To what extent does the model of processing sincereincomplete rankings affect the likelihood of the truncation paradox?

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Abstract

For a given voting rule, if voters can favor a preferred outcome by providing only a part of their sincere rankings on the competing candidates, rather than listing their entire preference rankings on all the competing candidates, this rule is said to be vulnerable to the truncation paradox. In this paper, we show that the way of dealing with the truncated ballot can really impact the occurrence of the paradox: this paradox never occurs with any one-shot scoring rules when truncated ballot a treated according the optimistic model. The optimistic model is, along with the pessimistic model and the averaged model, the three most common ways of dealing with truncated preferences. The few papers that assess the likelihood of the occurrence of this paradox implicitly assume the pessimistic model. In this paper, we assess the likelihood of the truncation paradox under the two other models for three-candidate elections and large electorates. We focus on whole families of one-shot scoring rules, iterative scoring rules both with one-by-one eliminations and with elimination by the average. This assessment confirms that the choice of model may really matter: under the optimistic model, all the one-shot scoring rules are immune to the truncation paradox, whereas it is is more likely to occur under the pessimistic model than under the averaged model; for each of the scoring runoff rules, we find that the likelihood of the truncation paradox is higher under the pessimistic model, and lower under the optimistic model. Our analysis is performed under the Impartial Anonymous Culture assumption.

Keywords: Truncation, Rankings, Scoring model, Probability, Paradox, Impartial and Anonymous Culture.
JEL Classification Number: D71, D72

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1 Introduction

Since the pioneering work of Arrow (1963), Gibbard (1973) and Satterthwaite (1975), it has been known that there is no "best" voting rule. Unless it is dictatorial, any voting rule is manipulable, i.e. a voter or a group of voters may have a vested interest in changing their sincere preference against an insincere preference and lead to a result that is favourable to them. The adoption or choice of a voting rule should therefore not be made in ignorance of the strategic behaviors to which the rule may be vulnerable. One way to discriminate between rules is to look at their vulnerability or the frequencies with which the rules are likely to lead to voting paradoxes.[1]

Alongside Gibbard-Satterthwaite-style strategic behaviour, Brams (1982) and Fishburn and Brams (1984) described another strategic behaviour in which voters try to profit "while maintaining their sincerity": sincere truncation. Many democratic organizations and societies have recourse to voting systems in which individuals vote by submitting a ranking of the options (candidates) offered up for assessment. Since producing a complete ranking on a large number of candidates is not always obvious or even possible, it is often permitted for individuals to produce partial rankings (even on a small number of candidates). According to Brams (1982) and Fishburn and Brams (1984), this possibility of producing truncated rankings could be used for strategic purposes: a voter or a group of voters can favor a preferred outcome by providing only a part of their sincere rankings on the competing candidates, rather than listing their (entire) sincere preference rankings on all the competing candidates. This is known as the *sincere truncation paradox* or simply, the truncation paradox. Without been exhaustive, it comes from the works of Brams (1982), Felsenthal (2012), Fishburn and Brams (1983, 1984), Nurmi (1999) that almost all the well-known voting rules are vulnerable to the truncation paradox; among these rules are the scoring rules, the iterative scoring rules and Condorcet consistent rules.[2] The few exceptions are the *Plurality rule*, *Plurality runoff* and *Approval voting*.

In this paper, we want to emphasize that the manipulability of the voting rules by sincere truncation may depend on how incomplete (sincere) preferences are treated. If this turns out to be the case, States or organizations that, in their collective decision-making process, resort to voting rules vulnerable to the paradox of truncation should frankly question the "best" way to deal with incomplete preferences.

Before going further, let us note that when the truncation paradox was introduced by Brams (1982) and Fishburn and Brams (1984), it has explicitly been alluded to the fact that when a voter truncates, only the candidate(s) mentioned on the ballot will receive points from that voter while the others will receive nothing. This way of proceeding with incomplete preferences is known in the literature as the *pessimistic model* (see Baumeister et al., 2012). This model is used for political elections in Slovenia and in Kiribati where the voting rule is

[1]For an overview of the vast literature of voting theory devoted to the evaluation of the probabilities of voting paradoxes, the reader may refer to the recent books by Gehrlein and Lepelley (2011, 2017) and Diss and Merlin (2021).

