Functional Calculus in the Algebra of Generalized Hyperfunctions on the Circle and Applications

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FUNCTIONAL CALCULUS IN THE ALGEBRA OF GENERALIZED HYPERFUNCTIONS ON THE CIRCLE AND APPLICATIONS

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Abstract. This paper deals with a functional calculus in the algebra $\mathcal{H}(\mathbb{T})$ of generalized hyperfunctions on the circle. This is done introducing an inductive family of complete ultrametric subalgebras. Power series expansions of classical functions such as the exponential, logarithm or power ones are considered. As an application, a nonlinear Cauchy problem involving fractional powers of generalized hyperfunctions is studied. [1]

1. Introduction

This paper aims to provide the algebra $\mathcal{H}(\mathbb{T})$ of generalized periodic hyperfunctions with a functional calculus based on elementary functions but with high nonlinearities. This becomes essential when dealing with nonlinear differential or functional equations. The algebra $\mathcal{H}(\mathbb{T})$ was introduced in [18] and its ultrametric topology in [17]. Earlier a first version was given in [16] involving real 2π-periodic smooth functions. Later on, using the framework of sequence spaces, see [5, 6, 7], the author and his collaborators have given a general topological description of various algebras of generalized functions including $\mathcal{H}(\mathbb{T})$. This descrition involves projective and inductive limits of locally convex spaces. It is well-known that contrary to projective limits inductive limits have a bad inheritance of completeness. Moreover it has never been proved that $\mathcal{H}(\mathbb{T})$ was a complete space or not. Then to overcome such a situation, we introduce an inductive family $(\mathcal{H}^r(\mathbb{T}))_{r>1}$ of complete ultrametric differential algebras in such a way that $\mathcal{H}(\mathbb{T}) = \mathrm{ind}\lim_{\mathrm{r}\to 1}\mathcal{H}^{\mathrm{r}}(\mathbb{T})$ in a set theoritical sense. Therefore it is shown that the induced inductive limit topology on $\mathcal{H}(\mathbb{T})$ is finer that its original one. Recall that the initial ultrametric topology of $\mathcal{H}(\mathbb{T})$ is given by $\omega(f, g) = \nu(f - g)$ where ν is the so-called *indicator* introduced in [17]. We point out that $\nu(\lambda) = 1$ for all nonzero complex number λ. It follows that $(\mathcal{H}(\mathbb{T}), \omega)$ is

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not a classical topological algebra over the field $\mathbb{C}$ of complex numbers since the multiplication by a nonzero complex number is not continuous. Nevertheless ν induces a complete ultrametric structure on the associated algebra $\mathcal{C}$ of generalized complex numbers over which $\mathcal{H}(\mathbb{T})$ is a classical topological algebra but it should be noticed that $\mathcal{C}$ is not a field nor a domain. In the same way the topology of each algebra $\mathcal{H}^r(\mathbb{T})$ is defined by an indicator ν_r. Endowed with the ultrametric ω_r such that $\omega(f,g) = \nu_r(f-g)$, $\mathcal{H}^r(\mathbb{T})$ is a complete algebras.
For the basic theory of Colombeau generalized functions, we refer to [3, 4, 9, 10, 13, 14]. Topological results on generalized functions can be found in [7, 13]. For the theory of periodic hyperfunctions we refer to [1, 2, 11, 12]. We notice that a product of hyperfunctions on the circle is defined in [8] in a more classical setting. This is done using conditions on Fourier coefficients. In the setting of Colombeau algebras, the first work on product of hyperfunctions has been done in [15].
The paper is organized as follows. Section 2 presents some preliminaries on the algebra $\mathcal{H}(\mathbb{T})$ which are useful for the sequel. References for this section are mainly [12, 17, 18]. In Section 3 we define and study the algebras $\mathcal{H}^r(\mathbb{T}), r > 1$. They are proved to be complete and the same is done for the algebra $\mathcal{C}$ of generalized numbers endowed with the ultrametric ω. In Section 4 we give necessary of sufficient conditions for the existence of $\log(h)$, $\exp(h)$ or $h^s, s \in \mathbb{R}$ where $h \in \mathcal{H}(\mathbb{T})$. Section 4 is concerned with the resolution of a nonlinear Cauchy problem in $\mathcal{H}(\mathbb{T})$ where the introduced functional calculus is used.

2. Preliminaries

2.1. **The algebra of generalized hyperfunctions on the circle.** For this section we refer mainly to [12, 17, 18]. For $r > 1$ let

$$C_r = \{z \in \mathbb{C}, 1/r < |z| < r\} \text{ and } \|f\|_r = \sup_{z \in C_r} |f(z)|$$

for every bounded continuous function f defined in C_r. We denote by $\mathcal{O}_r$ the Banach space of bounded holomorphic functions in C_r endowed with the norm $\|\cdot\|_r$. Then, the topological space of real analytic functions on the unit circle $\mathbb{T}$ is

$$\mathcal{A}(\mathbb{T}) = \mathrm{ind\,lim_{r \to 1}} \mathcal{O}_\mathrm{r}.$$

If $\mathcal{X}(\mathbb{T})$ is the set of sequences of functions $(f_n)_n$ with $f_n \in \mathcal{A}(\mathbb{T})$, we denote by $\mathcal{X}_e(\mathbb{T})$ the subset of $\mathcal{X}(\mathbb{T})$ whose elements $(f_n)_n$ satisfy:

$$\exists a > 0, \exists \eta \in \mathbb{N}, \exists r > 1, f_n \in \mathcal{O}_r, \|f_n\|_r \leqslant a^n,\ n > \eta.$$

We denote by $\mathcal{N}_e(\mathbb{T})$ the subset of $\mathcal{X}_e(\mathbb{T})$ constituted of elements $(f_n)_n$ satisfying:

$$\forall b \in (0,1), \exists \eta \in \mathbb{N}, \exists r > 1, f_n \in \mathcal{O}_r, \|f_n\|_r \leqslant b^n, \ n > \eta.$$

Clearly $\mathcal{X}_e(\mathbb{T})$ is an algebra for usual termwise operations and $\mathcal{N}_e(\mathbb{T})$ is an ideal of $\mathcal{X}_e(\mathbb{T})$.

Proposition 2.1. [18, Proposition 3.1] *If* $(f_n)_n \in \mathcal{X}(\mathbb{T})$*, then:*
(i) $(f_n)_n \in \mathcal{X}_e(\mathbb{T})$ *if and only if*

$$\exists a > 0, \exists \eta \in \mathbb{N}, \exists r > 1, |\widehat{f}_n(k)| \leqslant a^n r^{-|k|}, n > \eta, k \in \mathbb{Z}.$$

(ii) $(f_n)_n \in \mathcal{N}_e(\mathbb{T})$ *if and only if*

$$\forall b \in (0,1), \exists \eta \in \mathbb{N}, \exists r > 1, |\widehat{f}_n(k)| \leqslant b^n r^{-|k|}, \ n > \eta, k \in \mathbb{Z}.$$

The algebra of generalized hyperfunctions on $\mathbb{T}$ is the factor algebra

$$\mathcal{H}(\mathbb{T}) = \mathcal{X}_e(\mathbb{T})/\mathcal{N}_e(\mathbb{T})$$

The class of $(f_n)_n$ in $\mathcal{H}(\mathbb{T})$ will be denoted by $\mathrm{cl}(f_n)$.

Embedding of $\mathcal{B}(\mathbb{T})$ *and* $\mathcal{A}(\mathbb{T})$ *in* $\mathcal{H}(\mathbb{T})$*.* The space $\mathcal{B}(\mathbb{T})$ of periodic hyperfunctions is the topological dual of $\mathcal{A}(\mathbb{T})$. For $n \in \mathbb{N}$ we set

$$\varphi_n(z) = \sum_{|k| \leqslant n} z^k.$$

Then we have $\varphi_n * \varphi_n = \varphi_n$ and $\lim_{n\to\infty} \varphi_n = \delta$ in $\mathcal{B}(\mathbb{T})$ where δ is the periodic Dirac distribution. If $H \in \mathcal{B}(\mathbb{T})$, then $(H * \varphi_n)(z) = \sum_{|k|\leqslant n} \widehat{H}(k) z^k$ and $\lim_{n\to\infty} H * \varphi_n = H$ in $\mathcal{B}(\mathbb{T})$. Moreover, the maps $\mathbf{i} : \mathcal{B}(\mathbb{T}) \to \mathcal{X}_e(\mathbb{T})$ defined by $\mathbf{i}(H) = (H * \varphi_n)_n$ and $\mathbf{i}_0 : \mathcal{A}(\mathbb{T}) \to \mathcal{X}_e(\mathbb{T})$ defined by $\mathbf{i}_0(f) = (f_n)_n$ with $f_n = f$, satisfy the following:

(i) $\mathbf{i}$ and $\mathbf{i}_0$ are linear embeddings;
(ii) $\mathbf{i}_0$ is a morphism of algebras.

We denote by ∂_θ be the differential operator defined for $f \in \mathcal{O}_r$, by

$$\partial_\theta f = iz\frac{df}{dz}$$

where $z \in C_r$. It follows that for every $k \in \mathbb{Z}$,

$$\widehat{(\partial_\theta f)}(k) = ik\hat{f}(k).$$

Henceforth, $\mathcal{H}(\mathbb{T})$ is endowed with two structures of differential algebra defined by

$$\frac{df}{dz} = \mathrm{cl}\left(\frac{df_n}{dz}\right) \text{ and } \partial_\theta f = \mathrm{cl}(\partial_\theta f_n)$$

where $f \in \mathcal{H}(\mathbb{T})$ and $(f_n)_n$ is any representative of f. Passing to the quotient spaces we get a linear embedding $\bar{\mathbf{i}}$ and an injective morphism of algebras $\bar{\mathbf{i}}_{\mathbf{0}}$ such that $\bar{\mathbf{i}}|_{\mathcal{A}(\mathbb{T})} \approx \bar{\mathbf{i}}_{\mathbf{0}}$. For any $H \in \mathcal{B}(\mathbb{T})$ one has

$$\bar{\mathbf{i}}(\frac{dH}{dz}) = \frac{d}{dz}\left(\bar{\mathbf{i}}(H)\right) \text{ and } \bar{\mathbf{i}}(\partial_\theta H) = \partial_\theta\left(\bar{\mathbf{i}}(H)\right).$$

2.2. **The algebra of generalized numbers of exponential type.** Let $\mathcal{C}_e$ be the algebra of complex valued sequences $(z_n)_{n\geqslant 1}$ such that:

$$\exists a > 0,\ \exists \eta \in \mathbb{N}^*,\ \forall n \in E_\eta,\ |z_n| \leqslant a^n.$$

Elements of $\mathcal{C}_e$ are said to be of exponential growth. In the same way, we define $\mathcal{I}_e$ as the set of elements $(z_n)_n \in \mathcal{C}_e$ for which

$$\forall b \in (0,1),\ \exists \eta \in \mathbb{N}^*,\ \forall n \in E_\eta,\ \ |z_n| \leqslant b^n.$$

The elements of $\mathcal{I}_e$ are said to be of exponential decrease. It may be seen that $\mathcal{C}_e$ is a subalgebra of $\mathcal{C}$ and that $\mathcal{I}_e$ is an ideal of $\mathcal{C}_e$.

