u_-)\omega)d\omega = \frac{g(u) - g(u_-)}{u - u_-}.$$

Thus, the eigenvalues $\Lambda(\bar{u}_-, \bar{u})$ of the matrix $\mathcal{H}(\bar{u}_-, \bar{u})$ have the form

(63)
$$\Lambda^{\pm}(\bar{u}_-, \bar{u}) = \pm\sqrt{\frac{g(u) - g(u_-)}{u - u_-}}.$$

Therefore, the Hugoniot locus consists of two branches:

(64)
$$v - v_- = -\Lambda^{\pm}(\bar{u}_-, \bar{u})(u - u_-).$$

Obviously, to satisfy the assumption **A** it is enough to define v according to (64) for all states $u_{\pm}$, v_-.

Next the Liu E-condition (9) implies formally the inequalities, which are known as the Wendroff E-condition:

$$\frac{g(u_{i+1}) - g(u)}{u_{i+1} - u} \geq \frac{g(u_{i+1}) - g(u_i)}{u_{i+1} - u_i} \quad \text{or} \quad \frac{g(u_{i+1}) - g(u)}{u_{i+1} - u} \leq \frac{g(u_{i+1}) - g(u_i)}{u_{i+1} - u_i},$$

where the first inequality applies for 1-shocks ($\Lambda = \Lambda^-$), the second inequality applies for 2-shocks ($\Lambda = \Lambda^+$), and u is situated between u_i and u_{i+1}. However, the complete form of the assumption **B** eliminates the possibility of the existence of inflection points. So, $g(u)$ can be either a convex or a concave function. Obviously, the assumptions **B** and **B′** are equivalent in this case.

Furthermore, it is clear that the unique varying coefficient of the matrix $\mathcal{H}_{\alpha,\beta}(\bar{u}(\xi_\alpha), \sigma, \tau)$ has the form

$$h_{\alpha,\beta}(u(\xi_\alpha), \sigma, \tau) = \int_{-\infty}^{\infty} g'(u_{\alpha,\beta})\omega'(\eta)d\eta,$$

where the notation

$$u_\alpha = u(\xi_\alpha) + a_\alpha \omega(\eta) + a_\beta \omega(\eta - \sigma), \quad u_\beta = u(\xi_\alpha) + a_\beta \omega(\eta) + a_\alpha \omega(\eta + \sigma)$$

has been used. Obviously,

$$\Lambda_{\alpha,\beta}^{\pm}(\bar{u}(\xi_\alpha), \sigma, \tau) = \pm \sqrt{\int_{-\infty}^{\infty} g'(u_{\alpha,\beta}) \omega'(\eta) d\eta},$$

and

(65) $$A_{\alpha,\beta} = (a_{\alpha,\beta}, b_{\alpha,\beta}), \quad b_{\alpha,\beta} = -\Lambda_{\alpha,\beta}^{\pm}(\bar{u}(\xi_\alpha), \sigma, \tau) a_{\alpha,\beta}.$$

Therefore, the assumption $\mathbf{A}'$ can be transformed to the solvability condition for the equations:

(66) $$a_\alpha + a_\beta = u_\beta - u_\alpha,$$

$$\sqrt{g'(u_\alpha)} a_\alpha + \sqrt{g'(u_\beta)} a_\beta = \mp(v_\beta - v_\alpha).$$

EXAMPLE 2. A slightly more complicated example arises from the system

(67)
$$\frac{\partial u}{\partial t} - \frac{\partial g_1(v)}{\partial x} = 0,$$
$$\frac{\partial v}{\partial t} - \frac{\partial g_2(u)}{\partial x} = 0,$$

under the condition $g_1'(v) g_2'(u) > 0$.

Eigenvalues $\Lambda(\bar{u}_-, \bar{u})$ of the matrix $\mathcal{H}(\bar{u}_-, \bar{u})$ have the form now

(68) $$\Lambda^{\pm}(\bar{u}_-, \bar{u}) = \pm \sqrt{\frac{(g_1(v) - g_1(v_-))(g_2(u) - g_2(u_-))}{(v - v_-)(u - u_-)}}.$$

Therefore, the Hugoniot locus consists of two branches again

$$v - v_- = -\Lambda_{1,2}(\bar{u}_-, \bar{u})(u - u_-).$$

However, the assumption $\mathbf{A}$ is not trivial now but requires the solvability of the relation

$$\sqrt{\frac{(v - v_-)}{(u - u_-)} \frac{(g_2(u) - g_2(u_-))}{(g_1(v) - g_1(v_-))}} = \pm \frac{v - v_-}{u - u_-}$$

between u and v.

Furthermore, the Liu E-condition $\mathbf{B}$ takes the form now

$$\pm \sqrt{\frac{(g_1(v) - g_1(v_{i+1}))(g_2(u) - g_2(u_{i+1}))}{(v - v_{i+1})(u - u_{i+1})}}$$

$$\leq \pm\sqrt{\frac{\big(g_1(v_i) - g_1(v_{i+1})\big)\big(g_2(u_i) - g_2(u_{i+1})\big)}{\big(v_i - v_{i+1}\big)\big(u_i - u_{i+1}\big)}},$$

where v and u are situated between v_i, v_{i+1} and u_i, u_{i+1} respectively. Obviously, the condition **B** does not imply the automatic validity of **B**′.