[2]A Condorcet consistent rule is a voting system that always elects the Condorcet winner when he exists. A Condorcet winner is a candidate who defeats each of the other candidates in pairwise comparisons.

the Borda rule.[3] Although the pessimistic model seems to be the most prevalent in both the literature and in practice, other ways of dealing with truncated preferences exist, including the *optimistic model* (see Baumeister et al., 2012, Saari, 2008) and the *averaged model* (see Dummett, 1997).[4] Under the optimistic model, if out of m candidates in the running, a voter only ranks k of them, each of the $m - k$ other candidates will be awarded the points that would have been associated with the $k + 1$th position of the voter's ranking. The optimistic model is used for the election of the leader of the Irish Green Party. Under the averaged model, each non-ranked candidate is awarded a number of points equal to the average of the total points that all non-ranked candidates would have received if they had been ranked. According to Baumeister et al. (2012), the drawbacks of the pessimistic model is that it gives incentives for voters to rank only a single candidate so that the impact of the vote on the score of this candidate, relative to the scores of other candidates, is greatest; on the other hand, the optimistic model rewards the voters who rank more candidates: the more candidates one ranks, the more points (in relative terms) these candidates receive.

When considering the pessimistic model, the vulnerability of most voting rules is clearly established, as noted above. What do we have with the optimistic and the averaged models? To answer this question, we focus our attention on the family of one-shot scoring rules and on that of iterative scoring rules. We will show that (i) all these rules are vulnerable to the truncation paradox under the averaged model and that (ii) if the optimistic model is used to deal with truncated preferences, all the one-round scoring rules are immune to the truncation paradox; this is not the case for the iterative scoring rules.

Only a few papers have tried to assess the likelihood of the truncation paradox by only considering the pessimistic model. Using simulations based on the spatial model for three-candidate voting situations with an electorate size varying from ten to a million, Plassmann and Tideman (2014) evaluated the likelihood of the *strong* truncation paradox[5] by focusing, among other things, on certain scoring rules and iterative scoring rules. They found that the likelihood of the strong truncation paradox tends to decrease as the number of voters increases. For their part, Kilgour et al. (2019) assessed the significance of ballot truncation in ranked-choice elections with four, five, and six candidates using intensive simulations on real data under both spatial and random models of voter preferences. In a more recent paper, Kamwa and Moyouwou (2020) characterized for three-candidate elections and large electorates all the voting situations where the truncation paradox can occur for the whole family of one-shot scoring rules and runoff scoring rules, and for these families of rules they computed the likelihood of the truncation paradox under the impartial and anonymous

[3] With m competing candidates, the Borda rule is the scoring rule which gives $m-k$ points to a candidate each time she is ranked kth by a voter; the winner is the candidate with the greatest total number of points. In the Slovenian national elections, this rule is used for the reserved legislative seats for Hungarian and Italian ethnic minorities.

[4] For an overview of schemes for dealing with incomplete preferences in collective decision-making, the reader may refer among others to Baumeister et al. (2012), Kruger and Terzopoulou (2020), Menon and Larson (2017), Narodytska and Walsh (2014) and Terzopoulou and Endriss (2021, 2019).

[5] The strong truncation paradox occurs if one voter reports only part of his ranking, then a candidate will win whom the voter ranks higher than the candidate who would win if the voter reported his complete ranking of the candidates.

culture assumption (defined later).[6]

We believe that the way in which truncated preferences are handled may magnify or attenuate the occurrence of the truncation paradox. This paper aims to explore this. More, we think that using the pessimistic model for dealing with sincere truncated preferences leaves ample possibilities for strategic behavior. Therefore, it may come as no surprise that for a given voting rule, the probability of the truncation paradox is higher with the pessimistic model than with the optimistic or the averaged models. But in what proportion? Of what order of magnitude are these differences in probability? In attempting to provide answers to these questions, we consider three-candidate elections with large electorates and for each of three models for dealing with incomplete rankings, we provide a characterization of all the voting situations where the truncation paradox can occur for the whole family of scoring rules and the whole family of two types of iterative scoring rules. Then, for these families of rules, we compute the likelihood of the truncation paradox under the impartial and anonymous culture (defined later).

The rest of the paper is organized as follows: Section 2 is devoted to basic definitions. In Section 3, we derive some general results on the behaviour of scoring rules and iterative scoring rules with regard to the sincere truncation of preferences both under the pessimistic and the averaged models. Section 4 is devoted to specific results in the case of three-candidate elections: given an infinite number of voters having strict rankings, we characterize all the voting situations where the truncation paradox can occur; then we provide our computation results on the likelihood of the Truncation paradox under all the scoring rules and iterative scoring rules under each of the models. Section 5 concludes.

