Scoring Rules and Scoring Runoff Systems: Susceptibility to Sincere Truncation under three scoring models for incomplete preferences

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Keywords Truncation · Rankings · Scoring model · Probability · Paradox · Impartial and Anonymous Culture

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1 Introduction

Consider a voting situation in which voters are asked to rank (sincerely) the competing candidates from the most preferred to the least preferred and they are allowed to submit incomplete rankings. If there are some ballot configurations where at least one voter prefers the outcome obtained when he submits a sincere but incomplete ranking (truncated ranking) to the outcome obtained when he casts a complete sincere ranking, such configurations define the *truncation paradox* also known as the *sincere truncation of preferences*. This voting paradox was first introduced in the social choice literature by Brams (1982). In these configurations, all the candidates not ranked or listed on a ballot are assumed to be less preferred than all those who are ranked (Fishburn and Brams, 1983, 1984).

It is established that almost all the well-known voting rules are vulnerable to the truncation paradox. The few exceptions are the *Plurality rule*, *Plurality runoff* and *Approval voting*. For a non-exhaustive list of the voting rules vulnerable to the truncation paradox, the reader may refer to Felsenthal (2012), Nurmi (1999) and Fishburn and Brams (1984). The truncation paradox appears as a weak version of the no-show paradox; the no-show paradox occurs when a voter or a group of voter may do better to abstain than to vote since abstaining may result in the victory of a more preferable or desirable candidate. A voting rule that is vulnerable to the no-show paradox is also vulnerable to the truncation paradox, but the reverse is not necessary true (see Nurmi, 1999). The no-show paradox has been the subject of a fairly abundant literature; we refer to Kamwa et al. (2018) for a recent overview. Fishburn and Brams (1984, p.402) showed, as a consequence of Moulin's theorem (Moulin, 1988), that all the Condorcet consistent rules are sensitive to the truncation paradox. A Condorcet consistent rule always picks the Condorcet winner when she exists. A Condorcet winner, when he exists, is a candidate who defeats each of his opponents in pairwise comparisons.

Since the truncation paradox affects almost all the well-known voting rules, it would be interesting to consider its probabilities of occurrence. In social choice theory, the probabilities of occurrence of voting paradoxes can serve as a discriminating criterion between voting rules. This approach is complementary to the axiomatic approach which discriminates the voting rules on the basis of the normative properties which they satisfy or not. As part of the probabilistic approach, some papers have tried to analyze the impact of strategic manipulation by truncation. For the presentation of these works, we do it without being exhaustive. Baumeister et al. (2012), Menon and Larson (2017) and Narodytska and Walsh (2014) were interested in evaluating the feasibility and the complexity of the manipulation by truncation of a certain number of voting systems amongst other the family of scoring rules, scoring rules with runoff and some Condorcet consistent rules. Plassmann and Tideman (2014) evaluated the likelihood of the *strong truncation paradox* which occurs if one voter reports only part of his ranking, then a candidate will win whom the voter ranks higher than the candidate who will win if the voter reports his complete ranking of the candidates. For their analysis, Plassmann and Tideman (2014) used simulations based on the spatial model for drawing three-candidate voting situations with size of the electorates varying from ten to a million; they focused on some Condorcet consistent rules, some scoring rules and some iterative scoring rules. They found that the likelihood of the strong truncation paradox tends to decrease as the number of voters increases and that the Borda rule is less vulnerable than the Antiplurality rule. Kilgour et al. (2019) assessed the significance of ballot truncation in ranked-choice elections with four, five and six candidates using intensive simulations on real data under both spatial and random models of voter preferences. Kamwa and Moyouwou (2020) characterized for three-candidate elections and large electorates, all the voting situations where the truncation paradox can occur for the whole family of one-shot and runoff scoring rules and they computed for these family of rules, the likelihood of the truncation paradox under the impartial and anonymous culture (defined later).

It is worth noting that Plassmann and Tideman (2014), Kilgour et al. (2019) and Kamwa and Moyouwou (2020) in their respective assessments of the probabilities of the truncation paradox implicitly assume that when a voter submits a truncated ballot, only the candidate indicated on the ballot receives points from this voter while the others receive no points. This way of proceeding with incomplete preferences is known in the literature as the *pessimistic approach* (Baumeister et al., 2012). This approach is obviously not the only one possible; at least two other approaches are encountered in the literature: the *optimistic approach* (Baumeister et al., 2012, Saari, 2008) and the *averaged approach* inspired by Dummett (1997). For an overview of schemes for dealing with incomplete preferences in collective decision, the reader may refer among others to Baumeister et al. (2012), Kruger and Terzopoulou (2020), Menon and Larson (2017), Narodytska and Walsh (2014) and Terzopoulou and Endriss (2020, 2019). In this paper, we supplement the work of Kamwa and Moyouwou (2020) by characterizing for three-candidate elections and large electorates, all the voting situations where the truncation paradox can occur for the whole family of one-shot and runoff scoring rules under the averaged and the optimistic approach. Then, we computed for these family of rules, the likelihood of the truncation paradox under the impartial and anonymous culture for each of the schemes for dealing with truncated ballots. Our ambition is to highlight that the occurrence of the truncation paradox is strongly impacted by the model chosen to deal with incomplete preferences. This is indeed what we have achieved as we will see in our results.

The rest of the paper is organized as follows: Section 2 is devoted to basic definitions. Given a three-candidate election where voters have strict rankings we characterize, in Section 3, all the voting situations where the truncation paradox can occur for all the one-shot scoring rules under each of the three scoring models for incomplete preferences; then we compute the likelihood of the truncation paradox. We do the same job in Section 4 for all the runoff scoring rules Section 5 concludes. All the proof details are relegated to the appendices.

2 Notation and definitions

2.1 Preferences

Let N be a set of n voters ($n \geq 2$) and $A = \{a, b, c\}$ a set of three candidates. Individual preferences are linear orders, these are complete, asymmetric and transitive binary relations on A. With three candidates, there are exactly 6 linear orders $P_1, P_2, \ldots, P_6$ on A. A *voting situation* is a 6-tuple $\pi = (n_1, n_2, ..., n_t, ..., n_6)$ that indicates the total number n_t of voters casting each of the complete linear orders such that $\sum_{t=1}^{6} n_t = n$. We will simply write abc to denote the linear order on A according to which a is strictly preferred to b, b is strictly preferred to c; and by transitivity a is strictly preferred to c. Table 1 describes a voting situation on $A = \{a, b, c\}$.

Table 1 Possible strict rankings on $A = \{a, b, c\}$

$n_1 : abc$	$n_2 : acb$	$n_3 : bac$	$n_4 : bca$	$n_5 : cab$	$n_6 : cba$

Given $a, b \in A$ and a voting situation π, we denote by $n_{ab}(\pi)$ (or simply n_{ab}) the total number of voters who strictly prefer a to b. If $n_{ab} > n_{ba}$, we say that candidate a majority dominates candidate b; or equivalently, a beats b in a pairwise majority voting. In such a case, we will simply write $a\mathbf{M}b$.