Furthermore, we find the eigenvalues of the matrices $\mathcal{H}_{\alpha,\beta}\big(\bar{u}(\xi_\alpha),\sigma,\tau\big)$:

$$\Lambda_{\alpha,\beta}\big(\bar{u}(\xi_\alpha),\sigma,\tau\big) = \pm\sqrt{\int_{-\infty}^{\infty} g_1'(v_{\alpha,\beta}) g_2'(u_{\alpha,\beta}) \omega'(\eta) d\eta},$$

where

$$u_\alpha = u(\xi_\alpha) + a_\alpha\omega(\eta) + a_\beta\omega(\eta - \sigma), \quad u_\beta = u(\xi_\alpha) + a_\beta\omega(\eta) + a_\alpha\omega(\eta + \sigma),$$
$$v_\alpha = v(\xi_\alpha) + b_\alpha\omega(\eta) + b_\beta\omega(\eta - \sigma), \quad v_\beta = v(\xi_\alpha) + b_\beta\omega(\eta) + b_\alpha\omega(\eta + \sigma),$$

and we denote $A_{\alpha,\beta} = (a_{\alpha,\beta}, b_{\alpha,\beta})$. Therefore, to verify the assumption **A**′ we should investigate the complete system (43), (46) for A_α, A_β.

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PROPAGATION OF DELTA-SHOCKS IN
ONE SYSTEM OF CONSERVATION LAWS

Abstract. We study the propagation of δ-shock wave in a new type of system of conservation laws. The particular cases of this system are the system of nonlinear chromatography and the system for isotachophoresis.

1. Introduction

Even in the case of smooth initial data, in general, there exist no smooth and global in time solutions of this system. This fact leads to the necessity of introducing a notion of L^∞-*generalized solution* (weak solution) of the Cauchy problem in the sense of the integral identities. Moreover, there are "nonclassical" situations where, in contrast to Lax's and Glimm's classical results, the Cauchy problem for a system of conservation laws *either does not possess a weak L^∞-solution or possesses it for some particular initial data*. In order to solve the Cauchy problem in these "nonclassical" situations, it is necessary to seek solutions in the form of δ-*shocks*. Roughly speaking, a δ-*shock* is a solution whose components *contain Dirac delta functions*. Problems related to δ-shocks have been intensively studied recently (see [4, 5, 9], [12] and the references therein).

In numerous papers, δ-shocks were studied for the zero-pressure gas dynamics (see the above references). This system was used to describe the *formation of large-scale structures of the universe*; for modeling the *formation and evolution of traffic jams*; for modeling media which can be considered as *having no pressure* (for example, dusty gases, two-phase flows with solid particles or droplets). δ-Shocks arise in the model of non-classical shallow water flows [6], in the model of granular gases [7], in the system of *nonlinear chromatography* [11].

In [14], a new type of systems of conservation laws (admitting δ-shocks)

$$(1) \qquad (u_j)_t + \big(u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n)\big)_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0,$$

was studied, where $f_j(\cdot)$ is a smooth function, μ_j is a constant, $j = 1, 2, \ldots, n$. This class includes some *Temple type system* [15, 1]. In particular, the system of *nonlinear chromatography*: $f_j(v) = 1 + \frac{a_j}{1+v}$, $\mu_j = \pm 1$, where a_j is Henry's constant, $j = 1, 2, \ldots, n$ (see [11]); the system for *isotachophoresis* [2, (1.1.2),(1.1.3)]: $(\rho_j)_t + I\big(\frac{\mu_j \rho_j}{\sum_{s=0}^{n} \mu_s \rho_s}\big)_x = 0$, $j = 0, 1, 2, \ldots, n$, $\sum_{s=0}^{n} \rho_s = 0$, where ρ_j is the charge density of anions of the jth type ($j = 1, 2, \ldots, n$), ρ_0 is the corresponding value for cations, μ_j are the electrophoretic mobilities of the corresponding ions ($j = 0, 1, 2, \ldots, n,$); $\mu_0 < 0 < \mu_1 < \cdots < \mu_n$; $I =$ constant is the current.

In [14], we introduced integral identities, which give the definition of δ-shocks for system (1), and derived the corresponding Rankine–Hugoniot conditions. It was proved that "area" transport processes between the moving singular one-dimensional δ-shock wave front and the region outside the front are going on. The balance relations describing these processes were derived.

In this paper we describe the process of propagation of δ-shock wave in system (1), i.e., we construct a δ-shock wave type solution of the Cauchy problem (1), (12). First, by Theorem 2 *a weak asymptotic solution* of the problem is constructed. Next, in Theorem 3, we construct a δ-shock wave type solution of this problem as the weak limit (17) of the *weak asymptotic solution*.