2 Notation and definitions

2.1 Preferences

Let N be a set of n voters ($n \geq 2$) and $A = \{x, y, \dots\}$ a set of $m \geq 3$ candidates. Individual preferences are linear orders, and these are complete, asymmetric, and transitive binary relations on A. We assume that voters sincerely know their strict ranking on the candidates in A. With m candidates, there are exactly $m!$ linear orders $P_1, P_2, \dots, P_{m!}$ on A. For $x, y, z \in A$, we simply write xyz to denote the linear order on A according to which x is strictly preferred to y, y is strictly preferred to z; and by transitivity x is strictly preferred to z. A *voting situation* is a $m!$-tuple $\pi = (n_1, n_2, ..., n_t, ..., n_{m!})$ that indicates the total number n_t of voters casting the complete linear order of type t such that $\sum_{t=1}^{m!} n_t = n$.

For the particular case of three-candidate elections, Table 1 describes a voting situation

[6] As part of the probabilistic approach, some papers have tried to analyze the impact of strategic manipulation by truncation. Baumeister et al. (2012), Menon and Larson (2017) and Narodytska and Walsh (2014) sought to evaluate the feasibility of manipulation by truncation, and to assess how complicated such manipulation would be, for certain voting systems including the family of scoring rules and scoring rules with runoff rules.

on $A = \{x, y, z\}$.

Table 1: Possible strict rankings on $A = \{x, y, z\}$

Type 1: xyz (n_1)	Type 3: yxz (n_3)	Type 5: zxy (n_5)
Type 2: xzy (n_2)	Type 4: yzx (n_4)	Type 6: zyx (n_6)

Given $x, y \in A$ and a voting situation π, we denote by $n_{xy}(\pi)$ (or simply n_{xy}) the total number of voters who strictly prefer x to y. If $n_{xy} > n_{yx}$, we say that candidate x majority dominates candidate y; or equivalently x beats y in pairwise majority voting. In such a case, we will simply write $x\mathbf{M}y$.

2.2 Voting rules

In this paper we focus on the whole family of one-shot scoring rules and on the whole family of of scoring rules with eliminations. Let us define all these rules.

Scoring rules are voting systems that give points to candidates according to the position they have in voters' rankings. For a given scoring rule, the total number of points received by a candidate defines his score for this rule. The winner is the candidate with the highest score. In general, with m candidates and complete strict rankings, a scoring vector is a m-tuple $w = (w_1, w_2, \ldots, w_{m-1}, w_m)$ of real numbers such that $w_1 \geq w_2 \geq \ldots \geq w_{m-1} \geq w_m$ with $w_1 > w_m$. Given a voting situation π, each candidate receives w_k each time she is ranked k^{th} $(k = 1, 2, \ldots, m)$ by a voter. The score of a candidate $x \in A$ is the sum $S(\pi, w, x) = \sum_{t=1}^{m!} n_t w_{r(t,x)}$ where $r(t, x)$ is the rank of candidate x according to voters of type t.

In three-candidate elections, one can characterize the whole family of scoring rules by using the normalized scoring vector $w_\lambda = (1, \lambda, 0)$ with $0 \leq \lambda \leq 1$. For $\lambda = 0$, we obtain the *Plurality rule*. For $\lambda = 1$, we have the *Antiplurality rule* and for $\lambda = \frac{1}{2}$, we have the *Borda rule*. From now on, we will denote by $S(\pi, \lambda, x)$ the score of candidate x when the scoring vector is $w_\lambda = (1, \lambda, 0)$ and the voting situation is π; without loss of generality, F_λ will be used to refer to the voting rule.

Scoring rules with eliminations proceed via steps in which one or more candidates are eliminated at each step. We consider two main families of such rules: (i) at each step, the alternative with the lowest score is eliminated, and (ii) at each step, any candidate who obtains (strictly) less than the average of the scores is eliminated. The first family includes voting rules such as *Plurality elimination rule* (also called Instant Runoff Voting), *Negative plurality elimination rule* and the *Borda elimination rule*.

With three candidates, the first class of runoff systems involves two rounds of voting such that in the first round, the candidate with the smallest score is eliminated; and in the second round, a majority contest determines who is the winner. Given $w_\lambda = (1, \lambda, 0)$, if the runoff system is associated with $\lambda = 0$, we get the *Plurality elimination rule*; with $\lambda = 1$ we have the *Negative plurality elimination rule*; and for $\lambda = \frac{1}{2}$ we have the *Borda elimination rule*. The following scenarios are viable when eliminations proceed according to the average: (a) the ballot could stop in the first round if a single candidate has obtained more than the average of the scores; or (b) two candidates score more than the average and a majority contest in the second round determines the winner. If the iterative scoring rule in this class is associated with $\lambda = 1$, this defines the *Kim-Roush voting rule* (Kim and Roush, 1996), while we get the *Nanson rule* (Nanson, 1883) for $\lambda = \frac{1}{2}$.