Candidate a is said to be the *Condorcet winner* (resp. the *Condorcet loser*) if $a\mathbf{M}b$ and $a\mathbf{M}c$ (resp. $b\mathbf{M}a$ and $c\mathbf{M}a$). Table 2 gives the matrix of the pairwise comparisons on $A = \{a, b, c\}$ given the preferences of Table 1.

Table 2 Matrix of pairwise comparisons on $A = \{a, b, c\}$

vs	a	b	c
a	$-$	$n_1 + n_2 + n_5$	$n_1 + n_2 + n_3$
b	$n_3 + n_4 + n_6$	$-$	$n_1 + n_3 + n_4$
c	$n_4 + n_5 + n_6$	$n_2 + n_5 + n_6$	$-$

2.2 One-shot scoring rules and runoff scoring rules

Scoring rules are voting systems that give points to candidates according to the position they have in voters' rankings. For a given scoring rule, the total number of points received by a candidate defines his score for this rule. The winner is the candidate with the highest score. In general, with three candidates and complete strict rankings, a scoring vector is a 3-tuple $w = (w_1, w_2, w_3)$ of real numbers such that $w_1 \geq w_2 \geq w_3$ and $w_1 > w_3$. Given a voting situation π, each candidate receives w_k each time she is ranked k^{th} $(k = 1, 2, 3)$ by a voter. The score of a candidate $a \in A$ is the sum $S(\pi, w, a) = \sum_{t=1}^{6} n_t w_{r(t,a)}$ where $r(t, a)$ is the rank of candidate a according to voters of type t.

A normalized scoring vector has the shape $w_\lambda = (1, \lambda, 0)$ with $0 \leq \lambda \leq 1$. For $\lambda = 0$, we obtain the *Plurality rule*. For $\lambda = 1$, we have the *Antiplurality rule* and for $\lambda = \frac{1}{2}$, we have the *Borda rule*. From now on, we will denote by $S(\pi, \lambda, a)$ the score of candidate a when the scoring vector is $w_\lambda = (1, \lambda, 0)$ and the voting situation is π; without loss of generality, w_λ will be used to refer to the voting rule. Table 3 gives the score of each candidate in $A = \{a, b, c\}$ given the voting situation of Table 1.

Table 3 Scores of candidates

$S(\pi, \lambda, a) = n_1 + n_2 + \lambda(n_3 + n_5)$
$S(\pi, \lambda, b) = n_3 + n_4 + \lambda(n_1 + n_6)$
$S(\pi, \lambda, c) = n_5 + n_6 + \lambda(n_2 + n_4)$

In one-shot voting, the winner is just the candidate with the largest score. Runoff systems involve two rounds of voting: at the *first round*, the candidate with the smallest score is eliminated; at the *second round*, a majority contest determines who is the winner.

2.3 Dealing with truncated preferences: the scoring models

In our setting, we assume that voters sincerely provides complete strict rankings on the competing candidates. As there is only three candidates, when a voter of a given type wants to manipulate the outcome by truncation, he just states his top ranked candidate and erase the others. For example with Table 1, if some voters with the ranking abc truncate, this leads to a new voting situation π' in which these voters only state $a - -$ as their ranking. In the truncated ballot, all the candidates not ranked are assumed to be less preferred to the one who is ranked.

When the votes are top-truncated, we have to modify the point-assignment procedure in a certain way in order to deal with the truncated rankings. To have an overview on how to proceed in general, the reader may refer to Baumeister et al. (2012), Dummett (1997), Kruger and Terzopoulou (2020), Saari (2008), Terzopoulou and Endriss (2020, 2019). In our framework, the vector $w_\lambda = (1, \lambda, 0)$ needs to be modified for truncated rankings according to the following models:

- The *pessimistic scoring model*:

Assume that a voter with the ranking abc truncates and submits $a - -$. Under the pessimistic scoring model, considering this incomplete ranking, candidate a will receive 1 point in the new voting situation while b and c both receive zero point. Thus, for three-candidate elections, the scoring vector applied to truncated rankings under pessimistic scoring model is $w'_\lambda = (1, 0, 0)$.

- The *optimistic scoring model*:

Under the optimistic scoring model, if a voter with the ranking abc truncates and submits $a - -$, in the new voting situation candidate a will still receive 1 point while b and c both receive λ point. Thus, for three-candidate elections, the scoring vector applied to truncated rankings under the optimistic scoring model is $w'_\lambda = (1, \lambda, \lambda)$.

- The *averaged scoring model*:

Under this model, if for example a voter of type 1 with the ranking abc truncates and submits $a - -$, in the new voting situation candidate a will still receive 1 point while b and c both receive $\frac{\lambda}{2}$ point. Thus, the scoring vector applied to truncated rankings under the averaged scoring model is $w'_\lambda = (1, \frac{\lambda}{2}, \frac{\lambda}{2})$ in three-candidate elections.

According to Baumeister et al. (2012), among the models just listed, the pessimistic scoring model is the most popular one in practice; they also noticed that one of the drawbacks of the pessimistic scoring model is that it gives incentives for voters to rank only a single candidate so that the impact of the vote on the score of this candidate, relative to the scores of other candidates, is greatest. On the other hand, the optimistic model rewards the voters who rank more candidates: the more candidates one ranks, the more points (in relative terms) these candidates receive.

Note that in three-candidate elections, when some voters truncate, *(i)* under the pessimistic scoring model: only the scores of candidates ranked second by some of these voters are affected and diminish; *(ii)* under the optimistic scoring model: only the scores of candidates ranked last by some of these voters are affected and increase; *(iii)* under the averaged scoring model: the scores of candidates ranked second by some of these voters diminish while those of candidates ranked last by some of these voters increase. Moreover, truncation is only possible at the first round under runoff systems.

In our setting, we assume that ties among candidates will be broken alphabetically, e.g. a wins all ties against other candidates; while b wins all ties against c. Note that this special tie-breaking rule does not affect our results as we only deal with voting situations where the total number of voters tends to infinity.