2. δ-Shocks and the Rankine–Hugoniot conditions

Suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a set of curves lying in the first quadrant $\{(x,t) : x \geq 0, \ t \in [0,\infty)\}$ of $\mathbb{R}^2$ containing smooth arcs $\gamma_i = \{(x,t) : S_i(x,t) = 0\}$ of class C^1, $(S_i)_x \neq 0$, $i \in I$, and I is a finite set. Let I_0 be a subset of I such that the arcs γ_k for $k \in I_0$ start from points of the x-axis and let $\Gamma_0 = \{x_k^0 : k \in I_0\}$ be the set of initial points of the arcs γ_k, $k \in I_0$. It is clear that $-G = \frac{S_t}{|S_x|}$ is the *velocity of the moving point* $\Gamma_t = \{x \in \mathbb{R} : S(x,t) = 0\}$ in the direction $\nu = \frac{S_x}{|S_x|}$.

For system (1), we will use the δ-*shock type initial data*

(2) $$u^0 = (u_1^0, \ldots, u_n^0), \quad \text{where} \quad u_j^0(x) = \widehat{u}_j^0(x) + e_j^0 \delta(\Gamma_0),$$

and $\widehat{u}_j^0 \in L^\infty(\mathbb{R}; \mathbb{R})$. Also, $e_j^0 \delta(\Gamma_0) \stackrel{\text{def}}{=} \sum\limits_{k \in I_0} e_{j;k}^0 \delta(x - x_k^0)$, $e_{j;k}^0$ is a constant with $k \in I_0$, and $j = 1, 2, \ldots, n$.

DEFINITION 1 ([14]). A distribution $u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$ and a set of curves Γ, where

(3) $$u_j(x,t) = \widehat{u}_j(x,t) + e_j(x,t)\delta(\Gamma),$$

$\widehat{u}_j \in L^\infty(\mathbb{R} \times (0,\infty); \mathbb{R})$, $e_j(x,t)\delta(\Gamma) \stackrel{\text{def}}{=} \sum\limits_{i \in I} e_{j;i}(x,t)\delta(\gamma_i)$, $e_{j;i} \in C(\gamma_i)$ with $i \in I$, and $j = 1, 2, \ldots, n$, such that

(4) $$\sum_{j=1}^{n} \mu_j e_j(x,t) = 0,$$

is called a δ-*shock wave type solution* of the Cauchy problem (1), (2) if

(5)
$$\int_0^\infty \int \widehat{u}_j \left(\varphi_t + f_j(\mu_1 \widehat{u}_1 + \cdots + \mu_n \widehat{u}_n)\varphi_x \right) dx\,dt + \int \widehat{u}_j^0(x)\varphi(x,0)\,dx$$
$$+ \sum_{i \in I} \int_{\gamma_i} e_{j;i}(x,t) \frac{\delta\varphi(x,t)}{\delta t} \frac{dl}{\sqrt{1+G^2}} + \sum_{k \in I_0} e_{j;k}^0 \varphi(x_k^0, 0) = 0, \quad j = 1, 2, \ldots, n,$$

hold for all $\varphi \in \mathcal{D}(\mathbb{R} \times [0,\infty))$, where $\int_{\gamma_i} \cdot \, dl$ is the line integral over γ_i, and

$$(6) \qquad \left.\frac{\delta\varphi}{\delta t}\right|_{\gamma_i} = \left.\left(\frac{\partial\varphi}{\partial t} - \frac{(S_i)_t}{(S_i)_x}\frac{\partial\varphi}{\partial x}\right)\right|_{S_i(x,t)=0}$$

is the δ-*derivative* with respect to time [8, 5.2.(15)].

In the paper [14], the motivation of the above definition was given. One of the reasons is the following: in order to define the term $f_j(\mu_1 u_1 + \cdots + \mu_n u_n)$ in the sense of Schwartzian distributions, we need to assume that relation (4) holds. Another reason is given in the papers [4, 5, 13].

Here the δ-derivative (6) coincides with the Lagrangian derivative $\frac{D\varphi}{Dt}$ equal to $\left(\frac{\partial\varphi}{\partial t} + u_\delta\frac{\partial\varphi}{\partial x}\right)\big|_{\gamma_i}$, where

$$(7) \qquad \left. u_\delta(x,t)\right|_{\gamma_i} = Gv = -\left.\frac{(S_i)_t}{(S_i)_x}\right|_{\gamma_i},$$

is the velocity of a δ-shock on γ_i, $i \in I$; the delta function $\delta(\gamma_i)$ on the curve γ_i is defined in [8, 5.3.] as

$$(8) \qquad \langle \delta(S), \varphi(x,t)\rangle = \int_{\gamma_i} \varphi(x,t)\frac{dl}{\sqrt{1+G^2}}, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R} \times [0,\infty)).$$

If $\Gamma = \{\gamma_i : i \in I\}$, where $\gamma_i = \{(x,t) : x = \phi_i(t)\}$, $\phi_i(t) \in C^1$, $i \in I$, and $(\dot{\cdot}) = \frac{d}{dt}(\cdot)$, then

$$(9) \qquad \left.\frac{\delta\varphi}{\delta t}\right|_{\gamma_i} = \varphi_t(\phi_i(t),t) + \dot{\phi}_i(t)\varphi_x(\phi_i(t),t) = \frac{d\varphi(\phi_i(t),t)}{dt}.$$