3 The truncation paradox under the three models of dealing with truncate preferences

As stated before, under the pessimistic model, almost all the one-shot scoring rules (except the Plurality rule) and all the iterative scoring rules (except the Plurality elimination rule) are vulnerable to the truncation paradox. The ambition of this section is to help provide a comprehensive overview of the vulnerability to the paradox of truncation of scoring rules and iterative rules. Proposition 1 states what happen under the optimistic and the averaged models.

Proposition 1. *For all voting situation with at least three candidates and at least two voters,*

- *if that the optimistic model is used to deal with truncated preferences, all the one-shot scoring rules are immune to the truncation paradox; except the Plurality elimination rule, all the iterative scoring rules (one-by-one elimination or elimination according to the average) are vulnerable to the truncation paradox.*
- *If the averaged model is used to deal with truncated preferences, except the Plurality rule, all the one-shot scoring rules and except the Plurality elimination rule, all the iterative scoring rules (one-by-one elimination or elimination according to the average) are vulnerable to the truncation paradox.*

Proof. See Appendix A. □

Proposition 1 shows that, depending on voting rules, the choice of how to deal with incomplete preferences can be crucial. Thus, this choice should not be neglected. To support our argument, let us extend the analysis to the calculation of the frequencies of appearance of the truncation paradox. To do so, we will focus on the specific case of elections with three candidates.

4 Specific results in the case of three-candidate elections

In our setting, voters sincerely provide complete strict rankings on the competing candidates. With three candidates, when a voter of a given type truncates, he just states his top-ranked candidate and erases the others who are assumed to be less preferred. Then, the vector $w_\lambda = (1, \lambda, 0)$ has to be modified for truncated rankings according to three models. Assume that a voter with the ranking xyz truncates and submits $x - -$.

- with the pessimistic model, candidate x will receive 1 point in the new voting situation while y and z both receive zero points. Thus, for three-candidate elections, the scoring vector applied to truncated rankings is $w'_\lambda = (1, 0, 0)$.
- with the optimistic model, in the new voting situation candidate x will still receive 1 point while y and z both will receive λ points. Here, the scoring vector applied to truncated rankings is $w'_\lambda = (1, \lambda, \lambda)$.
- with the averaged model, candidate x will still receive 1 point while y and z will both receive $\frac{\lambda}{2}$ points. So, the scoring vector applied to truncated rankings is $w'_\lambda = (1, \frac{\lambda}{2}, \frac{\lambda}{2})$.

Notice that in three-candidate elections, when some voters truncate, *(i)* under the pessimistic scoring model: only the scores of candidates ranked second by some of these voters are affected and diminish; *(ii)* under the optimistic scoring model: only the scores of candidates ranked last by some of these voters are affected and increase; *(iii)* under the averaged scoring model: the scores of candidates ranked second by some of these voters diminish while those of candidates ranked last by some of these voters increase. Moreover, truncation is only possible at the first round under runoff systems.

Given π, let us denote by $\pi\left([xyz]\right)$ the voting situation obtained from π when all voters of type 1 with the ranking xyz truncate their preferences.

4.1 Characterization results

Consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{x, y, z\}$ and the one-shot rule F_λ with $0 < \lambda \leq 1$.

For one-shot scoring rules, Proposition 2 identifies all the voting situations in which the truncation paradox is possible under each of the models.

Proposition 2. *Consider a voting situation* $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ *on* $A = \{x, y, z\}$*, the one-shot scoring rule associated with* $0 < \lambda \leq 1$ *and a pair* $\{x, y\}$ *of candidates with* $A \setminus \{x, y\} = \{z\}$.

i) If x is the election winner at π, the truncation paradox is likely to occur at π in favor of y under the pessimistic model if and only if y is the election winner at $\pi\,([yxz, yzx])$ (Kamwa and Moyouwou, 2020).

ii) If x is the election winner at π, the truncation paradox is likely to occur at π in favor of y under the averaged model if and only if y is the election winner at $\pi\,([yxz, yzx])$.

iii) all the one-shot scoring rules are immune to manipulation by sincere truncation of preferences when the optimistic model is assumed (from Prop. 1).

The proof for the pessimistic model can been drawn from Kamwa and Moyouwou (2020). The proof for the average model follows the same scheme as that of the pessimistic model; so, we skip it.[7] The proof of (iii) comes from Prop. 1.

For each of the models of dealing with truncated preferences, Proposition 3 characterizes the voting situations where the truncation paradox is liable to occur under iterative scoring rules with one-by-one eliminations.