2.4 Probabilistic model: the impartial and anonymous culture assumption

The impartial and anonymous culture (IAC) is one of the most used assumptions used in social choice theory literature when computing the likelihood of voting events. Under IAC, first introduced by Kuga and Hiroaki (1974) and later developed by Gehrlein and Fishburn (1976), the likelihood of a given event is calculated with respect to the ratio between the number of voting situations in which the event is likely over the total number of possible voting situations. It is known that the total number of possible voting situations in three-candidate elections is given by the following five-degree polynomial in n: $C^n_{n+3!-1} = \frac{(n+5)!}{n!5!}$. The number of voting situations associated with a given event can be reduced to the solutions of a finite system of linear constraints with rational coefficients. As recently pointed out in the social choice literature, the appropriate mathematical tools to find these solutions are the Ehrhart polynomials. The background of this notion and its connection with the polytope theory can be found in Gehrlein and Lepelley (2017, 2011), Lepelley et al. (2008), and Pritchard and Wilson (2007). This technique has been widely used in numerous studies analyzing the probability of electoral events in the case of three-candidate elections under the IAC assumption. As we deal only with the probability with large electorates, we follow a procedure that was developed in Cervone et al. (2005) and recently used in many research papers such as Diss and Gehrlein (2015, 2012), Diss et al. (2020, 2018, 2012), Gehrlein et al. (2015), Kamwa (2019), Kamwa and Moyouwou (2020), Moyouwou and Tchantcho (2015) among others. This technique is based on the computation of polytopes' volumes. We say some few words on this technique in Appendix C.

3 The vulnerability of one-shot scoring rules to the truncation paradox

Consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{a, b, c\}$ and the one-shot rule with $0 < \lambda \leq 1$. Let $\pi([R_{j_1}, R_{j_2}, ...])$ stands for the voting situation obtained from π when all type $R_{j_1}, R_{j_2}, ...$ voters truncate their preferences. For example, $\pi[abc]$ differs from π only in the fact that at $\pi[abc]$, candidate a receives 1 point from each type 1 voter while the two other receive

0 point under the pessimistic scoring model, λ points under the optimistic scoring model and $\frac{\lambda}{2}$ points under the averaged scoring model. Similarly, from π to $\pi\,[abc, acb]$ the only change that occurs is that all type 1 voters and all type 2 voters now truncate their preferences to report $a \ldots$.

For one-shot scoring rules, Proposition 1 tells us that when operating under the optimistic scoring model, the truncation paradox vanishes.

Proposition 1 *For three-candidate elections, all the one-shot scoring rules are immune to manipulation by sincere truncation of preferences when the optimistic scoring model is assumed.*

Proof See Appendix A.

Proposition 2 identifies all the voting situations in which the truncation paradox is possible under both the pessimistic and the averaged scoring models; Propositions 3 and 4 provide the likelihood of the truncation paradox under these two scoring models. Let us notice that in Proposition 3 we obtain the same result as Kamwa and Moyouwou (2020) who implicitly dealt only with the pessimistic scoring model.

Proposition 2 *Consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{a, b, c\}$, the one-shot rule associated with $0 < \lambda \leq 1$ and a pair $\{x, y\}$ of candidates with $A \setminus \{x, y\} = \{z\}$.*

If x is the election winner at π, then the truncation paradox is likely to occur at π in favor of y under the pessimistic or the averaged scoring model if and only if y is the election winner at $\pi\,([yxz, yzx])$.

Proof See Appendix B.

Proposition 3 *Consider a one-shot scoring rule F_λ, $0 < \lambda \leq 1$. As the total number n of voters tends to infinity, the limit probability of observing a voting situation in which the truncation paradox may occur under the pessimistic scoring model is given by:*

$$\text{if } 0 < \lambda \leq \tfrac{1}{2}, \quad P_{Pe}(F_\lambda) = \frac{\begin{pmatrix} 10\,\lambda^{14} - 37\,\lambda^{13} - 179\,\lambda^{12} + 1310\,\lambda^{11} - 1778\,\lambda^{10} - 6319\,\lambda^{9} \\ +26773\,\lambda^{8} - 25735\,\lambda^{7} - 67880\,\lambda^{6} + 259941\,\lambda^{5} - 408078\,\lambda^{4} \\ +356643\,\lambda^{3} - 166536\,\lambda^{2} + 31833\lambda \end{pmatrix}}{6(3+\lambda)^2(3-2\,\lambda+\lambda^2)^2(\lambda-2)^2(2\,\lambda-3)^2(\lambda-1)(-3+5\,\lambda)}$$

$$\text{if } \tfrac{1}{2} < \lambda \leq 1, \quad P_{Pe}(F_\lambda) = \frac{\begin{pmatrix} 2\,\lambda^{13} + 50\,\lambda^{12} - 194\,\lambda^{11} - 190\,\lambda^{10} + 2548\,\lambda^{9} - 5560\,\lambda^{8} \\ -662\,\lambda^{7} + 26915\,\lambda^{6} - 62174\,\lambda^{5} + 73636\,\lambda^{4} - 48132\,\lambda^{3} \\ +16425\,\lambda^{2} - 3564\,\lambda + 324 \end{pmatrix}}{12(3+\lambda)^2(3-2\,\lambda+\lambda^2)^2(\lambda-2)^2\lambda^2(2\,\lambda-3)}$$

Proof See Appendix C for details of computations.

Proposition 4 *Consider a one-shot scoring rule F_λ, $0 < \lambda \leq 1$. As the total number n of voters tends to infinity, the limit probability of observing a voting situation in which the truncation paradox may occur under the averaged scoring model is given by:*

$$\text{if } 0 < \lambda \leq \tfrac{1}{2}, \quad P_{Av}(F_\lambda) = \frac{\begin{pmatrix} 84\,\lambda^{12} - 265\,\lambda^{11} - 591\,\lambda^{10} - 11351\,\lambda^{9} + 126949\,\lambda^{8} \\ -517521\,\lambda^{7} + 1185665\,\lambda^{6} - 1720474\,\lambda^{5} \\ +1636446\,\lambda^{4} - 999044\,\lambda^{3} + 356664\,\lambda^{2} - 56592\,\lambda \end{pmatrix}}{9(4-3\lambda)(-6+7\lambda)(-7\lambda+5\lambda^2+6)(-3+2\lambda)(-6+\lambda)(\lambda-1)^2(\lambda-2)(\lambda-4)}$$

$$\text{if } \tfrac{1}{2} \leq \lambda \leq 2-\sqrt{2}, \quad P_{Av}(F_\lambda) = \frac{\begin{pmatrix} 24\,\lambda^{12} + 106\,\lambda^{11} - 2836\,\lambda^{10} + 16430\,\lambda^{9} - 56040\,\lambda^{8} \\ +136282\,\lambda^{7} - 245600\,\lambda^{6} + 322203\,\lambda^{5} - 297754\,\lambda^{4} \\ +184484\,\lambda^{3} - 69752\,\lambda^{2} + 13632\,\lambda - 1152 \end{pmatrix}}{18\,\lambda^2(\lambda-1)(6-\lambda)(-4+3\,\lambda)(\lambda-2)(-7\,\lambda+5\,\lambda^2+6)(-3+2\,\lambda)(\lambda-4)}$$

$$\text{if } 2-\sqrt{2} \leq \lambda \leq 1, \quad P_{Av}(F_\lambda) = \frac{\begin{pmatrix} 12\,\lambda^{10} - 298\,\lambda^{9} + 3214\,\lambda^{8} - 15754\,\lambda^{7} + 40654\,\lambda^{6} - 62044\,\lambda^{5} \\ +59581\,\lambda^{4} - 36013\,\lambda^{3} + 13312\,\lambda^{2} - 3196\,\lambda + 336 \end{pmatrix}}{18\,\lambda^2(\lambda-5)(4-3\,\lambda)(-7\,\lambda+5\,\lambda^2+6)(-3+\lambda)(\lambda-2)}$$

Proof The proof follows the same scheme as that of Proposition 3.