THEOREM 1 (([14])). *Let us assume that* $\Omega \subset \mathbb{R}_+ \times (0,\infty)$ *is a region cut by a smooth curve* $\Gamma = \{(x,t) : S(x,t) = 0\}$ *into the left- and right-hand parts* $\Omega_{\mp}$. *Let* $u = (u_1,\ldots,u_n)$, *and let* Γ *be a* δ-*shock wave type solution of system* (1) *such that* $u_j(x,t) = \widehat{u}_j(x,t) + e_j(x,t)\delta(\Gamma)$ *are smooth in* $\Omega_{\pm}$ *and have one-sided limits* $\widehat{u}_{j\pm}$ *on* Γ, $j = 1,\ldots,n$. *Then the* Rankine–Hugoniot conditions *for the* δ-shock

$$(10) \qquad \begin{aligned} \frac{\delta e_j(x,t)}{\delta t} &= \left.\left([u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n)]_\Gamma - [u_j]_\Gamma u_\delta\right)\frac{S_x}{|S_x|}\right|_\Gamma, \\[2mm] u_\delta(x,t) &= \left.\frac{\sum_{j=1}^n \mu_j[u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n)]}{\sum_{j=1}^n \mu_j[u_j]}\right|_\Gamma, \quad j = 1,\ldots,n, \end{aligned}$$

hold along Γ, *where* $u_\delta(x,t)$ *is the velocity* (7) *of a* δ-*shock*, $[g(u)] \stackrel{\text{def}}{=} g(u_-) - g(u_+)$ *is the jump in a function* $g(u)$ *across the discontinuity curve* Γ.

If $\Gamma = \{(x,t) : x = \phi(t)\}$, $\phi(t) \in C^1(0,+\infty)$, then in view of (9), relations (10) take the form

$$(11) \qquad \begin{aligned} \dot{e}_j(t) &= \left.\left([u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n)] - [u_j]\dot{\phi}(t)\right)\right|_{x=\phi(t)}, \\[2mm] \dot{\phi}(t) &= \left.\frac{\sum_{j=1}^n \mu_j[u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n)]}{\sum_{j=1}^n \mu_j[u_j]}\right|_{x=\phi(t)}, \quad j = 1,\ldots,n. \end{aligned}$$

The right-hand side of relation (10) (or (11)) is called the *Rankine–Hugoniot deficit* in u_j, $j = 1, \ldots, n$.

In order to describe the propagation of a singular wave front (point) starting from the initial position $x_0 = 0$, we need to solve the Cauchy problem for system (1) with the initial data

(12) $$u^0 = \left(u_1^0, \ldots, u_n^0 \right), \quad u_j^0(x) = u_{j+}^0(x) + [u_j^0(x)]H(-x) + e_j^0 \delta(-x)$$

where $H(x)$ is the Heaviside function, $[u_j^0] = u_{j-}^0 - u_{j+}^0$ is the jump of the function $u_j^0(x)$ across the discontinuity point $x_0 = 0$; $u_j^0(x) = u_{j-}^0(x)$ if $x < 0$, and $u_j^0(x) = u_{j+}^0(x)$ if $x > 0$; $u_{j\pm}^0(x)$ are given smooth functions; e_j^0 are given constants, $j = 1, 2, \ldots, n$.

We will seek a δ-shock wave type solution of the Cauchy problem (1), (12) in the form of a vector-distribution $u = (u_1, \ldots, u_n)$, where

(13) $$u_j(x,t) = u_{j+}(x,t) + [u_j(x,t)]H(-x + \phi(t)) + e_j(x,t)\delta(-x + \phi(t)),$$

and the vector-function $u_\pm$ and functions e_j, $\phi(t)$ are to be found, $[u_j] = u_{j-} - u_{j+}$, $j = 1, 2, \ldots, n$.

We use the *overcompression condition*

(14) $$\lambda_j(u_+) \leq \dot\phi(t) \leq \lambda_j(u_-), \quad j = 1, 2, \ldots, n,$$

as the admissibility condition for δ-shock waves. Here $\lambda_j(u)$, $j = 1, \ldots, n$ are eigenvalues of the characteristic matrix of system (1), $\dot\phi(t)$ is the velocity of propagation of the δ-shock, $u_{j\mp}$ are the respective left- and right-hand values of u_j on the discontinuity curve $x = \phi(t)$. It means that all characteristics, which are outgoing on the left- and right-hand sides of the discontinuity, meet on the discontinuity curve.

3. Weak asymptotic solutions

To deal with strongly singular solutions to systems of conservation laws in [3], [4], [5], the *weak asymptotics method* was developed (see also [13, 5]). It is based on the construction of a *weak asymptotic solution* to the problem, which is used to construct a strongly singular solution to the problem.

Let $\alpha \in \mathbb{R}$. Denote by $O_{\mathcal{D}'}(\varepsilon^\alpha)$, $\varepsilon \to +0$ a collection of distributions $f_\varepsilon(\cdot, t) \in \mathcal{D}'(\mathbb{R}^n)$, $t \in [0, T]$, $\varepsilon > 0$ such that $\langle f_\varepsilon(\cdot, t), \psi(\cdot) \rangle = O(\varepsilon^\alpha)$, $\varepsilon \to +0$, for any $\psi(x) \in \mathcal{D}(\mathbb{R})$, where the function $\langle f_\varepsilon(\cdot, t), \psi(\cdot) \rangle$ is continuous in t, and the estimate $O(\varepsilon^\alpha)$ is *uniform* with respect to t in $[0, T]$. The notation $o_{\mathcal{D}'}(\varepsilon^\alpha)$, $\varepsilon \to +0$ is understood correspondingly.