Proposition 3. *Consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{x, y, z\}$ and a runoff rule F'_λ associated with the scoring vector $w_\lambda = (1, \lambda, 0)$. Assume that z is eliminated in the first round and that x wins against y in the second round.*

1. *under the pessimistic model and the averaged model:*
   - *(a) The truncation paradox can occur at π in favor of y if and only if at $\pi\,([yxz])$, x is eliminated in the first round and y wins the majority duel against z.*
   - *(b) The truncation paradox is liable to occur at π in favor of z if and only if z wins the majority duel against y and x is the first-round loser at $\pi\,([zxy])$; or if z wins the majority duel against x and y is the first-round loser at $\pi\,([zyx])$.*
2. *under the optimistic model:*
   - *(a) The truncation paradox can occur at π only in favor of y if and only if y wins the majority duel against z and x is the first-round loser at $\pi\,([yxz])$.*
   - *(b) The truncation paradox cannot occur at π in favor of z.*

Proof. See Appendix B. □

Although runoff rules based on one-by-one eliminations are more prevalent in practice than those based on average eliminations, the latter have been shown to be less vulnerable to the strategic misrepresentation of preferences (Favardin and Lepelley, 2006, Kim and Roush, 1996, Lepelley and Valognes, 2003) and less prone to the Borda paradoxes (Kamwa, 2019) than the former. Our study offers here the opportunity to test whether this performance of runoff rules based on average elimination remains valid when we extend the study to the

[7]Nonetheless, this is still available upon request.

truncation paradox. Before we turn to this test, let us first characterize the voting situations for which the truncation paradox is likely to occur when we assume elimination on the basis of the average.

For each of the models for dealing with truncated preferences, Proposition 4 characterizes the voting situations where the truncation paradox is possible.

Proposition 4. *Consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{x, y, z\}$ and a scoring rule $\tilde{F}'_\lambda$ associated with the scoring vector $w_\lambda = (1, \lambda, 0)$.*

1. *Assume that x wins in the first round (*i.e.*, only x's score is above the average of the scores):*

   (a) *The truncation paradox is not liable to occur under both the optimistic and the averaged models.*

   (b) *Under the pessimistic model, the truncation paradox is liable to occur in favor of one of the losers, say y if at $\pi([yxz])$, only y now scores above the average, or if y wins the second round against x or against z.*

2. *Assume that x wins in the second round against y (*i.e.*, only z's score is under the average):*

   (a) *Under both the optimistic and the averaged models, the truncation paradox is never possible in favor of z; it is possible at $\pi([yxz])$ in favor of y if after truncation, he is the only one to score above the average or if he wins the second round against z.*

   (b) *Under the pessimistic model, the truncation paradox is possible (i) in favor of y if and only if at $\pi([yxz])$, only y scores above the average or if y wins the second round against z; (ii) in favor of z if and only if at $\pi([zxy])$, z wins the second round against y.*

Proof. See Appendix C. □

4.2 Computation results

The impartial and anonymous culture (IAC) assumption introduced by Kuga and Hiroaki (1974) and Gehrlein and Fishburn (1976) is one of the most widely employed assumptions in social choice theory literature when computing the likelihood of voting events. Under IAC, the likelihood of a given event is calculated with respect to the ratio between the number of voting situations in which the event is likely over the total number of possible voting situations. It is known that the total number of possible voting situations in three-candidate elections is given by the following five-degree polynomial in n: $C^n_{n+3!-1} = \frac{(n+5)!}{n!5!}$. The number of voting situations associated with a given event can be reduced to the solutions of a finite system of linear constraints with rational coefficients. As recently pointed out, the appropriate mathematical tools to find these solutions are the Ehrhart polynomials Gehrlein and Lepelley (2011, 2017), Lepelley et al. (2008), Pritchard and Wilson (2007). This technique has been widely used in numerous studies analyzing the probability of electoral events in the case of three-candidate elections under the IAC assumption. As we are only dealing with large electorates, we follow a procedure that was developed in Cervone et al. (2005) and has recently been used in many research papers such as Diss and Gehrlein (2015, 2012), Diss et al. (2020, 2018, 2012), El Ouafdi et al. (2020), Gehrlein et al. (2015), Kamwa (2019), Kamwa and Moyouwou (2020), Moyouwou and Tchantcho (2015) among others. This technique is based on the computation of polytope volumes.

4.2.1 The case of scoring rules

Let us denote by $P_{\mathrm{Av}}(F_\lambda)$ (resp. by $P_{\mathrm{Op}}(F_\lambda)$) the limiting probability (with an infinite number of voters) of the truncation paradox under the average (resp. the optimistic) scoring model for a one-shot scoring rule F_λ.