Definition 2.1. *The algebra of complex generalized numbers of* exponential type*, is the quotient algebra* $\mathcal{C} = \mathcal{C}_e/\mathcal{I}_e$.

The complex number z is identified with a generalized number $\mathrm{cl}(z_n)$ where $z_n = z$ for all n. We denote by $\tilde{\mathbb{T}}$ the subalgebra of $\mathcal{C}$ constituted of elements z with a representative in $\mathbb{T}^{\mathbb{N}^*}$.

Definition 2.2. [18, Definition 3.3] *Let* $f \in \mathcal{H}(\mathbb{T})$ *and* $z \in \tilde{\mathbb{T}}$. *The value* $f(z)$ *of* f *at* z *is the generalized number* $f(z) = \mathrm{cl}\left(f_n(z_n)\right)$ *where* $f = \mathrm{cl}(f_n)$ *and* $z = \mathrm{cl}(z_n)$ *with* $(z_n)_n \in \mathbb{T}^{\mathbb{N}^*}$.

2.2.1. *Fourier coefficients of a generalized hyperfunction.*

Definition 2.3. *The Fourier coefficent of rank* $k \in \mathbb{Z}$ *of the generalized hyperfunction* f *is the generalized number*

$$\hat{f}(k) = \mathrm{cl}\left(\frac{1}{2i\pi}\int_{|z|=1} f_n(z) z^{-k-1} dz\right)$$

where $(f_n)_n$ *is an arbitrary representative of* f.

The Fourier coefficients do not depend on the chosen representative and we have the following:

Proposition 2.2. [18, Proposition 3.8] *If* $f \in \mathcal{H}(\mathbb{T})$, *then:*

(i) *There exists* $F \in \mathcal{H}(\mathbb{T})$ *such that* $\partial_\theta F = f$ *if and only if* $\hat{f}(0) = 0$.

(ii) *There exists* $F \in \mathcal{H}(\mathbb{T})$ *such that* $\dfrac{dF}{dz} = f$ *if and only if* $\hat{f}(-1) = 0$.

2.3. **Invertibility.** We denote by $\mathcal{C}^*$ the subset of invertible elements in $\mathcal{C}$. It follows from [18, Threorem 3.9], that $z \in \mathcal{C}^*$ if and only if z admits a representative $(z_n)_n$ such that

$$\exists b \in (0,1), \exists \eta \in \mathbb{N}^*, \forall n > \eta, |z_n| \geqslant b^n.$$

Let $\mathcal{H}^*(\mathbb{T})$ denote the subset of invertible elements of $\mathcal{H}(\mathbb{T})$. From [18, Theorem 3.10], we know that $f \in \mathcal{H}^*(\mathbb{T})$ if and only if it admits a representative $(f_n)_n$ for which there is $r > 1$ such that $f_n \in \mathcal{O}_r$ and:

$$\exists b \in (0,1), \exists \eta \in \mathbb{N}^*, \forall n > \eta, \inf_{z \in C_r} |f_n(z)| \geqslant b^n.$$

This means that the generalized number $\mathrm{cl}(\inf_{z \in C_r} |f_n(z)|)$ is invertible. Moreover this condition does not depend on the chosen representative.

2.4. **The topological structure of $\mathcal{H}(\mathbb{T})$.**

Definition 2.4. [17, Definition 3.1] *The* indicator of $f \in \mathcal{H}(\mathbb{T})$ is:

$$\nu(f) = \lim_{r \to 1} \left(\limsup_{n \to +\infty} \|f_n\|_r^{1/n} \right) \tag{1}$$

where $(f_n)_n$ is an arbitrary representative of f.

It is shown (c.f. [17, Proposition 3.6] that $\nu(f)$ is also given by

$$\nu(f) = \lim_{r \to 1} \left\{ \limsup_{n \to +\infty} \left[\sup_{k \in \mathbb{Z}} (r^{|k|} |\widehat{f}_n(k)|) \right]^{1/n} \right\}. \tag{2}$$

Then we have:

Proposition 2.3. [17, Proposition 3.1] *Let $f, g \in \mathcal{H}(\mathbb{T})$ and $\lambda \in \mathbb{C}^*$. Then the following holds.*

(i) $\nu(f) \geqslant 0$ *and* $\nu(f) = 0$ *iff* $f = 0$;
(ii) $\nu(\lambda f) = \nu(f)$;
(iii) $\nu(fg) \leqslant \nu(f)\nu(g)$;
(iv) $\nu(f+g) \leqslant \sup(\nu(f), \nu(g))$;
(v) $|\nu(f) - \nu(g)| \leqslant \nu(f-g)$;
(vi) $\nu(f^{-1}) \geqslant (\nu(f))^{-1}$ *if* $f \in \mathcal{H}^*(\mathbb{T})$.

Setting

$$\omega(f,g) = \nu(f-g), \ f, g \in \mathcal{H}(\mathbb{T}),$$

we define a translation invariant ultrametric distance on $\mathcal{H}(\mathbb{T})$. Moreover addition and multiplication are continuous mappings from $\mathcal{H}(\mathbb{T})^2$ to $\mathcal{H}(\mathbb{T})$ where $\mathcal{H}(\mathbb{T})^2$ is endowed with the ultrametric distance D defined by

$$D[(f,g),(u,v)] = \sup(\omega(f,u), \omega(g,v)).$$

The inverse fonction is a continuous operator of $\mathcal{H}^*(\mathbb{T})$ (see [17, Proposition 3.4 and Corollary 3.2]). We end this section by the following result.

Proposition 2.4. [17, Corollary 3.5] *The following holds:*

(i) *If $f \in \bar{\mathbf{i}}(\mathcal{B}(\mathbb{T}))$ and $f \neq 0$, then $\nu(f) = 1$.*
(ii) *The mapping ν is surjective from $\mathcal{H}(\mathbb{T})$ to $\mathbb{R}_+$.*

3. Completeness of basic subalgebras

3.1. Completeness of the ultrametric space $\mathcal{C}$.

The subalgebra $\mathcal{C}$ of $\mathcal{H}(\mathbb{T})$ is endowed with the restriction of ν and then with the restriction of the metric ω.

Theorem 3.1. *The ultrametric space $(\mathcal{C}, \omega)$ is complete. Then it is a closed subspace of $\mathcal{H}(\mathbb{T})$.*

Proof. Let $(\lambda_m)_m$ be a Cauchy sequence in $\mathcal{C}$; we denote by $(\lambda_{m,n})_n$ a representative of λ_m. Then we have:

$$\forall \varepsilon > 0, \exists m_0 \in \mathbb{N}^*, \forall p, q \in \mathbb{N}^*, p > q \geqslant m_0, \liminf_{n\to+\infty} |\lambda_{p,n} - \lambda_{q,n}|^{1/n} \leqslant \varepsilon/2.$$

Hence, for each (p, q) as above there exists $\eta > 0$ such that $|\lambda_{p,n} - \lambda_{q,n}|^{1/n} \leqslant \varepsilon$. It follows that we can define two sequences (m_k) and (η_k) of positive integers both strictly increasing and such that:

$$\forall k \in \mathbb{N}^*, \forall n \in \mathbb{N}^*, n \geqslant \eta_k, |\lambda_{m_{k+1},n} - \lambda_{m_k,n}| \leqslant \frac{1}{2^{kn}}. \quad (3)$$

We define the sequence $(\mu_m)_m$ in $\mathcal{C}$ by

$$\mu_{k,n} = \lambda_{m_k,n} \text{ if } n \geqslant \eta_k \text{ and } \mu_{k,n} = 0 \text{ if } n < \eta_k.$$

Since the sequence (η_k) is increasing, we have $\mu_{k+1,n} = 0$ if $n < \eta_k$. Then it follows that

$$\forall k \in \mathbb{N}^*, \forall n \in \mathbb{N}^*, |\mu_{k+1,n} - \mu_{k,n}| \leqslant \frac{1}{2^{kn}}. \quad (4)$$

Hence, we have

$$\sum_{k=1}^{+\infty} |\mu_{k+1,n} - \mu_{k,n}| \leqslant \sum_{k=1}^{+\infty} \left(\frac{1}{2^n}\right)^k = \frac{1}{2^n - 1}.$$

It follows that for each $n \in \mathbb{N}^*$, the sequence $(\mu_{k,n})_k$ converges to ζ_n where

$$\zeta_n = \mu_{1,n} + \sum_{k=1}^{+\infty} \mu_{k+1,n} - \mu_{k,n}.$$

This shows that (ζ_n) is a moderate element, and then we set $\zeta = \mathrm{cl}(\zeta_n)$. Using (4), we have for every $p \in \mathbb{N}^*$:

$$|\mu_{k+p,n} - \mu_{k,n}| \leqslant \sum_{j=0}^{p-1} |\mu_{k+j+1,n} - \mu_{k+j,n}| \leqslant \sum_{j=0}^{p-1} \left(\frac{1}{2^n}\right)^{k+j} \leqslant \left(\frac{1}{2^n}\right)^{k-1} \frac{1}{2^n - 1}.$$

Letting $p \to +\infty$, we get that

$$|\zeta_n - \mu_{k,n}| \leqslant \left(\frac{1}{2^n}\right)^{k-1} \frac{1}{2^n - 1},$$

from which it follows that

$$\limsup_{n\to+\infty} |\zeta_n - \mu_{k,n}|^{1/n} \leqslant \left(\frac{1}{2}\right)^k.$$

This means that $\nu(\mu_k - \zeta) \leqslant \left(\frac{1}{2}\right)^k$ showing that $(\mu_k)_k$ converges to ζ in $(\mathcal{C}, \omega)$. But since $\mu_{k,n} = \lambda_{m_k,n}$ for $n \geqslant \eta_k$, it follows that $\mu_k = \lambda_{m_k}$ which implies that $(\lambda_m)_m$ converges to ζ and concludes the proof. □

3.2. **The ultrametric algebras $\mathcal{H}^r(\mathbb{T})$.** For every $r > 1$ we set

$$\mathcal{X}_e^r(\mathbb{T}) = \{(f_n)_n \in \mathcal{X}_e(\mathbb{T}), \exists \eta \in \mathbb{N}, \forall n > \eta, f_n \in \mathcal{O}_r, \limsup_{n\to+\infty} \|f_n\|_r^{1/n} < +\infty\}$$

and we define

$$\mathcal{H}^r(\mathbb{T}) = \{f \in \mathcal{H}(\mathbb{T}), \exists (f_n)_n \in \mathcal{X}_e^r(\mathbb{T}), \mathrm{cl}(f_n) = f\}.$$

Therefore, if $\mathbb{R}_+ = [0, +\infty)$, we get a well defined mapping

$$\nu_r : \mathcal{H}^r(\mathbb{T}) \to \mathbb{R}_+$$

by setting

$$\nu_r(f) = \inf\{\limsup_{n\to+\infty} \|f_n\|_r^{1/n}, (f_n)_n \in \mathcal{X}_e^r(\mathbb{T}), \mathrm{cl}(f_n) = f\}. \tag{5}$$

Then, ν_r satisfies to the following.