DEFINITION 2. Let a vector-function $u_\varepsilon(x,t) = (u_{1\varepsilon}(x,t), \ldots, u_{n\varepsilon}(x,t))$ which is smooth as $\varepsilon > 0$, $t \in [0, T]$ be such that $u_{j\varepsilon}(x,t) = \widehat{u}_{j\varepsilon}(x,t) + \Delta_{j\varepsilon}(x,t)$, where the weak limit $\lim_{\varepsilon \to 0} \widehat{u}_{j\varepsilon} \in L^\infty\left(\mathbb{R} \times (0,T); \mathbb{R}\right)$ and the weak limit $\lim_{\varepsilon \to 0} \Delta_{j\varepsilon}$ can include

delta-functions. The vector-function $u_\varepsilon(x,t)$ is called a *weak asymptotic solution* of the Cauchy problem $(1),(2)$ if we have

$$(15) \qquad \lim_{\varepsilon \to 0} \sum_{j=1}^{n} \mu_j u_{j\varepsilon} \in L^\infty\big(\mathbb{R} \times (0,T); \mathbb{R}\big),$$

and

$$(16) \qquad \begin{aligned} \big(u_{j\varepsilon}\big)_t + \big(u_{j\varepsilon} f_j(\mu_1 \widehat{u}_{1\varepsilon} + \cdots + \mu_n \widehat{u}_{n\varepsilon})\big)_x &= o_{\mathcal{D}'}(1), \quad j = 1,2,\ldots,n, \\ u_\varepsilon(x,0) &= u^0(x) + o_{\mathcal{D}'}(1), \ \varepsilon \to +0, \end{aligned}$$

hold, where the first estimate is uniform in $t \in [0,T]$.

In Definition 2 the condition $\lim_{\varepsilon \to 0} \sum_{j=1}^{n} \mu_j u_{j\varepsilon} \in L^\infty\big(\mathbb{R} \times (0,T); \mathbb{R}\big)$ is in compliance with condition (4). In view of this condition, we have $\mu_1 u_1 + \cdots + \mu_n u_n = \mu_1 \widehat{u}_1 + \cdots + \mu_n \widehat{u}_n$. And the regularization of the latter distribution coincides with the function $\mu_1 \widehat{u}_{1\varepsilon} + \cdots + \mu_n \widehat{u}_{n\varepsilon}$. This definition admits passing to the limit in the weak sense as $\varepsilon \to +0$. A *viscosity solution* can be considered as a particular case of a *weak asymptotic solution* to the problem, and the term $o_{\mathcal{D}'}(1)$ on the right-hand side of the first relation in (16) can be interpreted as small viscosity.

Within the framework of the *weak asymptotics method*, we find a δ-shock wave type solution of the Cauchy problem $(1),(2)$ as the weak limit

$$(17) \qquad u(x,t) = \lim_{\varepsilon \to +0} u_\varepsilon(x,t),$$

where u_ε is a *weak asymptotic solution* to the problem (16).

In compliance with [4, 5, 13], a *weak asymptotic solution* of the Cauchy problem $(1),(12)$ is constructed in the form of a smooth ansatz

$$u_{j\varepsilon}(x,t) = \widetilde{u}_{j\varepsilon}(x,t) + R_j(x,t,\varepsilon), \quad \varepsilon > 0,$$

where the function $\widetilde{u}_{j\varepsilon}$ is a regularization of the distribution u_j (see (13)) with respect to singularities $H(-x+\phi(t))$, $\delta(-x+\phi(t))$; and the *corrections* $R_j(x,t,\varepsilon)$ are the desired functions which are assumed to admit the estimates:

$$(18) \qquad R_j(x,t,\varepsilon) = o_{\mathcal{D}'}(1), \quad \frac{\partial R_j(x,t,\varepsilon)}{\partial t} = o_{\mathcal{D}'}(1), \quad \varepsilon \to +0, \quad j = 1,2,\ldots,n.$$

Thus, according to our technique, we seek a weak asymptotic solution of the Cauchy problems $(1),(12)$ in the form

$$(19) \qquad \begin{aligned} u_{j\varepsilon}(x,t) &= u_{j+}(x,t) + [u_j(x,t)]H_j(-x+\phi(t),\varepsilon) \\ &\quad + e_j(t)\delta_j(-x+\phi(t),\varepsilon) + R_j(x,t,\varepsilon), \quad j = 1,2,\ldots,n, \end{aligned}$$

where

$$(20) \qquad \delta_j(x,\varepsilon) = \frac{1}{\varepsilon}\omega_{ej}\Big(\frac{x}{\varepsilon}\Big), \quad H_j(x,\varepsilon) = \omega_{0j}\Big(\frac{x}{\varepsilon}\Big) = \int_{-\infty}^{x/\varepsilon} \omega_j(\eta)\,d\eta$$

are regularizations of the δ-function and the Heaviside function $H(\xi)$, respectively; ω_{ej}, ω_j are mollifiers with the standard properties; $j = 1,2,\ldots,n$.