In Table 4, we report some numerical evaluations of $P_{\text{Pe}}(F_\lambda)$ and $P_{\text{Av}}(F_\lambda)$; Figure 1 give a complete overview of their behavior. It appears that as the number of voters tends to infinity, the limit probability, under the IAC assumption, of observing a voting situation in which the truncation paradox may occur with a one-shot scoring rule F_λ increases from 0 to 75% and from 0 to 34.03% respectively under the pessimistic model and the averaged model as the weight λ increases from 0 (the Plurality rule) to 1 (the Antiplurality rule). It should also be noted that for any $\lambda \in]0\ 1]$, while the paradox is not likely to occur under the optimistic scoring model, it is almost twice as likely to occur under the pessimistic scoring model as under the averaged scoring model. Thus, the model under which we operate does indeed have a significant impact on the probability of the paradox.

Fig. 1 Vulnerability of one-shot scoring rules to the truncation paradox

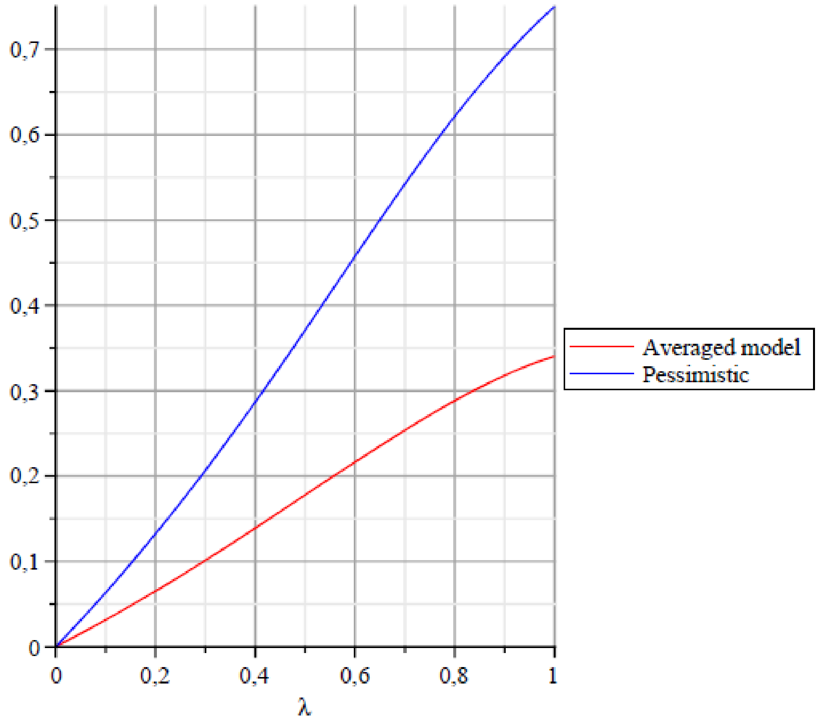

Table 4 Likelihood of the truncation paradox for one-shot scoring rules

	λ										
Models	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Pessimistic	–	0.06334	0.1322	0.2067	0.2866	0.3710	0.4575	0.5423	0.6215	0.6916	0.7500
Averaged	–	0.03145	0.06517	0.1011	0.1390	0.1778	0.2163	0.2537	0.2883	0.3180	0.3403
Optimistic	–	0	0	0	0	0	0	0	0	0	0

4 The vulnerability of scoring runoff rules to the truncation paradox

Consider the voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ and a runoff rule with $0 < \lambda \leq 1$. Assume that at π, z is eliminated at the first round and that x wins against y at the second round.

For simplicity, we say that x is the winner, y is the challenger and z is the (first-round) loser. To see how the truncation paradox arises under a runoff rule, recall that this paradox can be seen as a strategic behavior by some voters. Taking into account the specificity of runoff rules that combine both counting points at the first round and majority voting at the second round, successful truncations of preferences are either (i) in favor of the challenger when, by truncating their rankings, some voters make x lose at the first round and cause the loser to be beaten by the challenger at the second round; or (ii) in favor of the loser who defeats the winner or the challenger in the second round.

Proposition 5 *Consider a voting situation* $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ *on* $A = \{a, b, c\}$ *and a runoff rule with* $0 < \lambda \leq 1$. *Assume that* x *is the winner,* y *is the challenger and* z *is the first-round loser.*

1. *under the pessimistic or the averaged scoring models:*
   - *(a) The truncation paradox is liable to occur at* π *in favor of* y *if and only if* y *wins the majority duel against* z *and* x *is the first-round loser at* $\pi([yxz])$.
   - *(b) The truncation paradox is liable to occur at* π *in favor of* z *if and only if* z *wins the majority duel against* y *and* x *is the first-round loser at* $\pi([zxy])$*; or if* z *wins the majority duel against* x *and* y *is the first-round loser at* $\pi([zyx])$.
2. *under the optimistic scoring model:*
   - *(a) The truncation paradox is likely to occur at* π *in favor of* y *if and only if* y *wins the majority duel against* z *and* x *is the first-round loser at* $\pi([yxz])$.
   - *(b) The truncation paradox cannot occur at* π *in favor of* z.

Proof See Appendix D.

Proposition 5 completely describes all the possible scenarios that support possible occurrence of the truncation paradox given a voting situation. These conditions lead us to some sets of linear constraints that characterize all possible occurrence of the truncation paradox under a runoff rule. Details are available in Appendix D. Computing the volume of all the corresponding polytopes leads to Proposition 6. In this proposition, we obtain the same result as Kamwa and Moyouwou (2020).