Proposition 3.2. *Let $f, g \in \mathcal{H}^r(\mathbb{T})$ and $\lambda \in \mathbb{C}^*$. Then we have:*

(i) $\nu_r(\lambda) = \nu(\lambda)$;
(ii) $\nu_r(\lambda f) = \nu_r(f)$;
(iii) $\nu(f) \leqslant \nu_r(f)$;
(iv) $\nu_r(f) = 0$ *if and only if* $f = 0$;
(v) $\nu_r(fg) \leqslant \nu_r(f)\nu_r(g)$;
(vi) $\nu_r(f + g) \leqslant \max(\nu_r(f), \nu_r(g))$.

Proof. Assume that $\mathrm{cl}(\lambda_n)$ and $\mathrm{cl}(\mu_n)$ are two representatives of λ. Then, we have $(\lambda_n - \mu_n)_n \in \mathcal{N}_e$ and consequently for every $b \in (0,1)$ there is $\eta \in \mathbb{N}$ such that $|\lambda_n - \mu_n| < b^n$ for $n > \eta$. Therefore

$$|\lambda_n|^{1/n} \leqslant (|\mu_n| + b^n)^{1/n} \leqslant |\mu_n|^{1/n} + b$$

and then $\limsup_{n\to+\infty} |\lambda_n|^{1/n} \leqslant \limsup_{n\to+\infty} |\mu_n|^{1/n}$. It follows that $\limsup_{n\to+\infty} |\lambda_n|^{1/n} = \limsup_{n\to+\infty} |\mu_n|^{1/n}$ which shows that

$$\nu_r(\lambda) = \limsup_{n\to+\infty} |\lambda_n|^{1/n} = \nu(\lambda)$$

and proves (i). The proof of (ii) can be done following those of [17, Proposition 3.1], (see Proposition 2.3). To prove (iii), let $\alpha > \nu_r(f)$. Then, there exists a representative $(f_n)_n$ of f in $\mathcal{X}_e^r(\mathbb{T})$ such that $\limsup_{n\to+\infty} \|f_n\|_r^{1/n} < \alpha$. Since $\|f_n\|_\rho^{1/n} \leqslant \|f_n\|_r^{1/n}$ for $\rho < r$, it follows that $\nu(f) = \lim_{\rho\to 1}(\limsup_{n\to+\infty} \|f_n\|_\rho^{1/n}) < \alpha$. Thus, $\nu(f) \leqslant \nu_r(f)$. We see that (iv) follows from (iii). Now take $\beta > \nu(g)$ and choose a representative $(g_n)_n$ of g such that $\limsup_{n\to+\infty} \|g_n\|_r^{1/n} < \beta$. Since $\limsup_{n\to+\infty} \|f_n g_n\|_r^{1/n} \leqslant \limsup_{n\to+\infty} \|f_n\|_r^{1/n} \times \limsup_{n\to+\infty} \|g_n\|_r^{1/n}$, it follows that $\nu_r(fg) \leqslant \alpha\beta$ proving (v). Using the above notation, there exists $\eta \in \mathbb{N}$ such that $\|f_n\|_r < \alpha^n$and $\|g_n\|_r < \beta^n$ for $n > \eta$. It follows that

$$\|f_n + g_n\|_r^{1/n} \leqslant (\alpha^n + \beta^n)^{1/n}.$$

Assuming tha $\alpha \geqslant \beta$ we get

$$(\alpha^n + \beta^n)^{1/n} = \alpha\left(1 + \left(\frac{\beta}{\alpha}\right)^n\right)^{1/n} \to \alpha \text{ as } n \to +\infty$$

which proves (vi). The proof of the proposition is then complete.
□

Clearly $\mathcal{H}^r(\mathbb{T})$ is a subalgebra of $\mathcal{H}(\mathbb{T})$ and $\mathcal{H}^r(\mathbb{T}) \subset \mathcal{H}^s(\mathbb{T})$ if $r \geqslant s > 1$ since $\nu_r \geqslant \nu_s$. Moreover we have $\mathcal{H}(\mathbb{T}) = \cup_{r>1}\mathcal{H}^r(\mathbb{T})$. We introduce the ultrametric distances ω_r on $\mathcal{H}^r(\mathbb{T})$ and D_r on $\mathcal{H}^r(\mathbb{T})^2$ as follows:

$$\omega_r(f,g) = \nu_r(f-g) \text{ and } D_r((f,u),(g,v)) = \max(\omega_r(f,g), \omega_r(u,v)).$$

It is easily seen that addition and multiplication are continuous maps from $\mathcal{H}^r(\mathbb{T})^2$ to $\mathcal{H}^r(\mathbb{T})$, and the inverse map is a continuous operator on $\mathcal{H}^r(\mathbb{T})^*$ the group of invertible elements in $\mathcal{H}^r(\mathbb{T})$. Moreover, if $r \geqslant s > 1$ the embeddings $u_{s,r} : \mathcal{H}^r(\mathbb{T}) \to \mathcal{H}^r(\mathbb{T})$ and $u_r : \mathcal{H}^r(\mathbb{T}) \to \mathcal{H}(\mathbb{T})$ are continuous. It follows that

$$\mathcal{H}(\mathbb{T}) = \mathrm{ind}\lim\nolimits_{\mathrm{r}\to 1}\mathcal{H}^{\mathrm{r}}(\mathbb{T}),$$

can be endowed with the inductive limit topology of the spaces $\mathcal{H}^r(\mathbb{T})$ which will be denoted by $\mathcal{T}$. Then we have:

Proposition 3.3. *The inductive limit topology defined by the ultrametric spaces $\mathcal{H}^r(\mathbb{T})$ on $\mathcal{H}(\mathbb{T})$ is finer that the one induced by ν.*

Proof. Let V be an open set in $\mathcal{H}(\mathbb{T})$ for the topology defined by ν and take $f \in V$. Then, there exists an open ball centered at f such that $B(f,\alpha) \subset V$. If $r > 1$ is such that $f \in \mathcal{H}^r(\mathbb{T})$, the corresponding open ball $B_r(f,\alpha)$ for the topology induced by ν_r satisfies $B_r(f,\alpha) \subset B(f,\alpha)$ since $\nu \leqslant \nu_r$. It follows that $B_r(f,\alpha) \subset V \cap \mathcal{H}^r(\mathbb{T})$ which proves that $V \cap \mathcal{H}^r(\mathbb{T})$ is an open set in $\mathcal{H}^r$ for the topology induced by ν_r. Hence V is an open set for the topology $\mathcal{T}$, which concludes the proof. □

For any bounded function g on $\mathbb{T}$, we set

$$\|g\|_{\infty,\mathbb{T}} = \sup_{z\in\mathbb{T}} |g(z)|.$$

Then, the following holds:

Proposition 3.4. *Let $f \in \mathcal{H}(\mathbb{T})$. If $(f_n)_n$ and $(g_n)_n$ are two representatives of f, then*

$$\limsup_{n\to+\infty} \|f_n\|_{\infty,\mathbb{T}}^{1/n} = \limsup_{n\to+\infty} \|g_n\|_{\infty,\mathbb{T}}^{1/n}.$$

Proof. Since $(f_n - g_n)_n \in \mathcal{N}_e(\mathbb{T})$, then for every $b \in (0,1)$ there are $r > 1$ and $\eta \in \mathbb{N}$ such that $f_n, g_n \in \mathcal{O}_r$ and $\|f_n - g_n\|_r < b^n$ if $n > \eta$. Thus we have: $\forall b \in (0,1), \exists r > 1, \exists \eta \in \mathbb{N}, \forall n > \eta$,

$$\|f_n - g_n\|_{\infty,\mathbb{T}} < b^n,\ n > \eta.$$

It folows that $\|f_n\|_{\infty,\mathbb{T}} \leqslant \|g_n\|_{\infty,\mathbb{T}} + b^n$ for $n > \eta$ and then

$$\limsup_{n\to+\infty} \|f_n\|_{\infty,\mathbb{T}}^{1/n} \leqslant \max(\limsup_{n\to+\infty} \|g_n\|_{\infty,\mathbb{T}}^{1/n}, b).$$

- If $\limsup_{n\to+\infty} \|g_n\|_{\infty,\mathbb{T}}^{1/n} = 0$, then $\limsup_{n\to+\infty} \|f_n\|_{\infty,\mathbb{T}}^{1/n} \leqslant b$ for every $b \in (0,1)$ which implies that $\limsup_{n\to+\infty} \|f_n\|_{\infty,\mathbb{T}}^{1/n} = 0$.
- If $\limsup_{n\to+\infty} \|g_n\|_{\infty,\mathbb{T}}^{1/n} > 0$, taking $b < \limsup_{n\to+\infty} \|f_n\|_{\infty,\mathbb{T}}^{1/n}$ gives $\limsup_{n\to+\infty} \|f_n\|_{\infty,\mathbb{T}}^{1/n} \leqslant \limsup_{n\to+\infty} \|g_n\|_{\infty,\mathbb{T}}^{1/n}$.
We have proved that in any case we have

$$\limsup_{n\to+\infty} \|f_n\|_{\infty,\mathbb{T}}^{1/n} \leqslant \limsup_{n\to+\infty} \|g_n\|_{\infty,\mathbb{T}}^{1/n}.$$

The converse inequality can be shown to be true in the same way. □

This allows us to define

$$\nu_1(f) = \limsup_{n\to+\infty} \|f_n\|_{\infty,\mathbb{T}}^{1/n} \tag{6}$$

where $(f_n)_n$ is any representative of f. It is easy to see that properties (i), (iii) and (vi) of Proposition 3.2 are satisfied for $r = 1$ and $\nu_1 \leqslant \nu$.