From Proposition 2, we derive a system of linear inequalities which we solve by using the same technique as in Cervone et al. (2005). From the solutions of this system of inequalities, we reach Proposition 5 which provides what we obtain as the form of $P_{\mathrm{Av}}(F_\lambda)$ and recall $P_{\mathrm{Pe}}(F_\lambda)$ as obtained by Kamwa and Moyouwou (2020) for the pessimistic model.

Proposition 5. *For three-candidate elections and the IAC assumption, the likelihood of the truncation paradox for a given scoring rule F_λ ($\lambda \in [0\ 1]$) under respectively the optimistic, the average, and the pessimistic models is given by:*

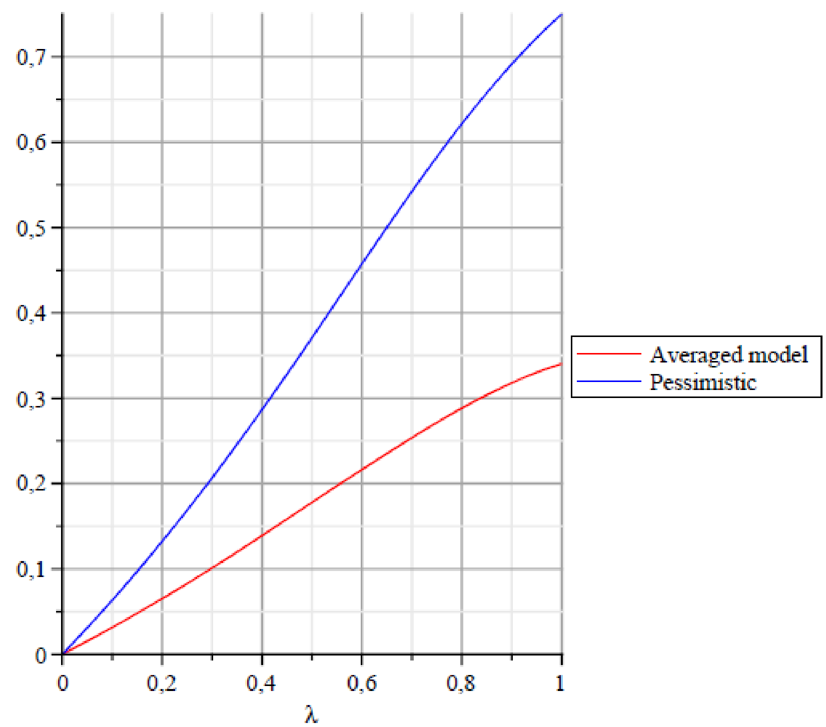
$$P_{Op}(F_\lambda) = \ 0, \quad \textit{if} \ \ 0 \leq \lambda \leq 1.$$

$$P_{Av}(F_\lambda) = \begin{cases} \dfrac{\begin{pmatrix} 84\,\lambda^{12} - 265\,\lambda^{11} - 591\,\lambda^{10} - 11351\,\lambda^{9} + 126949\,\lambda^{8} \\ -517521\,\lambda^{7} + 1185665\,\lambda^{6} - 1720474\,\lambda^{5} \\ +1636446\,\lambda^{4} - 999044\,\lambda^{3} + 356664\,\lambda^{2} - 56592\,\lambda \end{pmatrix}}{9(4-3\lambda)(-6+7\lambda)(-7\lambda+5\lambda^2+6)(-3+2\lambda)(-6+\lambda)(\lambda-1)^2(\lambda-2)(\lambda-4)}, & \textit{if} \ \ 0 < \lambda \leq \frac{1}{2}; \\[2ex] \dfrac{\begin{pmatrix} 24\,\lambda^{12} + 106\,\lambda^{11} - 2836\,\lambda^{10} + 16430\,\lambda^{9} - 56040\,\lambda^{8} \\ +136282\,\lambda^{7} - 245600\,\lambda^{6} + 322203\,\lambda^{5} - 297754\,\lambda^{4} \\ +184484\,\lambda^{3} - 69752\,\lambda^{2} + 13632\,\lambda - 1152 \end{pmatrix}}{18\,\lambda^2(\lambda-1)(6-\lambda)(-4+3\,\lambda)(\lambda-2)(-7\,\lambda+5\,\lambda^2+6)(-3+2\,\lambda)(\lambda-4)}, & \textit{if} \ \ \frac{1}{2} \leq \lambda \leq 2-\sqrt{2}; \\[2ex] \dfrac{\begin{pmatrix} 12\,\lambda^{10} - 298\,\lambda^{9} + 3214\,\lambda^{8} - 15754\,\lambda^{7} \\ 40654\,\lambda^{6} - 62044\,\lambda^{5} + 59581\,\lambda^{4} - 36013\,\lambda^{3} \\ +13312\,\lambda^{2} - 3196\,\lambda + 336 \end{pmatrix}}{18\,\lambda^2(\lambda-5)(4-3\,\lambda)(-7\,\lambda+5\,\lambda^2+6)(-3+\lambda)(\lambda-2)}, & \textit{if} \ \ 2-\sqrt{2} \leq \lambda \leq 1. \end{cases}$$