Proposition 6 *Consider the scoring runoff rule* F'_λ *associated with the scoring vector* $w_\lambda = (1, \lambda, 0)$ *with* $0 < \lambda \leq 1$. *As the total number* n *of voters tends to infinity, the limit probability* $P_{Pe}(F'_\lambda)$ *of observing a voting situation in which the truncation paradox may occur under the pessimistic scoring model is given by :*

If $0 \leq \lambda \leq \frac{1}{2}$,

$$P_{Pe}(F'_\lambda) = -\frac{\begin{pmatrix} 996\,096\lambda^{20} - 25\,010\,368\lambda^{19} + 286\,101\,152\lambda^{18} - 2000\,804\,220\lambda^{17} \\ +9664\,972\,152\lambda^{16} - 34\,453\,144\,125\lambda^{15} + 94\,322\,255\,778\lambda^{14} \\ -203\,353\,434\,975\lambda^{13} + 350\,716\,379\,871\lambda^{12} - 488\,312\,722\,095\lambda^{11} \\ +551\,142\,449\,552\lambda^{10} - 504\,159\,008\,281\lambda^{9} + 372\,136\,194\,567\lambda^{8} \\ -219\,653\,377\,992\lambda^{7} + 102\,140\,474\,607\lambda^{6} - 36\,558\,733\,185\lambda^{5} \\ +9711\,109\,602\lambda^{4} - 1801\,641\,852\lambda^{3} + 208\,222\,083\lambda^{2} - 11\,278\,359\lambda \end{pmatrix}}{96(\lambda-1)^2(\lambda-2)^2(2\lambda-3)^2(4\lambda-3)^2(5\lambda-3)^2(-2\lambda+\lambda^2+3)(-5\lambda+\lambda^2+3)^2(-4\lambda+2\lambda^2+3)(-7\lambda+3\lambda^2+3)}$$

If $\frac{1}{2} \leq \lambda \leq 1$,

$$P_{Pe}(F'_\lambda) = \frac{\begin{pmatrix} 132\lambda + 9346\lambda^2 - 55\,961\lambda^3 + 161\,587\lambda^4 - 283\,660\lambda^5 \\ +330\,502\lambda^6 - 265\,921\lambda^7 + 149\,437\lambda^8 - 57\,766\lambda^9 \\ +14\,560\lambda^{10} - 2112\lambda^{11} + 128\lambda^{12} - 180 \end{pmatrix}}{288\lambda^3(\lambda-2)^2(3-2\lambda)(-2\lambda+\lambda^2+3)(-4\lambda+2\lambda^2+3)}$$

Proof See Appendix E for further details on the computation.

Proposition 7 *Consider the scoring runoff rule* F'_λ *associated with the scoring vector* $w_\lambda = (1, \lambda, 0)$ *with* $0 < \lambda \leq 1$ *and the pessimistic scoring model. As the total number* n *of voters tends* *to infinity, the limit probability $P_{Pe}(F'_\lambda)$ of observing a voting situation in which the truncation paradox may occur under the optimistic scoring model is given by :*

If $0 \leq \lambda \leq \frac{1}{2}$,

$$P_{Op}(F'_\lambda) = \frac{\begin{pmatrix} 284150\,\lambda^6 - 457914\,\lambda^5 + 442197\,\lambda^4 - 100941\,\lambda^7 + 20832\,\lambda^8 - 8168\,\lambda^9 \\ -10611\lambda - 253752\,\lambda^3 + 79848\,\lambda^2 + 320\,\lambda^{12} + 6627\,\lambda^{10} - 2592\,\lambda^{11} \end{pmatrix}}{-864(\lambda^2-2\,\lambda+3)(\lambda-2)(-3+2\,\lambda)(5\,\lambda-3)(\lambda-1)^3(3-5\,\lambda+\lambda^2)}$$

If $\frac{1}{2} \leq \lambda \leq 1$,

$$P_{Op}(F'_\lambda) = \frac{\begin{pmatrix} -22742\,\lambda^4 + 15531\,\lambda^3 + 4423\,\lambda^7 + 21028\,\lambda^5 + 64\,\lambda^9 \\ -126 + 1482\,\lambda - 12287\,\lambda^6 - 864\,\lambda^8 - 6665\,\lambda^2 \end{pmatrix}}{-864(\lambda^2-2\,\lambda+3)(\lambda-2)(-3+2\,\lambda)\lambda^3}$$

Proposition 8 *Consider the scoring runoff rule F'_λ associated with the scoring vector $w_\lambda = (1, \lambda, 0)$ with $0 < \lambda \leq 1$. As the total number n of voters tends to infinity, the limit probability $P_{Av}(F'_\lambda)$ of observing a voting situation in which the truncation paradox may occur under the averaged scoring model is given by :*

If $\leq \lambda \leq \frac{1}{2}$,

$$P_{Av}(F'_\lambda) = \frac{\begin{pmatrix} \lambda\Big(29756136329786421\,\lambda^7 + 229734452559443301\,\lambda^{11} - 178272672122561787\,\lambda^{10} \\ +117070741750497903\,\lambda^9 - 1295375725376735\,\lambda^{20} + 4697132676107282\,\lambda^{19} \\ -14007896683551062\,\lambda^{18} + 34679171642310785\,\lambda^{17} - 71889114422947871\,\lambda^{16} \\ -63900465840 - 251541492982706718\,\lambda^{12} + 336960000\,\lambda^{27} \\ +112293876000\,\lambda^{25} - 8652816000\,\lambda^{26} + 8156911894790\,\lambda^{23} \\ -1050941832560\,\lambda^{24} + 291624916827179\,\lambda^{21} - 53593644332126\,\lambda^{22} \\ -64653635448103662\,\lambda^8 + 1627625218824\,\lambda - 19744908073848\,\lambda^2 \\ -831277990913364\,\lambda^4 + 151845798562668\,\lambda^3 + 3448576501074018\,\lambda^5 \\ -11268232559588502\,\lambda^6 + 125657219231572896\,\lambda^{15} - 186150918901235109\,\lambda^{14} \\ +234488280310373037\,\lambda^{13} \Big) \end{pmatrix}}{\begin{pmatrix} -864\left(3\,\lambda^2 - 7\,\lambda + 3\right)\left(13\,\lambda^2 - 18\,\lambda + 6\right)\left(2\,\lambda^2 - 4\,\lambda + 3\right)(-3 + 4\,\lambda)^2 \\ (-6 + 5\,\lambda)(-3 + \lambda)(-4 + 3\,\lambda)\left(9\,\lambda^2 - 16\,\lambda + 6\right)\left(5\,\lambda^2 - 10\,\lambda + 6\right) \\ (-3 + 5\,\lambda)^2(\lambda - 2)(\lambda - 1)^3(-3 + 2\,\lambda)\left(\lambda^2 - 2\,\lambda + 3\right)\left(3 - 5\,\lambda + \lambda^2\right) \end{pmatrix}}$$

If $\frac{1}{2} \leq \lambda \leq \frac{2}{3}$,

$$P_{Av}(F'_\lambda) = \frac{\begin{pmatrix} 2099520 - 18230665471\,\lambda^7 - 43292910605\,\lambda^{11} + 55648570181\,\lambda^{10} \\ -53946322373\,\lambda^9 + 624000\,\lambda^{18} - 15121600\,\lambda^{17} + 155807560\,\lambda^{16} \\ +25686340889\,\lambda^{12} + 38189483083\,\lambda^8 - 14475024\,\lambda + 5218992\,\lambda^2 - 863638434\,\lambda^4 \\ +234422424\,\lambda^3 + 614152332\,\lambda^5 + 4456679443\,\lambda^6 - 949706470\,\lambda^{15} \\ +3906190562\,\lambda^{14} - 11586749657\,\lambda^{13} \end{pmatrix}}{4320(3-2\,\lambda)(\lambda^2-2\,\lambda+3)(\lambda-2)\lambda^3(-4+3\,\lambda)(5\,\lambda^2-10\,\lambda+6)(-6+5\,\lambda)(\lambda-1)(-3+\lambda)(1+4\,\lambda)(2\,\lambda^2-4\,\lambda+3)}$$