4. Propagation of δ-shock wave

4.1. Construction of a weak asymptotic solution.

In order to construct a *weak asymptotic solution* of the Cauchy problem (1), (12) in the form (19), we choose corrections in the form

$$(21) \qquad R_j(x,t,\varepsilon) = R_j(t)\frac{1}{\varepsilon}\Omega_j''\Big(\frac{-x+\phi(t)}{\varepsilon}\Big),$$

where $R_j(t)$ is a continuous function, $\varepsilon^{-3}\Omega_j''(x/\varepsilon)$ is a regularization of the distribution $\delta''(x)$, $\Omega_j(\eta)$ is a mollifier, $j = 1, 2, \ldots, n$. It is clear that estimates (18) hold. Reformulating the following lemma [4, Lemma 2.2], [5, Lemma 14], we obtain:

LEMMA 1. *If $f_j(u)$ is a smooth function, $u_{j\varepsilon}(x,t)$ is given by* (19), (20), (21), $v_\varepsilon(x,t) = \sum_{j=1}^n \mu_j \widehat{u}_{j\varepsilon}(x,t) = \sum_{j=1}^n \mu_j\big(u_{j+}(x,t) + [u_j(x,t)]H_j(-x+\phi(t),\varepsilon)\big)$, *then*

$$u_{j\varepsilon}(x,t)f_j\big(v_\varepsilon(x,t)\big) = u_{j+}(x,t)f_j\big(v_+(x,t)\big) + \big[u_j(x,t)f_j\big(v(x,t)\big)\big]H(-x+\phi(t))$$

$$(22) \qquad + \Big\{e_j(t)a_j(t) + R_j(t)c_j(t)\Big\}\delta(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0,$$

where $v(x,t) = \sum_{j=1}^n \mu_j\widehat{u}_j(x,t) = \sum_{j=1}^n \mu_j\big(u_{j+}(x,t) + [u_j(x,t)]H(-x+\phi(t))\big)$,

$$a_j(t) = \int f_j\Big(\sum_{j=1}^n \mu_j\big(u_{j-}(x,t)\omega_{0j}(\eta) + u_{j+}(x,t)(1-\omega_{0j}(\eta)))\big)\Big)\Big|_{x=\phi(t)} \omega_{ej}(\eta)\,d\eta,$$

$$c_j(t) = \int f_j\Big(\sum_{j=1}^n \mu_j\big(u_{j-}(x,t)\omega_{0j}(\eta) + u_{j+}(x,t)(1-\omega_{0j}(\eta)))\big)\Big)\Big|_{x=\phi(t)} \Omega_j''(\eta)\,d\eta.$$

THEOREM 2. *Let* (14) *hold for $t = 0$. Then there exist $T > 0$ and a zero neighborhood $K \subset \mathbb{R}$ such that for $(x,t) \in K \times [0,T)$, the Cauchy problem* (1), (12) *has a weak asymptotic solution* (19), (21) *if and only if*

$$(23) \qquad \begin{aligned} (u_{j\pm})_t &+ \big(u_{j\pm}f_j\big(\mu_1 u_{1\pm} + \cdots + \mu_n u_{n\pm}\big)\big)_x = 0, \quad \pm x > \pm\phi(t), \\ \dot{\phi}(t) &= \frac{\sum_{j=1}^n \mu_j[u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n)]}{\sum_{j=1}^n \mu_j[u_j]}\Big|_{x=\phi(t)}, \\ \dot{e}_j(t) &= \big([u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n)] - [u_j]\dot{\phi}(t)\big)\big|_{x=\phi(t)}, \end{aligned}$$

$$(24) \qquad R_j(t) = \frac{e_j(t)}{c_j(t)}\big(\dot{\phi}(t) - a_j(t)\big), \quad j = 1, \ldots, n,$$

where $a_j(t)$, $c_j(t)$ are defined in Lemma 1. *The initial data for system* (23), (24) *are defined from* (12), *and $\phi(0) = 0$, $R_j(0) = \frac{e_j^0}{c_j(0)}\big(\dot{\phi}(0) - a_j(0)\big)$.*

Proof. It is clear that (19), (21) implies that in the weak sense we have

$$\lim_{\varepsilon \to 0} \sum_{j=1}^{n} \mu_j u_{j\varepsilon} = \sum_{j=1}^{n} \mu_j \big(u_{j+}(x,t) + [u_j(x,t)]H(-x+\phi(t)) \big) + \delta(-x+\phi(t)) \sum_{j=1}^{n} \mu_j e_j(x,t),$$

i.e., the first condition in (15) is satisfied if and only if (4) holds.