Theorem 3.5. *For every $r > 1$ and for every $f \in \mathcal{H}^r(\mathbb{T})$ we have:*
(i) $\nu_r(f) \leqslant \max(\nu_r(f'), \nu_1(f))$;
(ii) $\nu_1(f') \leqslant \sqrt{\nu_1(f)\nu(f)}$.

Proof. For $z \in C_r$ set $z' = z/|z|$. If $(f_n)_n$ is a representative of f, we have

$$f_n(z) = \int_{[z',z]} f_n'(\xi)d\xi + f_n(z')$$

and then

$$|f_n(z)| \leqslant |z - z'| \|f_n'\|_r + \|f_n\|_{\infty,\mathbb{T}}.$$

Since $|z - z'| \leqslant \max(r - 1, 1 - 1/r) = r - 1$, it follows that

$$|f_n(z)| \leqslant (r - 1)\|f_n'\|_r + \|f_n\|_{\infty,\mathbb{T}}.$$

Finally we obtain

$$\limsup_{n\to+\infty} \|f_n\|_r^{1/n} \leqslant \max(\limsup_{n\to+\infty} \|f_n'\|_r^{1/n}, \limsup_{n\to+\infty} \|f_n\|_{\infty,\mathbb{T}}^{1/n})$$

from which (i) follows.
Now let $a \in \mathbb{T}$ and choose $s > 0$ such $\overline{D(a,s)} \subset C_r$ where $D(a,s) = \{z \in \mathbb{C}, |z - a| < s\}$. Recall that the remainder after the term of degree m in the Taylor expansion of f_n about a is

$$R_{n,m}(z) = \frac{(z-a)^{m+1}}{2i\pi} \int_{\Gamma_s} \frac{f_n(\xi)d\xi}{(\xi - z)(\xi - a)^{m+1}}$$

where $\Gamma_s = \{\xi \in \mathbb{C}, |\xi - a| = s\}$. It follows that if $|z - a| \leqslant \rho < s$, then

$$|R_{n,m}(z)| \leqslant \frac{s}{s-\rho}\left(\frac{\rho}{s}\right)^{m+1} \|f_n\|_r.$$

Thus, if $|z - a| = \rho$ and $z \in \mathbb{T}$, writting $f_n(z) = f_n(a) + (z - a)f_n'(a) + R_{n,1}(z)$ and using the above inequality with $m = 1$ gives

$$\|f_n'\|_{\infty,\mathbb{T}} \leqslant \frac{2\|f_n\|_{\infty,\mathbb{T}}}{\rho} + \frac{\rho}{s(s-\rho)}\|f_n\|_r. \tag{7}$$

Set $\rho = ts$ with $t \in (0,1)$. Therefore (7) becomes

$$\|f_n'\|_{\infty,\mathbb{T}} \leqslant \frac{1}{s}\left(\frac{2\|f_n\|_{\infty,\mathbb{T}}}{t} + \frac{t}{1-t}\|f_n\|_r\right). \tag{8}$$

Let $\alpha = 2\|f_n\|_{\infty,\mathbb{T}}$ and $\beta = \|f_n\|_r$. We let φ denote the function

$$\varphi(t) = \frac{\alpha}{t} + \frac{\beta t}{1-t}$$

where $t \in (0,1)$. A simple calculation gives

$$\varphi'(t) = \frac{(\beta - \alpha)t^2 + 2\alpha t - \alpha}{t^2(1-t)^2}.$$

For $\beta-\alpha \neq 0$, the value of the reduced discriminant of the polynomials $(\beta-\alpha)t^2+2\alpha t-\alpha$ being equal to $\sqrt{\alpha\beta}$, we find that it has two roots t_0 and t_1 given by

$$t_0 = \frac{-\alpha-\sqrt{\alpha\beta}}{\beta-\alpha} \text{ and } t_1 = \frac{-\alpha+\sqrt{\alpha\beta}}{\beta-\alpha}.$$

If $\beta>\alpha$, we find that

$$t_0 < 0 \text{ and } t_1 = \frac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}},$$

If $\beta<\alpha$, we find that

$$t_0 > 1 \text{ and } t_1 = \frac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}.$$

If $\alpha=\beta$, $\varphi'(t)$ vanishes for $t=\dfrac{1}{2}$ and $\varphi(\dfrac{1}{2}) = 3\alpha$.

Therefore, in any case $\varphi(t)$ reaches its minimum at $t = \dfrac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}$ in $(0,1)$ and we find that

$$\varphi\left(\frac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}\right) = \alpha+2\sqrt{\alpha\beta}.$$

This equality is also true when $\beta=\alpha$. Finally we obtain

$$\|f_n'\|_{\infty,\mathbb{T}} \leqslant \frac{2}{s}(2\|f_n\|_{\infty,\mathbb{T}} + \sqrt{2\|f_n\|_{\infty,\mathbb{T}}\cdot\|f_n\|_r}).$$

It follows that

$$\nu_1(f') \leqslant \max(\nu_1(f), \sqrt{\nu_1(f)}\sqrt{\limsup_{n\to+\infty}\|f_n\|_r^{1/n}}).$$

Making $r\to 1$ and using $\nu(f) = \lim_{r\to 1}(\limsup_{n\to+\infty}\|f_n\|_r^{1/n})$ gives (ii) and concludes the proof. □

Using Theorem 3.5, (ii) we get straightforwardly:

Corollary 3.6. *Let $f\in\mathcal{H}(\mathbb{T})$. If $\nu_1(f)=0$, then for every $m\in\mathbb{N}^*$ we have $\nu_1(f^{(m)})=0$.*

3.3. **Continuity of the differential operators d/dz and ∂_θ.** To establish the continuity of these differential operators we state and prove the following.

Theorem 3.7. *Let $f\in\mathcal{H}^r(\mathbb{T})$ for some $r>1$. The following holds:*

(i) $\nu_\rho(\partial_\theta f) = \nu_\rho(f') \leqslant \nu_r(f), \forall\rho\in(1,r)$;
(ii) $\nu(\partial_\theta f) = \nu(f') \leqslant \nu(f)$;
(iii) *If* $\hat{f}(0)=0$, *then* $\nu(\partial_\theta f) = \nu(f') = \nu(f)$.

Proof. Let $(f_n)_n$ denote a representative of f in $\mathcal{X}_e^r(\mathbb{T})$ and let $z \in C_\rho$ with $\rho \in (1, r)$. We have $(\partial_\theta f)(z) = izf'(z)$ with $\frac{1}{\rho} \leqslant |z| \leqslant \rho$, and then

$$\frac{1}{\rho}\|f_n'\|_\rho \leqslant \|\partial_\theta f_n\|_\rho \leqslant \rho\|f_n'\|_\rho$$

which gives

$$\limsup_{n\to+\infty} \|\partial_\theta f_n\|_\rho^{1/n} = \limsup_{n\to+\infty} \|f_n'\|_\rho^{1/n}.$$

It follows that $\nu_\rho(\partial_\theta f) = \nu_\rho(f')$ and $\nu(\partial_\theta f) = \nu(f')$.
Let $\rho \in (1, r)$ and take r' such that $\rho < r' < r$. Hence, for all $z \in C_\rho$ we have

$$f_n(z) = \frac{1}{2i\pi}\int_{|\xi|=r'} \frac{f_n(\xi)d\xi}{\xi - z} - \frac{1}{2i\pi}\int_{|\xi|=1/r'} \frac{f_n(\xi)d\xi}{\xi - z}$$

and then

$$f_n'(z) = \frac{1}{2i\pi}\int_{|\xi|=r'} \frac{f_n(\xi)d\xi}{(\xi - z)^2} - \frac{1}{2i\pi}\int_{|\xi|=1/r'} \frac{f_n(\xi)d\xi}{(\xi - z)^2}.$$

It follows that

$$|f_n'(z)| \leqslant \frac{r'\|f_n\|_{r'}}{(r'-\rho)^2} + \frac{\frac{1}{r'}\|f_n\|_{r'}}{(\frac{1}{\rho} - \frac{1}{r'})^2}.$$

Simple calculation gives

$$|f_n'(z)| \leqslant \frac{r' + r'\rho^2}{(r'-\rho)^2}\|f_n\|_{r'}$$

and then

$$\|f_n'\|_\rho \leqslant \frac{r' + r'\rho^2}{(r'-\rho)^2}\|f_n\|_{r'}.$$

Using $\|f_n\|_{r'} \leqslant \|f_n\|_r$ and letting $r' \to r$ yields

$$\|f_n'\|_\rho \leqslant \frac{r + r\rho^2}{(r-\rho)^2}\|f_n\|_r.$$

It follows that $\nu_\rho(\partial_\theta f) = \nu_\rho(f') \leqslant \nu_r(f)$ and $\nu(\partial_\theta f) = \nu(f') \leqslant \nu(f)$ which proves (i) and (ii).
Since $\widehat{(\partial_\theta f_n)}(k) = ik\widehat{f}_n(k)$ for all $k \in \mathbb{Z}$, it follows from (2) that

$$\nu(f') = \lim_{\rho\to 1}\left\{\limsup_{n\to+\infty}\left[\sup_{k\in\mathbb{Z}}(\rho^{|k|}|k||\widehat{f}_n(k)|)\right]^{1/n}\right\}.$$

Hence, if $\hat{f}(0) = 0$, we can choose $(f_n)_n$ such that $\widehat{f}_n(0) = 0$ for every n and we will have

$$\sup_{k\in\mathbb{Z}}(\rho^{|k|}|k||\widehat{f}_n(k)|) \geqslant \sup_{k\in\mathbb{Z}}(\rho^{|k|}|\widehat{f}_n(k)|).$$

This leads to $\nu(\partial_\theta f) \geqslant \nu(f)$ and then $\nu(\partial_\theta f) = \nu(f)$, proving (iii). □

Thus, the following corollary is a straightforward consequence of Theorem 3.7.