If $\frac{2}{3} \leq \lambda \leq 1$,

$$P_{Av}\left(F'_{\lambda}\right)=\frac{\left(\begin{array}{c}18720000\,\lambda^{18}-264992000\,\lambda^{17}+1517810000\,\lambda^{16}-4612688020\,\lambda^{15}\\+7219079938\,\lambda^{14}-701040792\,\lambda^{13}-23566887867\,\lambda^{12}+58714236393\,\lambda^{11}\\-80453992152\,\lambda^{10}+71388488778\,\lambda^{9}-41360020859\,\lambda^{8}+13989995429\,\lambda^{7}\\-1120219921\,\lambda^{6}-1234798607\,\lambda^{5}+524869929\,\lambda^{4}-42508179\,\lambda^{3}\\-24266682\,\lambda^{2}+6888564\,\lambda-524880\end{array}\right)}{4320(2\,\lambda^2-4\,\lambda+3)(-2\,\lambda+1)(-5+9\,\lambda)(5\,\lambda-1)(-3+5\,\lambda)^2(1+4\,\lambda)(4+\lambda)(-3+2\,\lambda)(\lambda^2-2\,\lambda+3)(\lambda-2)\lambda^3}$$

The proofs of Propositions 7 and 8 follow the same scheme as that of Proposition 6.

We report in Table 5, some numerical evaluations of $P_{\mathrm{Pe}}\left(F'_{\lambda}\right)$, $P_{\mathrm{Op}}\left(F'_{\lambda}\right)$ and $P_{\mathrm{Av}}\left(F'_{\lambda}\right)$; Figure 2 give for an overview of their behavior. We notice that as the total number n of voters tends to infinity, the limit probability, under the IAC assumption, of observing a voting situation in which the truncation paradox may occur given a one-shot scoring rule F'_{λ} increases from 0 to 15.97%, from 0 to 13.38% and from 0 to 9.03% respectively under the pessimistic, the averaged and the optimistic models as the weight λ increases from 0 (the Plurality rule) to 1 (the Antiplurality rule). For any $\lambda\in]0\ 1]$, the paradox is almost more likely to occur under the pessimistic scoring model than under the averaged and the optimistic scoring models. Also, for $0.52<\lambda<0.57$ the probabilities $P_{\mathrm{Op}}\left(F'_{\lambda}\right)$ and $P_{\mathrm{Av}}\left(F'_{\lambda}\right)$ are quite close (but not equal) while still having $P_{\mathrm{Op}}\left(F'_{\lambda}\right)<P_{\mathrm{Av}}\left(F'_{\lambda}\right)$. We conclude that the model under which we operate does indeed have a significant impact on the probability of the truncation paradox for scoring runoff rules with three candidates.

Fig. 2 Vulnerability of runoff scoring rules to the truncation paradox

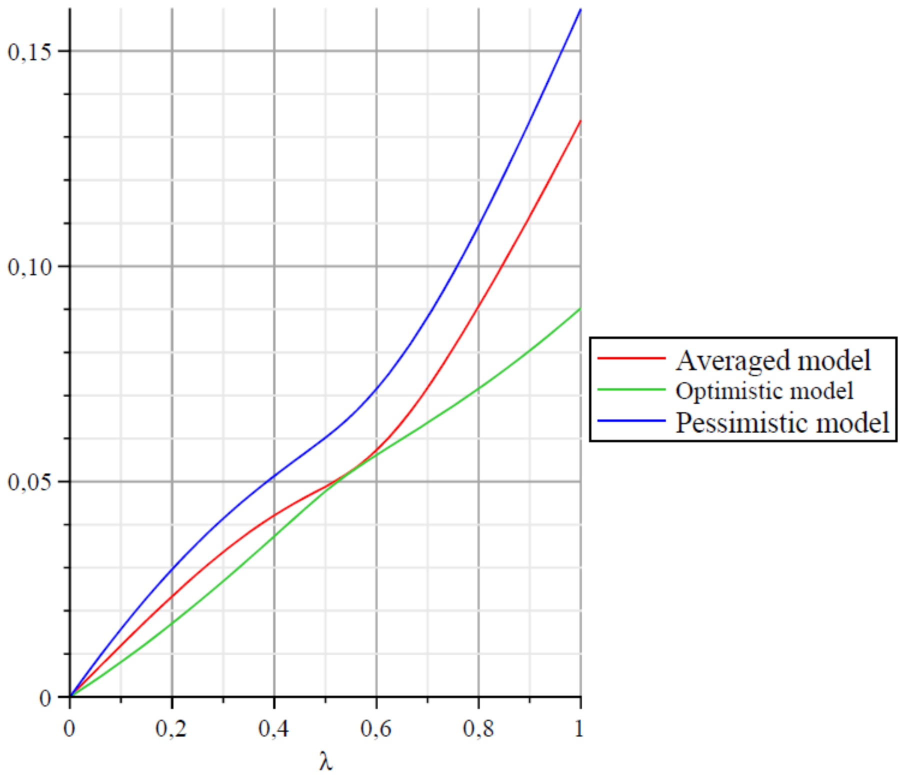

Table 5 Likelihood of the truncation paradox for runoff scoring rules

	λ										
Models	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Pessimistic	–	0.01575	0.02963	0.04142	0.05127	0.06018	0.07151	0.08817	0.10942	0.13374	0.15972
Averaged	–	0.01195	0.02335	0.03361	0.01742	0.04884	0.05728	0.07176	0.09071	0.11162	0.13389
Optimistic	–	0.00806	0.01706	0.02686	0.03728	0.04774	0.05615	0.06367	0.07161	0.08042	0.09028

5 Concluding remarks

The truncation paradox was first introduced by Brams (1982). It has been established that almost all the well-known voting rules are sensitive to this paradox and that the few exceptions are the Plurality rule, Plurality runoff and Approval voting. Only a small number of papers (Kamwa and Moyouwou, 2020, Kilgour et al., 2019, Plassmann and Tideman, 2014) has been interested in the assessment of the probabilities of occurrence of this paradox. This work is part of this same approach. For three-candidate elections, we have characterized all the voting situations under which the truncation paradox is likely to occur for one-shot scoring rules and scoring runoff rules under three scoring model that can be assumed for incomplete (truncated) preferences: the pessimistic, the optimistic and the averaged scoring rules. We have computed the limiting probability of the truncation paradox for each of the scoring models. It came that for any one-shot rule such that $\lambda \in]0\ 1]$, the truncation paradox never occurs under the optimistic scoring model while it is almost twice as likely to occur under the pessimistic scoring model as under the averaged scoring model. For scoring runoff rules, we found that any $\lambda \in]0\ 1]$, the truncation paradox is almost more likely to occur under the pessimistic scoring model than under the average scoring model and more than under the optimistic scoring model. The lesson we draw from our analysis is that the occurrence of the truncation paradox is highly dependent on the model that is applied to truncated preferences. Thus, for ballots using the scoring rules and where voters must provide a complete ranking of the candidates, it is always necessary indicate in advance the model which governs the truncated preferences. In such an objective, we will avoid the pessimistic model which, compared to other models, leaves the door open to a large margin of manipulation by truncation; the optimistic model limits this room for maneuver.