Substituting ansatz (19) and weak asymptotic (22) into system (1), and taking into account the estimates (18), we obtain with accuracy up to $O_{\mathcal{D}'}(\varepsilon)$ the following relations

$$\big(u_{j\varepsilon} \big)_t + \big(u_{j\varepsilon} f_j(\mu_1 \widehat{u}_{1\varepsilon} + \cdots + \mu_n \widehat{u}_{n\varepsilon}) \big)_x = \big(u_{j+} \big)_t + \big(u_{j+} f_j(\mu_1 u_{1+} + \cdots + \mu_n u_{n+}) \big)_x$$

$$+ \Big\{ \big([u_j]\big)_t + \big([u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n)]\big)_x \Big\} H(-x+\phi(t))$$

$$+ \Big\{ [u_j]\dot{\phi}(t) + \dot{e}_j(t) - [u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n)] \Big\} \delta(-x+\phi(t))$$

$$(25) \qquad + \Big\{ e_j(t)\dot{\phi}(t) - e_j(t)a_j(t) - c_j(t)R_j(t) \Big\} \delta'(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon),$$

where $a_j(t)$, $c_j(t)$ are defined in Lemma 1. Setting the right-hand side of (25) equal to zero, we obtain the necessary and sufficient conditions for the first equality in (16): first, third equations in (23), and (24). Here we choose the mollifiers $\omega_{0j}(\xi)$, $\Omega_j(\xi)$ such that $c_j(t) \neq 0$. Since (4) is satisfied, the second equation in (23) implies that the second equation in (23) holds.

Let us prove that system (23), (24) has a solution. Consider the Cauchy problem $(\widehat{u}_j)_t + \big(\widehat{u}_j f_j(\mu_1 \widehat{u}_1 + \cdots + \mu_n \widehat{u}_n) \big)_x = 0$, $j = 1, 2, \ldots, n$, $\widehat{u}(x,0) = \widehat{u}^0(x)$, where $\widehat{u}^0(x) = u_+^0(x)$ for $x > 0$ and $\widehat{u}^0(x) = u_-^0(x)$ for $x < 0$. Following the scheme from [10, Ch.4.2.], we extend the vector-function $u_+^0(x)$ $(u_-^0(x) = u_+^0(x) + [u^0(x)])$ to the set $x \leq 0$ $(x \geq 0)$ in a bounded C^1 fashion and continue to denote the extended vector-functions by $u_\pm^0(x)$. Let $u_\pm(x,t) = (u_{1\pm}, \ldots, u_{n\pm})$ be C^1 solutions of the Cauchy problem

$$(u_j)_t + \big(u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n) \big)_x = 0, \ j = 1, 2, \ldots, n, \quad u_\pm(x,0) = u_\pm^0(x),$$

which exist for small enough time interval $[0, T_1]$ and are determined by integration along the characteristics. The vector-functions $u_\pm(x,t)$ determine a two-sheeted covering of the plane (x,t). Next, we define the discontinuity curve $x = \phi(t)$ as a solution of the problem

$$\dot{\phi}(t) = \frac{\sum_{j=1}^{n} \mu_j \big[u_j(x,t) f_j\big(\mu_1 u_1(x,t) + \cdots + \mu_n u_n(x,t)\big) \big]}{\sum_{j=1}^{n} \mu_j \big[u_j(x,t) \big]} \Bigg|_{x=\phi(t)}, \quad \phi(0) = 0.$$

Obviously, there exists a unique function $\phi(t)$ for sufficiently short times $[0, T_2]$. Now, for $T = \min(T_1, T_2)$ we define a unique solution

$$\widehat{u}(x,t) = \begin{cases} u_+(x,t), & x > \phi(t), \\ u_-(x,t), & x < \phi(t). \end{cases}$$

of the Cauchy problem $(\widehat{u}_j)_t + (\widehat{u}_j f_j(\mu_1\widehat{u}_1 + \cdots + \mu_n\widehat{u}_n))_x = 0$, $j = 1, 2, \ldots, n$, $\widehat{u}(x, 0) = \widehat{u}^0(x)$ for $t \in [0, T)$. This solution defines a *unique* solution $u_\pm(x, t)$, $\phi(t)$ of the first two equations in system (23). Now substituting $u_\pm(x, t)$, $\phi(t)$ into the third equation in (23), we define $e_j(t)$ and $u_j(x, t) = \widehat{u}_j(x, t) + e_j(t)\delta(-x + \phi(t))$. Thus we have constructed the *unique* solution of system (23), thereby we defined $u_\pm(x, t)$, $e_j(t)$, $\phi(t)$ *uniquely*.

It is clear that for any set $u_\pm(x, t)$, $e_j(t)$, $\phi(t)$, $t \in [0, T)$, there exist functions $R_j(t)$, which are defined by equation (24).

It remains to note that since the initial data $u_\pm|_{t=0}$ and $\dot{\phi}(0)$ are such that inequality (14) holds for $t = 0$, then there exists $T^* > 0$ such that, for $0 \le t \le T^*$, inequality (14) holds. Hence the values $u_\pm$ on the curve $x = \phi(t)$, $0 \le t \le T^*$, are determined. Theorem 2 is thus proved. $\qquad\qquad\square$

4.2. Construction of a δ-shock type solution.

Using the weak asymptotic solution of the Cauchy problem (1), (12) constructed by Theorem 2 we obtain a δ-shock wave type solution of this problem.