Corollary 3.8. *The differential operators d/dz and ∂_θ are continuous in each of the following cases:*

(i) *from $\mathcal{H}(\mathbb{T})$ to $\mathcal{H}(\mathbb{T})$;*
(ii) *from $\mathcal{H}^r(\mathbb{T})$ to $\mathcal{H}(\mathbb{T})$;*
(iii) *from $\mathcal{H}^r(\mathbb{T})$ to $\mathcal{H}^s(\mathbb{T})$ with $1 < s < r$.*

Consequently $\mathcal{H}(\mathbb{T})$ is a topological differential algebra.

3.4. Completeness of the topological algebras $\mathcal{H}^r(\mathbb{T})$.

Theorem 3.9. *The ultrametric algebra $(\mathcal{H}^r(\mathbb{T}), \omega_r)$ is a complete one.*

Proof. Let $(F_m)_m$ be a Cauchy sequence in $\mathcal{H}^r(\mathbb{T})$. It follows from the definition of ν_r that there exist $m_1, m_2 \in \mathbb{N}^*$ with $m_2 > m_1$ and two representatives $(F^{[1]}_{m_1,n})_n$ and $(F^{[1]}_{m_2,n})_n$ of F_{m_1} and F_{m_2} respectively such that:

$$\limsup_{n\to+\infty} \|F^{[1]}_{m_2,n} - F^{[1]}_{m_1,n}\|_r^{1/n} < \frac{1}{2^1}. \tag{9}$$

Then, we set

$$F_{m_1,n} = F^{[1]}_{m_1,n} \text{ and } F_{m_2,n} = F^{[1]}_{m_2,n}. \tag{10}$$

In the same way we get $m_3 \in \mathbb{N}^*$ with $m_3 > m_2$ and two representatives $(F^{[2]}_{m_2,n})_n$ and $(F^{[2]}_{m_3,n})_n$ of F_{m_2} and F_{m_3} respectively such that:

$$\limsup_{n\to+\infty} \|F^{[2]}_{m_3,n} - F^{[2]}_{m_2,n}\|_r^{1/n} < \frac{1}{2^2}.$$

Then, for each $n \in \mathbb{N}^*$, we set

$$F_{m_3,n} = F^{[2]}_{m_3,n} - F^{[2]}_{m_2,n} + F_{m_2,n}.$$

Hence, by induction, we get a subsequence $(F_{m_k})_k$ along with representatives $(F^{[k]}_{m_{k+1},n})_n$ and $(F^{[k]}_{m_k,n})_n$ of $F^{[k]}_{m_{k+1}}$ and $F^{[k]}_{m_k}$ respectively such that for every $k \in \mathbb{N}^*$,

$$\limsup_{n\to+\infty} \|F^{[k]}_{m_{k+1},n} - F^{[k]}_{m_k,n}\|_r^{1/n} < \frac{1}{2^k}. \tag{11}$$

Then, for every $(k,n) \in \mathbb{N}^* \times \mathbb{N}^*$ we set

$$F_{m_{k+1},n} = F^{[k]}_{m_{k+1},n} - F^{[k]}_{m_k,n} + F_{m_k,n}. \tag{12}$$

It follows that

$$F_{m_{j+1},n} - F_{m_j,n} = F^{[j]}_{m_{j+1},n} - F^{[j]}_{m_j,n}$$

for $1 \leqslant j \leqslant k$, and summing up we find that for every $k \geqslant 2$:

$$F_{m_{k+1},n} = F^{[k]}_{m_{k+1},n} + \sum_{j=2}^{k} (F^{[j-1]}_{m_j,n} - F^{[j]}_{m_j,n}). \tag{13}$$

Since $(F^{[j-1]}_{m_j,n})_n$ and $(F^{[j]}_{m_j,n})_n$ are both representatives of $F_{m_j,n}$, it follows that $\left(\sum_{j=2}^{k}[F^{[j-1]}_{m_j,n} - F^{[j]}_{m_j,n}]\right)_n \in \mathcal{N}_e(\mathbb{T})$ and then $(F_{m_{k+1},n})_n$ is a representative of $F_{m_{k+1}}$. Using (12), we get $F_{m_{k+1}} - F_{m_k} = F^{[k]}_{m_{k+1},n} - F^{[k]}_{m_k,n}$ and then using (11) we find

$$\limsup_{n\to+\infty} \|F_{m_{k+1},n} - F_{m_k,n}\|_r^{1/n} < \frac{1}{2^k}. \tag{14}$$

Then, there exists a sequence $(\eta_k)_k$ of positive integers which is strictly increasing and such that

$$\forall (k,n) \in \mathbb{N}^* \times \mathbb{N}^*, n \geqslant \eta_k, \|F_{m_{k+1},n} - F_{m_k,n}\|_r \leqslant \left(\frac{1}{2^k}\right)^n. \tag{15}$$

For each $k \in \mathbb{N}^*$, we define the sequence of functions $(G_{k,n})_n$ as follows:

$$G_{k,n} = F_{m_k,n} \text{ if } n \geqslant \eta_k \text{ and } G_{k,n} = 0 \text{ otherwise.}$$

It follows that $(G_{k,n})_n$ is a moderate sequence, and if $G_k = [(G_{k,n})]$, then $G_k = F_{m_k}$. We also have:

$$\forall (k,n) \in \mathbb{N}^* \times \mathbb{N}^*, \|G_{k+1,n} - G_{k,n}\| \leqslant \left(\frac{1}{2^n}\right)^k.$$

Using successively the above inequality, we get for every $p \in \mathbb{N}^*$:

$$\begin{aligned}
\|G_{k+p,n} - G_{k,n}\|_r &\leqslant \|G_{k+p,n} - G_{k+p-1,n}\|_r + \cdots + \|G_{k+1,n} - G_{k,n}\|_r \\
&\leqslant \left(\tfrac{1}{2^n}\right)^{k+p-1} + \cdots + \left(\tfrac{1}{2^n}\right)^k \\
&\leqslant \left(\tfrac{1}{2^n}\right)^k \left[\left(\tfrac{1}{2^n}\right)^{p-1} + \cdots + 1\right] \\
\|G_{k+p,n} - G_{k,n}\|_r &\leqslant \left(\tfrac{1}{2^n}\right)^{k-1} \tfrac{1}{2^n-1}.
\end{aligned}$$

It follows that for each $n \in \mathbb{N}^*$, the sequence $(G_{k,n})_k$ is a Cauchy sequence in $\mathcal{O}_r$ and then it converges to an element g_n in $\mathcal{O}_r$. Letting $p \to +\infty$ in the above inequality gives

$$\|g_n - G_{k,n}\|_r \leqslant \left(\frac{1}{2^n}\right)^{k-1} \frac{1}{2^n - 1}. \tag{16}$$

This shows that (g_n) is a moderate element; in fact we have:

$$\|g_n\|_r \leqslant \|G_{k,n}\|_r + \left(\frac{1}{2^{k-1}}\right)^n.$$

Then we set $g = [(g_n)]$. Using (16), we have for every $p \in \mathbb{N}^*$:

$$\|g_n - G_{k,n}\|_r^{1/n} \leqslant \left(\frac{1}{2}\right)^{k-1} \left(\frac{1}{2^n - 1}\right)^{1/n}$$

which gives

$$\nu_r(g - G_k) \leqslant \limsup_{n\to+\infty} \|g_n - G_{k,n}\|_r^{1/n} \leqslant \left(\frac{1}{2}\right)^{k}$$

and proves that

$$\lim_{k\to+\infty} \nu_r(g - G_k) = 0.$$

Hence, $(F_{m_k})_k$ converges to g in $\mathcal{H}(\mathbb{T})$, and since $(F_m)_m$ is a Cauchy sequence, it converges to g which concludes the proof. □

4. Functional calculus and applications

All the results stated in this section for the algebra $\mathcal{H}(\mathbb{T})$ are also true for the subalgebras $\mathcal{H}^r(\mathbb{T})$ and $\mathcal{C}$.

4.1. **Exponential, logarithm and power functions.**

4.1.1. *The exponential of a generalized hyperfunction.* Let $u \in \mathcal{H}(\mathbb{T})$ and let (u_n) be a representative of u such that $u_n \in \mathcal{O}_r$ for some $r > 1$. If $z \in C_r$, then $|\exp(u_n(z))| = \exp(\Re u_n(z))$ and consequently

$$\|\exp(u_n)\|_r = \exp(\sup_{z\in C_r} \Re u_n(z)).$$

It follows that (u_n) satisfies $\|\exp(u_n)\|_r \leqslant a^n$ for some positive constant a if and only if $\sup_{z\in C_r} \Re u_n(z) \leqslant n \ln a$.

Definition 4.1. *A generalized hyperfunction u is said to be* real sublinear *if it admits a representative $(u_n)_n$ such that $u_n \in \mathcal{O}_r$ for some $r > 1$ and $\sup_{z\in C_r} \Re u_n(z) \leqslant \lambda n$ for a real constant λ and n large enough.*

We have the following:

Proposition 4.1. *For a generalized hyperfunction u, the condition to be real sublinear does not depend on the chosen representative.*

Proof. Let $(u_n)_n$ and $(v_n)_n$ be two representatives of u where $(u_n)_n$ is real sublinear; we set

$$\alpha_n = \sup_{z\in C_r} \Re u_n(z) \text{ and } \beta_n = \sup_{z\in C_r} \Re u_n(z).$$

It follows that

$$|e^{\beta_n} - e^{\alpha_n}| = |\|e^{v_n}\|_r - \|e^{u_n}\|_r| \leqslant \|e^{v_n} - e^{v_n}\|_r$$

and then using $|e^z - 1| \leqslant |z|e^{|z|}$, we get

$$\begin{aligned} |e^{\beta_n} - e^{\alpha_n}| &\leqslant \|e^{u_n}(e^{v_n - u_n} - 1)\|_r \\ &\leqslant \|e^{u_n}\|_r \|e^{v_n - u_n} - 1\|_r \\ &\leqslant e^{\alpha_n} e^{\|v_n - u_n\|_r} \|v_n - u_n\|_r. \end{aligned}$$

Since $(v_n - u_n)_n$ is negligible, for every $\varepsilon > 0$ there exists $\eta_1 \in \mathbb{N}$ such that $e^{\|v_n-u_n\|_r}\|v_n - u_n\|_r \leqslant \varepsilon$ if $n > \eta_1$. It follows that $e^{\beta_n} \leqslant (1+\varepsilon)e^{\alpha_n}$ for $n > \eta_1$. Hence, if $\alpha_n \leqslant \lambda n$ for $n > \eta > \eta_1$, then we have $\beta_n \leqslant [\lambda + \ln(1+\varepsilon)]n$ for $n > \eta$ which proves the proposition □