Appendices

A. Proof of Proposition 1

Let us consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ and a one-shot scoring rule F_λ with $0 < \lambda \leq 1$. Assume that a is the winner for a given λ. This means that $S(\pi, w, a) > S(\pi, w, b)$ and $S(\pi, w, a) > S(\pi, w, c)$.

Assume without loss of generality that voters of type 3 want to favor b by truncation under the optimistic scoring model. With the truncation, only candidate c's score is affected, it increases while those of candidates a and b remain unchanged for all λ: it comes that a's score is still greater than that of b; thus, by truncation, it is not possible to favor b. We reach the same conclusion if voters of type 6 wanted to favor b. So, under the optimistic scoring model, there is no one-shot scoring rules manipulable by sincere truncation.

B. Proof of Proposition 2

The proof provided here is for the pessimistic scoring model; that of the averaged scoring model can be easily adapted.

Consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{a, b, c\}$, the one-shot rule associated with $0 < \lambda \leq 1$ and a pair $\{x, y\}$ of candidates. Let z be the third candidate.

Necessity. Assume that x is the election winner at π, and that the truncation paradox is liable to occur in π in favor of y. Then by truncating their true preferences, a coalition of voters, say S, favors the election of y. Moreover each voter in S strictly prefers y to x. Since the truncation operation only affects the second-ranked candidates of each voter in S, then the preferences of each voter in S is yxz or yzx. At the new voting situation π', y wins. Without loss of generality, we denote by $n_{xyz}(\pi)$ the total number of voters in π who rank x first, y second and z last at π. Note that $|S| \leq n_{yxz}(\pi) + n_{yzx}(\pi)$. Then from π' to $\pi([yxz])$, the score of y increases, the scores of both x and z decrease. Hence y also wins in $\pi([yxz, yzx])$.

Sufficiency. Assume that x is the election winner at π while y wins in $\pi([yxz, yzx])$. Clearly, the truncation paradox is liable to occur in π in favor of y since all voters who truncate their preferences in $\pi([yxz, yzx])$ prefers y to x.

C. Computation details for Proposition 3

Let T_x denote the set of all voting situations in which x is the election winner while the truncation paradox is liable to occur; and T_{xy} the subset of T_x that consists of all voting situations in which truncating preferences may favor the election of y. Note for example that

$$T_a = T_{ab} \cup T_{ac} \text{ and } |T_a| = |T_{ab}| + |T_{ac}| - |T_{ab} \cap T_{ac}|.$$

By Proposition 2, $\pi \in T_{ab}$ if and only if $S(\pi, \lambda, a) \geq S(\pi, \lambda, b)$, $S(\pi, \lambda, a) \geq S(\pi, \lambda, c)$, $S(\pi[bac, bca], \lambda, b) > S(\pi[bac, bca], \lambda, a)$ and $S(\pi[bac, bca], \lambda, b) \geq S(\pi[bac, bca], \lambda, c)$. Equivalently,[1]

$$\pi \in T_{ab} \Longleftrightarrow \begin{cases} (\lambda-1)n_1 - n_2 + (1-\lambda)n_3 + n_4 - \lambda n_5 + \lambda n_6 \leq 0 \\ -n_1 + (\lambda-1)n_2 - \lambda n_3 + \lambda n_4 + (1-\lambda)n_5 + n_6 \leq 0 \\ (1-\lambda)n_1 + n_2 - n_3 - n_4 + \lambda n_5 - \lambda n_6 < 0 \\ -\lambda n_1 + \lambda n_2 - n_3 - n_4 + n_5 + (1-\lambda)\, n_6 \leq 0 \end{cases}$$

Clearly, each of the six possible sets T_{xy} with $x, y \in A$ can be similarly described by a set of four linear constraints as with T_{ab} above. As n tends to infinity, $vol\,(P_{xy})$ is the 5 -dimensional volume of the polytope P_{xy} obtained from the characterization of T_{xy} by replacing each n_j by $p_j = \frac{n_j}{n}$. Note that some inequalities in the characterization of P_{xy} may be strict. We simply ignore this while evaluating $vol\,(P_{xy})$ by considering the closure of P_{xy} obtained from the characterization of P_{xy} by turning each strict inequality ($<$) to its larger form ($\leq$); by doing so, we simply move from P_{xy} to its closure without changing the volume. Taking into account that T_a, T_b and T_c are disjoint sets of voting situations, and since by symmetries, all the six possible T_{xy} generates polytopes of equal volume, the limit probability $P\,(F_\lambda, TP, IAC)$ under the IAC assumption, of observing a voting situation in which the truncation paradox may occur is[2]

$$P_\bullet\,(F_\lambda) = \frac{vol(P_a) + vol(P_b) + vol(P_c)}{vol(P)} = 720 vol\,(P_{ab}) - 360 vol\,(P_{ab} \cap P_{ac})$$

where P is the simplex $P = \{(p_1, p_2, \ldots, p_6) : \sum_{t=1}^{6} p_j = 1 \text{ with } p_j \geq 0, j = 1, 2, \ldots, 6\}$. Given $0 < \lambda \leq 1$, computing $vol\,(P_{ab})$ and $vol\,(P_{ab} \cap P_{ac})$, one obtains the result of Proposition 2. All volume computations performed in this paper use the same technique as in Cervone et al. (2005).[3] Roughly, one needs for example to determine all vertices of the given polytope and then triangulate the set of those vertices into simplices. More details are presented in Moyouwou and Tchantcho (2015) and Gehrlein and Lepelley (2011); further illustrations are available in Gehrlein et al. (2015) or more recently in El Ouafdi et al. (2020). A Maple procedure is also available from

[1] The inequality system is that of the pessimistic scoring model; that of the averaged scoring model can be easily adapted.

[2] The symbol $\bullet$ stands for the scoring model.