THEOREM 3. *Let* (14) *hold for* $t = 0$. *Then there exist* $T > 0$ *and a zero neighborhood* $K \subset \mathbb{R}$ *such that for* $(x, t) \in K \times [0, T)$, *the Cauchy problem* (1), (12) *has a unique solution* (13), *which satisfies the integral identities* (5), *where* $\Gamma = \{(x, t) : x = \phi(t), t \in [0, T)\}$, *the distributions* $u_{j\pm}(x, t)$, *and the functions* $\phi(t)$, $e_j(t)$ *are defined by the system of equations*

$$
\begin{aligned}
(u_{j\pm})_t &+ \left(u_{j\pm}f_j(\mu_1 u_{1\pm} + \cdots + \mu_n u_{n\pm})\right)_x = 0, \quad \pm x > \pm\phi(t), \\
(26) \qquad \dot{e}_j(t) &= \left([u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n)] - [u_j]\dot{\phi}(t)\right)\big|_{x=\phi(t)}, \\
\dot{\phi}(t) &= \frac{\sum_{j=1}^n \mu_j[u_j f_j(\mu_1 u_1 + \cdots + \mu_n u_n)]}{\sum_{j=1}^n \mu_j[u_j]}\bigg|_{x=\phi(t)}, \quad j = 1, \ldots, n,
\end{aligned}
$$

with the initial data defined from (12), $\phi(0) = 0$.

Proof. Substituting relation (24) into the asymptotic formula (22), we obtain

$$
u_{j\varepsilon}(x, t)f_j(v_\varepsilon(x, t)) = u_{j+}(x, t)f_j(v_+(x, t)) + [u_j(x, t)f_j(v(x, t))]H(-x + \phi(t))
$$

$$
(27) \qquad\qquad + e_j(t)\dot{\phi}(t)\delta(-x + \phi(t)) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0,
$$

where

$$
\begin{aligned}
v_\varepsilon(x, t) &= \sum_{j=1}^n \mu_j\left(u_{j+}(x, t) + [u_j(x, t)]H_j(-x + \phi(t), \varepsilon)\right), \\
v(x, t) &= \sum_{j=1}^n \mu_j\left(u_{j+}(x, t) + [u_j(x, t)]H(-x + \phi(t))\right).
\end{aligned}
$$

By Theorem 2, $(u_{j\varepsilon})_t + (u_{j\varepsilon}f_j(\mu_1\widehat{u}_{1\varepsilon} + \cdots + \mu_n\widehat{u}_{n\varepsilon}))_x = o_{\mathcal{D}'}(1)$, where $j = 1, 2, \ldots, n$. Multiplying both sides of these relations by a test function $\varphi \in \mathcal{D}(\mathbb{R} \times [0, T))$, taking into account that $u_{j\varepsilon}$, $f_j(\mu_1\widehat{u}_{1\varepsilon} + \cdots + \mu_n\widehat{u}_{n\varepsilon})$ are smooth, and integrating by parts their left-hand sides, we obtain

$$
\langle u_{j\varepsilon}, \varphi_t(x, t)\rangle + \left\langle u_{j\varepsilon}f_j\left(\sum_{j=1}^n \mu_j\widehat{u}_{j\varepsilon}\right), \varphi_x(x, t)\right\rangle + \langle u_{j\varepsilon}(x, 0), \varphi(x, 0)\rangle = o(1).
$$

Substituting (19), (21), and (27) into the latter relations, passing to the limit as $\varepsilon \to +0$ in each of the functionals, and taking into account (8), we obtain

$$\langle \widehat{u}_j, \varphi_t(x,t) \rangle + \langle \widehat{u}_j f_j(\mu_1 \widehat{u}_1 + \cdots + \mu_n \widehat{u}_n), \varphi_x(x,t) \rangle$$

$$+ \langle \delta(-x + \phi(t)), e_j(t) \varphi_t(x,t) \rangle + \langle \delta(-x + \phi(t)), e_j(t) \dot{\phi}(t) \varphi_x(x,t) \rangle$$

$$+ \langle \widehat{u}_j(x,0), \varphi(x,0) \rangle + \langle \delta(-x + \phi(0)), e_j(0) \varphi(x,0) \rangle$$

$$= \int_0^\infty \int \widehat{u}_j \left(\varphi_t + f_j(\mu_1 \widehat{u}_1 + \cdots + \mu_n \widehat{u}_n) \varphi_x \right) dx\, dt + \int \widehat{u}_j^0(x) \varphi(x,0)\, dx$$

$$+ \int_\Gamma e_j(t) \left(\varphi_t(\phi(t),t) + \dot{\phi}(t) \varphi_x(\phi(t),t) \right) \frac{dl}{\sqrt{1 + (\dot{\phi}(t))^2}} + e_j^0 \varphi(0,0) = 0,$$

$j = 1, 2, \ldots, n$. Thus, taking into account (9), one can see that distributions (13) satisfy the integral identities (5).

According to the proof of Theorem 2, system (26), which determines the components $u_{j\pm}(x,t)$, $\phi(t)$, $e_j(t)$ of a δ-shock wave type solution (12), has a *unique* solution. Thus the Cauchy problem (1), (12) has a *unique* solution. $\square$

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