We notice that if u is bounded, i.e. $\|u_n\|_r \leqslant \alpha$ for some $\alpha > 0$ for n large enough, then it is real sublinear. Clearly, if u is real sublinear then λu is also real sublinear if λ is a nonnegative real number. It is easily seen that if $u, v \in \mathcal{H}(\mathbb{T})$, then

$$\exp(u + v) = \exp u \times \exp v.$$

Moreover, since $\sup_{z\in C_r}(-\Re u_n(z)) = -\inf_{z\in C_r} \Re u_n(z)$, it follows that $(-u)$ is real sublinear if and only if $\inf_{z\in C_r} \Re u_n(z) \geqslant \mu n$ for some $\mu \in \mathbb{R}$ when n is large enough. Thus u and $(-u)$ are both real sublinear if and only if there are $\lambda, \mu \in \mathbb{R}$ such that

$$\mu n \leqslant \inf_{z\in C_r} \Re u_n(z) \leqslant \sup_{z\in C_r}(-\Re u_n(z)) \leqslant \lambda n.$$

Under this condition $\exp(u)$ and $\exp(-u)$ are invertible with

$$[\exp(u)]^{-1} = \exp(-u).$$

4.1.2. *The exponential of u for $\nu(u) < 1$.*

Theorem 4.2. *If $u \in \mathcal{H}(\mathbb{T})$ is such that $\nu(u) < 1$, then $\exp(u)$ is well defined in $\mathcal{H}(\mathbb{T})$ and is given by*

$$\exp(u) = \sum_{k=0}^{+\infty} \frac{u^k}{k!}.$$

Proof. Let $u \in \mathcal{H}(\mathbb{T})$ satisfy $\nu(u) < 1$ and choose any representative $(u_n)_n$ of u. Then we have:

$$\nu(u) = \lim_{r\to 1}(\limsup_{n\to\infty} \|u_n\|_r^{1/n}) < 1.$$

Hence, for every α such that $\nu(u) < \alpha < 1$, there exists $\rho > 1$ such that

$$\nu_\rho(u) = \limsup_{n\to\infty} \|u_n\|_\rho^{1/n} < \alpha,$$

and there exists $n_0 \in \mathbb{N}^*$ such that for every $n \geqslant n_0$:

$$\|u_n\|_\rho < \alpha^n < 1.$$

Hence, $(\|u_n\|_\rho)_n$ is bounded and then $\exp(u)$ is well defined. Moreover, since $\nu_\rho(\frac{u^k}{k!}) = \nu_\rho(u^k)$, if p and q are two integers such that $p > q$, it follows from $\nu_\rho(u) < 1$, that:

$$\nu_\rho\left(\sum_{k=q+1}^{p} \frac{u^k}{k!}\right) \leqslant \max_{q+1\leqslant k\leqslant p} \nu_\rho(\frac{u^k}{k!}) \leqslant [\nu_\rho(u)]^{q+1}.$$

Hence, we have $\lim_{q\to+\infty}[\nu_\rho(u)]^{q+1} = 0$ and then,

$$\lim_{p,q\to+\infty} \nu_\rho\left(\sum_{k=q+1}^{p} \frac{u^k}{k!}\right) = 0$$

showing that $\left(\sum_{k=0}^{m} \frac{u^k}{k!}\right)_m$ is a Cauchy sequence in $\mathcal{H}^\rho(\mathbb{T})$. Since $\mathcal{H}^\rho(\mathbb{T})$ is complete and the embedding $u_\rho : \mathcal{H}^\rho(\mathbb{T}) \to \mathcal{H}(\mathbb{T})$ is continuous, it follows that the series $\sum_{k\geqslant 0} \frac{u^k}{k!}$ converges in $\mathcal{H}(\mathbb{T})$ to $\exp(u)$. □

4.1.3. *The logarithm function.* Let $u \in \mathcal{H}(\mathbb{T})$ admit a representative (u_n) such that $u_n(C_r) \cap \mathbb{R}_- = \varnothing$ for $n > n_0$ for some $n_0 \in \mathbb{N}^*$. Then $\log(u_n)$ is holomorphic in C_r and for every $z \in C_r$, we have

$$\log(u_n(z)) = \ln|u_n(z)| + i\arg(u_n(z))$$

where arg denotes the principal determination of the argument function. If $\|u_n\|_r \leqslant a^n$ for $n > \eta$ for some $a > 1$ and $\eta \in \mathbb{N}^*$, then we have $\|\ln|u_n|\|_r \leqslant \ln\|u_n\|_r \leqslant n\ln a$. It follows that

$$\|\log(u_n)\|_r \leqslant n\ln a + 2\pi.$$

This shows that $(\log u_n)$ is a moderated sequence and $\log u = \mathrm{cl}(\log u_n)$ is real sublinear. Consequently, $\exp(\log u)$ is well defined, and one gets

$$\exp(\log u) = u. \tag{17}$$

The condition $u_n(C_r) \cap \mathbb{R}_- = \varnothing$ for the existence of $\log u$ depends on the chosen representative (u_n). Then it is necessary to get a sufficient one depending only on u.

Proposition 4.3. *Let $u \in \mathcal{H}(\mathbb{T})$ and let (u_n) denote a representative of u in some $O_r^{\mathbb{N}^*}$. Define*

$$d_r(u_n) = dist(u_n(C_r), \mathbb{R}_-) = \inf_{z\in C_r, \lambda\in\mathbb{R}_-} |u_n(z) - \lambda|. \tag{18}$$

Then $(d_r(u_n)) \in \mathcal{C}_e$ and $d_r(u) = \mathrm{cl}(d_r(u_n))$ is independent on the representative (u_n), and $d_r(u) \leqslant d_s(u)$ if $s < r$. Moreover if $d_r(u) \in \mathcal{C}^*$, then $\log u$ is well defined .

Proof. For every $z \in C_r$ and $\lambda \in \mathbb{R}_-$, we have

$$d_r(u_n) \leqslant |u_n(z) - \lambda|$$

and then $(d_r(u_n)) \in \mathcal{C}_e$. Let (g_n) denote another representative of u in $O_r^{\mathbb{N}^*}$. For every $z \in C_r$ and $\lambda \in \mathbb{R}_-$, writting $(g_n(z) - \lambda) - (u_n(z) - \lambda) = g_n(z) - u_n(z)$, gives

$$||g_n(z) - \lambda| - |u_n(z) - \lambda|| \leqslant |g_n(z) - u_n(z)|.$$

It follows that

$$d_r(g_n) \leqslant |g_n(z) - u_n(z)| + |u_n(z) - \lambda|$$

which leads to

$$|d_r(g_n) - d_r(u_n)| \leqslant |g_n(z) - u_n(z)| \leqslant \|g_n - u_n\|_r.$$

Whence $(d_r(g_n) - d_r(u_n)) \in \mathcal{I}_e$ i.e. $\mathrm{cl}(d_r(g_n)) = \mathrm{cl}(d_r(u_n))$. This shows that $\mathrm{cl}(_rd(u_n))$ does not depend on the representative (u_n) and then $d_r(u) = \mathrm{cl}(d_r(u_n))$ is well defined. Since $\{(z, \lambda) \in C_s \times \mathbb{R}_-\} \subset \{(z, \lambda) \in C_r \times \mathbb{R}_-\}$ if $s < r$, it follows that $d_r(u) \leqslant d_s(u)$. Now assume that $d_r(u)$ is an invertible element of $\mathcal{C}$. This means that:

$$\exists c \in (0,1), \exists n_0 \in \mathbb{N}^*, \forall n > n_0 : \mathrm{dist}(u_n(C_r), \mathbb{R}_-) \geqslant c^n. \tag{19}$$

Since $\mathrm{dist}(u_n(C_r), \mathbb{R}_-) > 0$ for $n > n_0$, it follows that $u_n(C_r) \cap \mathbb{R}_- = \varnothing$ for $n > n_0$ and then $\log u$ is well defined. □

Corollary 4.4. *Let $u \in \mathcal{H}(\mathbb{T})$. If $d_r(u)$ is invertible for some $r > 1$, then u is invertible.*

Proof. Let (u_n) denote a representative of u such that $u_n \in \mathcal{O}_r$. Since $\{(z, 0); z \in C_r\} \subset \{(z, \lambda) \in C_r \times \mathbb{R}_-\}$, it follows that $\inf_{z \in C_r} |u_n(z)| \geqslant d_r(u_n)$. Hence, if $d_r(u)$ is invertible, $\mathrm{cl}(\inf_{z \in C_r} |u_n(z)|)$ is invertible which means that u is invertible (see Section 2.3). □

Remark 4.1. *If $\xi \in \mathcal{C}$, we set $d(\xi) = \inf_{\lambda \in \mathbb{R}_-} |\xi - \lambda| = d_r(\xi)$ for any $r > 1$, ξ being considered as a constant generalized hyperfunction.*

4.1.4. *Series expansion of* $\log(1 + u)$ *for* $\nu(u) < 1$.

Theorem 4.5. *Let $u \in \mathcal{H}(\mathbb{T})$ be such that $\nu(u) < 1$. Then $\log(1 + u)$ is well defined in $\mathcal{H}(\mathbb{T})$ and is given by*

$$\log(1 + u) = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} u^k.$$

Proof. Let $u \in \mathcal{H}(\mathbb{T})$ satisfy $\nu(u) < 1$. It follows that there exists $\rho > 1$ such that $l = \limsup_{n\to+\infty} \|u_n\|_\rho^{1/n} < 1$. Taking α such that $l < \alpha < 1$, there exists $n_0 \in \mathbb{N}^*$ such that $\|u_n\|_\rho < \alpha^n$ for $n > n_0$. Hence, for every $z \in C_\rho$ and every $\lambda \in \mathbb{R}_-$, if $n > n_0$ we have

$$|(1 + u_n(z)) - \lambda| \geqslant (1 - \lambda) - \|u_n\|_\rho \geqslant (1 - \alpha)^n.$$

Hence, $1 + u \in \mathcal{H}^\rho(\mathbb{T})$ and $d_\rho(1 + u) \in \mathcal{C}^*$ for $n \geqslant n_0$. It follows from Proposition 4.3 that $\log(1 + u)$ is well defined. Since $\nu_\rho(\frac{(-1)^{k+1}u^k}{k}) = \nu_\rho(u^k)$, we can proceed as in the proof of Theorem 4.2 to show that $\left(\sum_{k=1}^{m} \frac{(-1)^{k+1}u^k}{k}\right)_m$ is a Cauchy sequence in $\mathcal{H}^\rho(\mathbb{T})$. Hence, the series $\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}u^k}{k}$ converges in $\mathcal{H}(\mathbb{T})$ to $\log(1 + u)$. □

4.1.5. *Power functions.* Let $h \in \mathcal{H}(\mathbb{T})$ such that $\log h$ exists and let $s \in \mathcal{H}(\mathbb{T})$. If $s \log h$ is real sublinear, we can calculate $\exp(s \log h)$, then we define

$$h^s = \exp(s \log h).$$

Let (s_n) and (h_n) be respective representatives of s and h in some $\mathcal{O}_r^{\mathbb{N}^*}$ with $d_r(h)$ invertible. If $\Re s_n = a_n$ and $\Im s_n = b_n$ then we have

$$\Re(s_n \log h_n) = a_n \ln |h_n| - b_n \arg h_n.$$

For instance if (a_n) is bounded and $b_n = O(n)$ then $s \log h$ is real sublinear. We note that if $s \in \mathbb{C}$, then $s \log h$ is always real sublinear and h^s is well defined.