$$P_{Pe}\left(F_\lambda\right)=\begin{cases}\dfrac{\left(\begin{array}{c}10\,\lambda^{14}-37\,\lambda^{13}-179\,\lambda^{12}+1310\,\lambda^{11}-1778\,\lambda^{10}-6319\,\lambda^{9}\\+26773\,\lambda^{8}-25735\,\lambda^{7}-67880\,\lambda^{6}+259941\,\lambda^{5}-408078\,\lambda^{4}\\+356643\,\lambda^{3}-166536\,\lambda^{2}+31833\lambda\end{array}\right)}{6(3+\lambda)^2(3-2\,\lambda+\lambda^2)^2(\lambda-2)^2(2\,\lambda-3)^2(\lambda-1)(-3+5\,\lambda)}, & \textit{if } 0<\lambda\leq\frac{1}{2};\\[2ex]\dfrac{\left(\begin{array}{c}2\,\lambda^{13}+50\,\lambda^{12}-194\,\lambda^{11}-190\,\lambda^{10}+2548\,\lambda^{9}-5560\,\lambda^{8}\\-662\,\lambda^{7}+26915\,\lambda^{6}-62174\,\lambda^{5}+73636\,\lambda^{4}-48132\,\lambda^{3}\\+16425\,\lambda^{2}-3564\,\lambda+324\end{array}\right)}{12(3+\lambda)^2(3-2\,\lambda+\lambda^2)^2(\lambda-2)^2\lambda^2(2\,\lambda-3)}, & \textit{if } \frac{1}{2}<\lambda\leq 1.\end{cases}$$

For reasons of space we skip the computation details for Proposition 5, but these are available upon request. In Table 2, we report some numerical evaluations of $P_{\text{Pe}}\left(F_\lambda\right)$ and $P_{\text{Av}}\left(F_\lambda\right)$; Figure 1 give a complete overview of their behavior.

Figure 1: Vulnerability of one-shot scoring rules to the truncation paradox



It appears that as the number of voters tends to infinity, the limit probability, under the IAC assumption, of observing a voting situation in which the truncation paradox may occur with a one-shot scoring rule F_λ increases from 0 to 75% and from 0 to 34.03% respectively under the pessimistic model and the averaged model as the weight λ increases from 0 (the Plurality rule) to 1 (the Antiplurality rule). It should also be noted that for any $\lambda \in]0\ 1]$, while the paradox is not likely to occur under the optimistic scoring model, it is almost twice as likely to occur under the pessimistic model as under the averaged model. Thus, the model under which we operate does indeed have a significant impact on the probability of the paradox.

Table 2: Likelihood of the truncation paradox for one-shot scoring rules

	λ										
Models	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Pessimistic	–	0.06334	0.1322	0.2067	0.2866	0.3710	0.4575	0.5423	0.6215	0.6916	0.7500
Averaged	–	0.03145	0.06517	0.1011	0.1390	0.1778	0.2163	0.2537	0.2883	0.3180	0.3403
Optimistic	–	0	0	0	0	0	0	0	0	0	0

The lesson that can be drawn therefore is that when the voting rule is a one-shot scoring rule, any manipulation by truncation is futile if we resort to the optimistic model to deal with the possible truncated preferences. As for the averaged model, the probability of its being manipulated is half that of the pessimistic model. Thus, adopting the optimistic model would be an effective way to discourage strategic truncation behavior in an electoral system where voters have to rank candidates in order to elect a winner based on a one-shot scoring rule.

4.2.2 The case of iterative scoring rules with one-by-one eliminations

The conditions in Proposition 3 completely describe all the possible scenarios that support possible occurrences of the truncation paradox given a voting situation. From these conditions we draw some sets of linear constraints in order to characterize all possible occurrences of the truncation paradox under a rule (F'_λ) under the optimistic and the averaged models. By solving the system made by the sets of linear constraints with the same technique as in Cervone et al. (2005), we obtain Proposition 6.[8] In this proposition, we first recall the probability obtained by Kamwa and Moyouwou (2020) for the pessimistic model.