[3] This technique has recently been used in many research papers, such as Diss and Gehrlein (2015, 2012), Gehrlein et al. (2015), Kamwa et al. (2018), Kamwa and Valognes (2017), Moyouwou and Tchantcho (2015) and Kamwa (2019) among others.

authors upon request. Of course, there is an abundant literature on volume computations with very efficient algorithms and packages such as Büeler et al. (2000) and Lawrence (1991) for Maple users or Bruns and Ichim (2010) and Bruns et al. (2019, 2018).

D. Proof of Proposition 5

Consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{a, b, c\}$ and the runoff rule associated with $0 < \lambda \leq 1$. Assume that x is the winner, y is the challenger and z is the first-round loser.

1. **under the pessimistic or the averaged scoring model:**
   (a) The truncation paradox is liable to occur at π in favor of y if and only if y wins the majority duel against z and x is the first-round loser at $\pi([yxz])$.
   - *Necessity.* First assume that the truncation paradox is liable to occur at π in favor of y. Then by truncating their true preferences, a coalition of voters, say S, diminishes the score of x in such a way that x is now ruled out at the first round and y wins against z at the second round. Each voter in S strictly prefers y to x. The truncation operation by such a voter is only intended to diminish the score of x at the first round. Thus the preferences of each voter in S is yxz. In the new voting situation π', y wins. Since from π' to $\pi([yxz])$, the score of y does not decrease, the score of x does not increase, the score of z is unchanged and the second round duel is not affected by the truncation operation, then y also wins in $\pi([yxz])$ against z at the second round.
   - *Sufficiency.* Assume that y wins the majority duel against z and x is the first-round loser at $\pi([yxz])$. Then under the corresponding runoff rule, y wins in $\pi([yxz])$ against z at the second round. Hence, the truncation paradox occurs.

   (b) The truncation paradox is liable to occur at π in favor of z if and only if z wins the majority duel against y and x is the first-round loser at $\pi([zxy])$; or if z wins the majority duel against x and y is the first-round loser at $\pi([zyx])$.
   - *Necessity.* Assume that the truncation paradox is liable to occur at π in favor of z. By truncating their true preferences, members of some coalition, say S, favor the election of z whom they strictly prefer to x. In the new voting situation π', z wins the majority duel against x or against y. First suppose that z wins in π' against x at the second round. Then y is the first-round loser at π'. Moreover, voters in S all strictly prefer z to x; and the truncation operation is intended, at the first round in π', to diminish the score of y. Thus the preference of each voter in S is zyx. Hence $|S| \leq n_{zyx}(\pi)$. Therefore, in $\pi([zyx])$, z also wins against x and y is the first round loser. Finally, suppose that z wins in π' against y at the second round. Then x is the first-round loser at π'. Voters in S all strictly prefer z to y; and the truncation operation is intended, at the first round in π', to diminish the score of x. The preference of each voter in S is then zxy. This implies that $|S| \leq n_{zxy}(\pi)$. In $\pi([zyx])$, z also wins against y and x is the first round loser.
   - *Sufficiency.* Assume that z wins the majority duel against y and x is the first-round loser in $\pi([zxy])$. Then under the corresponding runoff rule, z wins in $\pi([zxy])$ against y at the second round. In the same way, suppose that z wins the majority duel against x and y is the first-round loser in $\pi([zyx])$. Then under the corresponding runoff rule, z wins in $\pi([zyx])$ against x at the second round. In both cases, the truncation paradox occurs.

2. **under the optimistic scoring models:**
   (a) The truncation paradox only liable to occur at π in favor of y if and only if y wins the majority duel against z and x is the first-round loser at $\pi([yxz])$.
   - *Necessity.* First assume that the truncation paradox is liable to occur at π in favor of y. Then by truncating their true preferences, a coalition of voters, say S, increases

the score of z while those of x and y remain unchanged in such a way that x is now ruled out at the first round and y wins against z at the second round. Each voter in S strictly prefers y to x. The truncation operation by such a voter is only intended to rule candidate x out at the first round. Thus the preferences of each voter in S is yxz. In the new voting situation π', y wins. Since from π' to $\pi([yxz])$, the scores of x and y do not increase nor decrease while the score of z increases and the second round duel is not affected by the truncation operation, then y also wins in $\pi([yxz])$ against z at the second round.

- *Sufficiency.* Assume that y wins the majority duel against z and x is the first-round loser at $\pi([yxz])$. Then under the corresponding runoff rule, y wins in $\pi([yxz])$ against z at the second round. Hence, the truncation paradox occurs.

(b) The truncation paradox cannot occur at π in favor of z.

- *Necessity and Sufficiency.* By truncating their true preferences, members of some coalition, say S may want to favor the election of z whom they strictly prefer to x. The preferences of each voter in S is xzy. In the new voting situation π', the scores of x and z do not increase nor decrease while the score of y increases. Since in π the score of z was lower than those of candidates x and y, this is also the case under π'. Thus, it is not possible to favor z.

E. Computations details for Proposition 6

Given $0 < \lambda \leq 1$, let R_{xy} denote the set of all voting situations in which the truncation paradox is liable to occur in favor of some candidate u under the runoff rule associated with the weight λ while x and y are respectively the election winner and the challenger. Let z be the first-round loser in each voting situation in R_{xy}. Denote by R_{xyy} the subset of R_{xy} that consists of all voting situations in which truncating preferences may favor the election of y; by R_{xyz} the subset of R_{xy} that consists of all voting situations in which truncating preferences may favor the election of z against x at the second round; and by R'_{xyz} the subset of R_{xy} that consists of all voting situations at which truncating preferences may favor the election of z against y at the second round. Then by Proposition 5

$$R_{ab} = R_{abb} \cup R_{abc} \cup R'_{abc}.$$

Note that R_{abb} and R_{abc} are disjoint sets of voting situations since y wins the majority duel against z in each voting situation in R_{abb} while the converse holds in each voting situation in R_{abc}. Therefore

$$|R_{ab}| = |R_{abb}| + |R_{abc}| + |R'_{abc}| - |R_{abb} \cap R'_{abc}| - |R_{abc} \cap R'_{abc}|.$$

Note that by Proposition 5, R_{abb}, R_{abc} and R'_{abc} are each defined by some set of linear constraints. Therefore the probability that the corresponding runoff rule exhibits the truncation paradox is derived by computing the volume of the polytopes P_{abb}, P_{abc} and P'_{abc} associated to R_{abb}, R_{abc} and R'_{abc} respectively. More precisely, by considering the six possible sets R_{xy} for all two ordered pairs (x, y) from $\{a, b, c\}$ and taking into account possible symmetries, the limit probability $P_{\mathrm{Pe}}(F'_\lambda)$, under the IAC assumption, of observing a voting situation with three candidates in which the truncation paradox may occur is

$$P_{\bullet}(F'_\lambda) = 720\left[vol(P_{abb}) + vol(P_{abc}) + vol(P'_{abc}) - vol(P_{abb} \cap P'_{abc}) - vol(P_{abc} \cap P'_{abc})\right]$$

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