Proposition 4.6. *Let $s \in \mathbb{R}$ such that $|s| \geqslant 1$ and $h \in \mathcal{H}(\mathbb{T})$. If $\log h$ exists, then the equation*

$$u^s = h \tag{20}$$

has a solution $u \in \mathcal{H}(\mathbb{T})$ given by $u = h^{1/s} = \exp(\frac{1}{s} \log h)$.

Proof. Since $\log h$ exists and $s \neq 0$, then $h^{1/s} = \exp(\frac{1}{s} \log h)$ is well defined. We show that $u = h^{1/s}$ is a solution to (20). Let (h_n) be a representative of h in some $\mathcal{O}_r^{\mathbb{N}^*}$ such that $h_n(C_r) \cap \mathbb{R}_- = \varnothing$ for every $n \in \mathbb{N}^*$. We have

$$\begin{aligned}
\exp(\tfrac{1}{s} \log h_n) &= \exp\left(\tfrac{1}{s}(\ln |h_n| + i \arg h_n)\right); \\
&= \exp(\tfrac{1}{s} \ln |h_n|) \exp\left(\tfrac{i}{s} \arg h_n\right); \\
&= |h_n|^{1/s} \exp\left(\tfrac{i}{s} \arg h_n\right).
\end{aligned}$$

Since $|s| \geqslant 1$, it follows that $\frac{1}{s} \arg h_n \in (-\pi, \pi)$ and then

$$\arg\left(\exp(\frac{1}{s} \log h_n)\right) = \frac{1}{s} \arg h_n.$$

Thus $\exp(\frac{1}{s}\log h_n)(C_r)\cap\mathbb{R}_- = \varnothing$ for every n and then $\log\left[\exp\left(\frac{1}{s}\log h_n\right)\right]$ is well defined and

$$\begin{aligned}\log\left[\exp\left(\tfrac{1}{s}\log h_n\right)\right] &= \tfrac{1}{s}\ln|h_n| + \tfrac{i}{s}\arg h_n;\\ &= \tfrac{1}{s}\log h_n.\end{aligned}$$

It follows that $s\log\left[\exp(\frac{1}{s}\log h_n)\right] = \log h_n$. Then we have

$$\exp\left[s\log\left(\exp(\frac{1}{s}\log h_n)\right)\right] = h_n$$

which gives

$$\begin{aligned}(h^{1/s})^s &= \exp(s\log h^{1/s});\\ &= \exp\left[s\log\left(\exp(\tfrac{1}{s}\log h)\right)\right] = h\end{aligned}$$

and proves the result. □

Let $\mathcal{Z}$ denote the subring of generalized integers, that is

$$\mathcal{Z} = \{\tilde{z}\in\mathcal{C}, \exists(z_n)_n\in\mathbb{Z}^{\mathbb{N}^*}\cap\mathcal{C}_e : \mathrm{cl}(z_n) = \tilde{z}\}.$$

Then, we have the following.

Proposition 4.7. *Let $s\in(-1,1)$ and $h\in\mathcal{H}(\mathbb{T})$ such that $\log h$ exists. Then, there exists a generalized hyperfunction p valued in $\mathcal{Z}$ and such that*

$$(h^{1/s})^s = (e^{-is\pi})^{2p}h. \tag{21}$$

Proof. Keep the notation of Proposition 4.6 and set

$$\frac{\arg h_n}{s} = 2p_{n,s}\pi + \frac{\theta_{n,s}}{s} \tag{22}$$

where $p_{n,s}(z)\in\mathbb{Z}$ and $|\theta_{n,s}(z)| < |s|\pi$ for $z\in C_r$. Since

$$\frac{1}{s}\log h_n = \ln|h_n|^{1/s} + i\frac{\arg h_n}{s},$$

it follows that

$$\exp(\frac{1}{s}\log h_n) = |h_n|^{1/s}\exp\left(\frac{i\theta_{n,s}}{s}\right).$$

Thus we have

$$\ln\left(\exp(\frac{1}{s}\log h_n)\right) = \frac{1}{s}\ln|h_n| + \frac{i\theta_{n,s}}{s}$$

and then

$$\begin{aligned}s\log\left[\exp\left(\tfrac{1}{s}\log h_n\right)\right] &= \ln|h_n| + i\theta_{n,s};\\ &= \ln|h_n| + i\arg h_n - 2isp_{n,s}\pi,\end{aligned}$$

that is

$$s\log\left[\exp\left(\frac{1}{s}\log h_n\right)\right] = \log h_n - 2isp_{n,s}\pi. \qquad (23)$$

The above equality gives

$$p_{n,s} = \frac{\log h_n - s\log\left(\exp(\frac{1}{s}\log h_n)\right)}{2is\pi}$$

which shows that $p_{n,s}$ is a holomorphic function in C_r. Since $p_{n,s}$ takes its values in $\mathbb{Z}$ and C_r is a connected space, it follows that for each $n \in \mathbb{N}^*$, $p_{n,s}$ is constant. The above equality also shows that $(p_{n,s})_n$ is moderated, but using (22) yields

$$p_{n,s} = \frac{\arg h_n - \theta_{n,s}}{2s\pi}.$$

Then, since $|\arg h_n| < \pi$ and $|\theta_{n,s}| < |s|\pi$, we obtain precisely that

$$\|p_{n,s}\|_r \leqslant \frac{1+|s|}{2|s|}$$

which shows that $(p_{n,s})_n \in \mathcal{X}_e^r$ and allows us to define

$$p = \mathrm{cl}(p_{n,s}).$$

Equality (23) also gives

$$\exp\left[s\log\left(\exp(\frac{1}{s}\log h_n)\right)\right] = (e^{-is\pi})^{2p_{n,s}}h_n.$$

It follows from $|s\pi| < \pi$ that $e^{-is\pi}$ has a logarithm and then $(e^{-is\pi})^{2p}$ is well defined as mentioned at the beginning of Section 4.1.5. Hence, we have

$$(h^{1/s})^s = \exp(s\log h^{1/s}) = (e^{-is\pi})^{2p}h.$$

The proposition is thus proved. □

The proof of Proposition 4.6 shows that the invertibility of $d_r(h)$ implies that $\exp(\frac{1}{s}\log h_n)(C_r)\cap\mathbb{R}_- = \varnothing$ for n is large enough. In fact, we have:

Proposition 4.8. *Let $h \in \mathcal{H}(\mathbb{T})$ such that $d_r(h)$ is invertible for some $r > 1$. If s is a real number such that $|s| \geqslant 1$, then $d_r\left(\exp(\frac{1}{s}\log h)\right)$ is also invertible.*

Proof. Let (h_n) be a representative of h in $\mathcal{O}_r^{\mathbb{N}^*}$. We have

$$d_r{}^2\left(\exp(\frac{1}{s}\log h_n)\right) = \inf_{z\in C_r,\lambda\in\mathbb{R}_-}\left|\left(\exp(\frac{1}{s}\log h)\right) - \lambda\right|^2.$$

For z fixed in C_r, set $\rho_n = |h_n(z)|^{1/s}$ and $\theta_n = \arg h_n(z)$. Then we get

$$\begin{aligned} {d_r}^2\left(\exp(\tfrac{1}{s}\log h_n)\right) &= \inf_{z\in C_r,\lambda\in\mathbb{R}_-}\left|\rho_n\cos\left(\tfrac{\theta_n}{s}\right)-\lambda+i\rho_n\sin\left(\tfrac{\theta_n}{s}\right)\right|^2 \\ {d_r}^2\left(\exp(\tfrac{1}{s}\log h_n)\right) &= \inf_{z\in C_r,\lambda\in\mathbb{R}_-}\{\left(\lambda-\rho_n\cos\left(\tfrac{\theta_n}{s}\right)\right)^2+{\rho_n}^2\sin^2\left(\tfrac{\theta_n}{s}\right)\}. \end{aligned}$$

Set $f(\lambda) = \left(\lambda-\rho_n\cos\left(\frac{\theta_n}{s}\right)\right)^2+{\rho_n}^2\sin^2\left(\frac{\theta_n}{s}\right)$ where $\lambda \leqslant 0$. Then f is a derivable function of λ and $f'(\lambda) = 2\left(\lambda-\rho_n\cos\left(\frac{\theta_n}{s}\right)\right)$.

If $\cos\left(\frac{\theta_n}{s}\right) \geqslant 0$, then $f'(\lambda) \leqslant 0$ and f reaches its minimum ${\rho_n}^2$ at $\lambda = 0$;

If $\cos\left(\frac{\theta_n}{s}\right) < 0$, then f reaches its minimum ${\rho_n}^2\sin^2\left(\frac{\theta_n}{s}\right)$ at $\lambda = \rho_n\cos\left(\frac{\theta_n}{s}\right)$.

The condition $\cos\left(\frac{\theta_n}{s}\right) < 0$ implies that $\frac{\pi}{2} < \left|\frac{\theta_n}{s}\right| < \frac{\pi}{|s|}$ and then $\sin^2\left(\frac{\theta_n}{s}\right) > \sin^2\left(\frac{\pi}{s}\right)$. It follows that in any case,

$$\inf_{\lambda\in\mathbb{R}_-} f(\lambda) \geqslant {\rho_n}^2\sin^2\left(\frac{\pi}{s}\right)$$

and then

$$d_r\left(\exp(\frac{1}{s}\log h_n)\right) \geqslant \sin\left(\frac{\pi}{|s|}\right)\inf_{z\in C_r}|h_n(z)|^{1/s}. \tag{24}$$

We notice that $\sin\left(\frac{\pi}{|s|}\right) \neq 0$ if $|s| > 1$. Since $d_r(h)$ is invertible, it follows from Corollary 4.4 that h is invertible, which means that there are $e \in (0,1)$ and $n_0 \in \mathbb{N}^*$ such that $\inf_{z\in C_r}|h_n(z)| \geqslant e^n$ if $n > n_0$. If $s > 1$, using (24), we have that $d_r\left(\exp(\frac{1}{s}\log h_n)\right) \geqslant (b^{1/s})^n$ for some $b \in (0,1)$ and n large enough. If $s < -1$, since h^{-1} is invertibe with (h_n^{-1}) as representative and

$$\inf_{z\in C_r}|h_n(z)|^{1/s} = \inf_{z\in C_r}|h_n^{-1}(z)|^{1/|s|},$$

it follows that $d_r\left(\exp(\frac{1}{s}\log h_n)\right) \geqslant (c^{1/|s|})^n$ for some $c \in (0,1)$ and n large enough. Thus $d_r\left(\exp(\frac{1}{s}\log h)\right)$ is invertible for $|s| > 1$.

If $s = 1$, we have $\exp(\log h) = h$ which is invertible. If $s = -1$, since

$$-\log h_n(z) = \ln|h_n(z)|^{-1} + i\arg h_n^{-1}(z) = \log h_n^{-1}(z),$$

it follows that $\exp(-\log h) = \exp(\log h^{-1}) = h^{-1}$ which is invertible. The proposition is thus proved. □

4.2. **Application to nonlinear differential equations.** Consider the nonlinear ordinary differential equation:

$$\partial_\theta h - uh^s = 0. \tag{25}$$

Proposition 4.9. *Assume that $u \in \mathcal{H}(\mathbb{T})$ satisfies $\hat{u}(0) = 0$ and $s \in (-\infty, 0] \cup [2, +\infty)$. If $U \in \mathcal{H}(\mathbb{T})$ is a primitive of u with respect to ∂_θ, there exists $\rho > 1$ and $\mu \in \mathcal{C}^*$ such that $d_\rho((1-s)U + \mu) \in \mathcal{C}^*$, and*

$$h = ((1-s)U + \mu)^{1/(s-1)}$$

is a solution to (25).

Proof. Since $\hat{u}(0) = 0$, Proposition 2.2 implies that there exists $U \in \mathcal{H}(\mathbb{T})$ such that $\partial_\theta U = u$. Then (25) is formally equivalent to

$$\frac{\partial_\theta h}{h^s} = \partial_\theta U. \tag{26}$$

On the other hand, we have:

$$\partial_\theta \left(\frac{1}{h^{s-1}}\right) = \frac{(1-s)\partial_\theta h}{h^s}$$

which gives

$$\partial_\theta (h^{s-1} - (1-s)U) = 0.$$

Thus, there exists a constant $\mu \in \mathcal{C}$ such that

$$h^{s-1} = (1-s)U + \mu. \tag{27}$$

Let $a > \nu(U) + 1, \nu(U) < b < a, \alpha > 0$ and take $\mu = \mathrm{cl}(\mu_n)$ with $\mu_n = a^n + \alpha^n$. If (U_n) is any representative of U, there are $\rho > 0$ and $\eta \in \mathbb{N}^*$ such that:

$$\|U_n\|_\rho < b^n,\ n > \eta.$$

For every $\lambda \in \mathbb{R}_-$ and $z \in C_\rho$, if $n > \eta$, we have

$$\begin{aligned}
|(1-s)U_n(z) + \mu_n - \lambda| &\geqslant \mu_n - \lambda - |1-s||U_n(z)| \\
&\geqslant \mu_n - \lambda - |1-s|\|U_n\|_\rho \\
&\geqslant a^n + \alpha^n - \lambda - |1-s|b^n \\
&\geqslant (a^n - |1-s|b^n - \lambda) + \alpha^n.
\end{aligned}$$

It follows from the hypotheses that $a^n - |1-s|b^n - \lambda \geqslant 0$ for n large enough which implies that $|(1-s)U_n(z) + \mu_n - \lambda| \geqslant \alpha^n$ for such n. Then we have

$$d_\rho((1-s)U + \mu) \in \mathcal{H}^*(\mathbb{T}).$$

Thus, $\log((1-s)U + \mu)$ is well defined. Using Proposition 4.6 and (27), we get that

$$h = ((1-s)U + \mu)^{1/(s-1)}$$

is effectively a solution to (25). □

Now consider the nonlinear Cauchy problem:

$$\left\{\begin{array}{rcl}\partial_\theta h - uh^s & = & 0\\ h(\zeta) & = & \tau\end{array}\right. \tag{28}$$

where $\zeta \in \tilde{\mathbb{T}}$, $\tau \in \mathcal{C}$ and $d(\tau) \in \mathcal{C}^*$. Then, we have:

Theorem 4.10. *Let $u \in \mathcal{H}(\mathbb{T})$ satisfy $\hat{u}(0) = 0$ and $s \in (-\infty, 0] \cup [2, +\infty)$. Assume that there exist $c > |1-s|\pi$ and $\varepsilon > 0$ such that*

$$d(\tau^{s-1}) - c\gamma_\varepsilon(u) \geqslant \alpha \tag{29}$$

for some positive real $\alpha \in \mathcal{C}^*$ where $\gamma_\varepsilon(u) = \mathrm{cl}((\nu(u)+\varepsilon)^n)$. Then, (28) has a solution in $\mathcal{H}(\mathbb{T})$.

Proof. We keep the notation of Proposition 4.9 and we set

$$w = ((1-s)U + \beta)^{1/(s-1)} \tag{30}$$

where $\beta \in \mathcal{C}$. We show that β can be chosen for w to be a solution to (28). Recall that $\zeta \in \tilde{\mathbb{T}}$ means that it has a representative $(\zeta_n)_n$ in $\mathbb{T}^{\mathbb{N}^*}$. Set $\zeta_n = e^{i\theta_n}$ and take $r > 1$ such that

$$(r-1)r < \frac{c}{|1-s|} - \pi. \tag{31}$$

For $\theta \in [-\pi, \pi]$, we set $z' = e^{i\theta} \in \mathbb{T}$ and $z = \rho e^{i\theta}$ where ρ varies in $(1/r, r)$; thus we have $z \in C_r$. We denote by κ_z the path from ζ_n through z' arriving at z whose image is the union of the circle arc $\widehat{\zeta_n, z'}$ and the line segment $[z'z]$. Let $(U_n)_n$ be a representative of U; then we have

$$\begin{array}{rcl}U_n(z) - U_n(\zeta_n) & = & \int_{\kappa_z} U_n'(\xi)d\xi\\ & = & \int_{\widehat{\zeta, z'}} U_n'(\xi)d\xi + \int_{[z',z]} U_n'(\xi)d\xi.\end{array}$$

If $u_n(z) = \partial_\theta U_n(z)$, then $(u_n)_n$ is a representative of u and

$$U_n'(\xi) = -i\frac{\partial_\theta U_n(\xi)}{\xi} = -i\frac{u_n(\xi)}{\xi},$$

whence we find that

$$U_n(z) - U_n(\zeta_n) = -i\int_{\widehat{\zeta, z'}} \frac{u_n(\xi)}{\xi}d\xi - i\int_{[z',z]} \frac{u_n(\xi)}{\xi}d\xi.$$

The length $|\theta - \theta_n|$ of $\widehat{\zeta, z'}$ will be chosen such that $|\theta - \theta_n| \leqslant \pi$. We notice that $|z - z'| < \max(1 - \frac{1}{r}, r-1) = r-1$ since $r > 1$. Then, using $1/|\xi| < r$ if $\xi \in [z', z]$ and $|\xi| = 1$ if $\xi \in \mathbb{T}$, we find that

$$\begin{array}{rcl}|U_n(z) - U_n(\zeta_n)| & \leqslant & |\theta - \theta_n| \sup_{\xi \in \widehat{\zeta, z'}} \left|\frac{u_n(\xi)}{\xi}\right| + |z - z'| \sup_{\xi \in [z', z]} \left|\frac{u_n(\xi)}{\xi}\right|\\ & \leqslant & |\theta - \theta_n| \sup_{\xi \in \mathbb{T}} |u_n(\xi)| + (r-1)r\|u_n\|_r.\end{array}$$

Thus we get

$$|U_n(z) - U_n(\zeta_n)| \leqslant (\pi + (r-1)r)\|u_n\|_r. \qquad (32)$$

Let $(\tau_n)_n$ be a representative of τ. Writting $w(\zeta) = \tau$, we find that

$$\beta = -(1-s)U(\zeta) + \tau^{s-1}$$

and then, for every $\lambda \in \mathbb{R}_-$,

$$\begin{aligned} |(1-s)U_n(z) + \beta_n - \lambda| &= |(1-s)U_n(z) - (1-s)U(\zeta_n) + \tau_n^{s-1} - \lambda| \\ &= |(1-s)(U_n(z) - U(\zeta_n)) + \tau_n^{s-1} - \lambda| \end{aligned}$$

where $\beta_n = -(1-s)U(\zeta_n) + \tau_n^{s-1}$. It follows that

$$|(1-s)U_n(z) + \beta_n - \lambda| \geqslant |\tau_n^{s-1} - \lambda| - |1-s|(\pi + (r-1)r)\|u_n\|_r$$

and then

$$d_r((1-s)U_n + \beta_n) \geqslant d(\tau_n^{s-1}) - |1-s|(\pi + (r-1)r)\|u_n\|_r.$$

There exists $n_0 \in \mathbb{N}^*$ such that $\|u_n\|_r < (\nu(u) + \varepsilon)^n$ if $n > n_0$, whence

$$d_r((1-s)U_n + \beta_n) \geqslant d(\tau_n^{s-1}) - |1-s|(\pi + (r-1)r)(\nu(u) + \varepsilon)^n$$

for $n > n_0$. It follows from (31) that $|1-s|(\pi+(r-1)r) < c$. Then, using (29) we get that $d_r((1-s)U+\beta) \geqslant \alpha$ which shows that $d_r((1-s)U+\beta)$ is invertible. Thus w is well defined by (30) and is a solution to (28). □

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