Proposition 6. *Consider the scoring runoff rule F'_λ associated with the scoring vector $w_\lambda = (1, \lambda, 0)$ with $0 < \lambda \leq 1$. As the total number n of voters tends to infinity, the limiting probability $P_{.}(F'_\lambda)$ of observing a voting situation in which the truncation paradox may occur is given by :*

$$P_{Pe}(F'_\lambda) = \begin{cases} -\dfrac{\begin{pmatrix} 996\,096\lambda^{20} - 25\,010\,368\lambda^{19} + 286\,101\,152\lambda^{18} - 2000\,804\,220\lambda^{17} \\ +9664\,972\,152\lambda^{16} - 34\,453\,144\,125\lambda^{15} + 94\,322\,255\,778\lambda^{14} \\ -203\,353\,434\,975\lambda^{13} + 350\,716\,379\,871\lambda^{12} - 488\,312\,722\,095\lambda^{11} \\ +551\,142\,449\,552\lambda^{10} - 504\,159\,008\,281\lambda^{9} + 372\,136\,194\,567\lambda^{8} \\ -219\,653\,377\,992\lambda^{7} + 102\,140\,474\,607\lambda^{6} - 36\,558\,733\,185\lambda^{5} \\ +9711\,109\,602\lambda^{4} - 1801\,641\,852\lambda^{3} + 208\,222\,083\lambda^{2} - 11\,278\,359\lambda \end{pmatrix}}{\begin{pmatrix} 96\left(\lambda-1\right)^2\left(\lambda-2\right)^2\left(2\lambda-3\right)^2\left(4\lambda-3\right)^2\left(5\lambda-3\right)^2\left(-2\lambda+\lambda^2+3\right) \\ \left(-5\lambda+\lambda^2+3\right)^2\left(-4\lambda+2\lambda^2+3\right)\left(-7\lambda+3\lambda^2+3\right) \end{pmatrix}} & \text{if } 0 \leq \lambda \leq \frac{1}{2} \\ \\ \dfrac{\begin{pmatrix} 132\lambda + 9346\lambda^2 - 55\,961\lambda^3 + 161\,587\lambda^4 - 283\,660\lambda^5 \\ +330\,502\lambda^6 - 265\,921\lambda^7 + 149\,437\lambda^8 - 57\,766\lambda^9 \\ +14\,560\lambda^{10} - 2112\lambda^{11} + 128\lambda^{12} - 180 \end{pmatrix}}{288\lambda^3(\lambda-2)^2(3-2\lambda)(-2\lambda+\lambda^2+3)(-4\lambda+2\lambda^2+3)} & \text{if } \frac{1}{2} \leq \lambda \leq 1 \end{cases}$$

[8]The computation details are available upon request.

$$
P_{Op}\left(F'_\lambda\right)=\begin{cases}
\dfrac{\left(\begin{array}{c}284150\,\lambda^6-457914\,\lambda^5+442197\,\lambda^4-100941\,\lambda^7\\+20832\,\lambda^8-8168\,\lambda^9-10611\lambda-253752\,\lambda^3\\+79848\,\lambda^2+320\,\lambda^{12}+6627\,\lambda^{10}-2592\,\lambda^{11}\end{array}\right)}{-864(\lambda^2-2\,\lambda+3)(\lambda-2)(-3+2\,\lambda)(5\,\lambda-3)(\lambda-1)^3(3-5\,\lambda+\lambda^2)} & \mathit{if}\ \ 0\le\lambda\le\frac{1}{2}\\[2ex]
\dfrac{\left(\begin{array}{c}-22742\,\lambda^4+15531\,\lambda^3+4423\,\lambda^7+21028\,\lambda^5+64\,\lambda^9\\-126+1482\,\lambda-12287\,\lambda^6-864\,\lambda^8-6665\,\lambda^2\end{array}\right)}{-864(\lambda^2-2\,\lambda+3)(\lambda-2)(-3+2\,\lambda)\lambda^3} & \mathit{if}\ \ \frac{1}{2}\le\lambda\le 1
\end{cases}
$$

$$
P_{Av}\left(F'_\lambda\right)=\begin{cases}
\dfrac{\left(\begin{array}{c}\lambda\Big(29756136329786421\,\lambda^7+229734452559443301\,\lambda^{11}\\-178272672122561787\,\lambda^{10}+117070741750497903\,\lambda^9\\-1295375725376735\,\lambda^{20}+4697132676107282\,\lambda^{19}\\-14007896683551062\,\lambda^{18}+34679171642310785\,\lambda^{17}\\-71889114422947871\,\lambda^{16}-63900465840\\-251541492982706718\,\lambda^{12}+336960000\,\lambda^{27}\\+112293876000\,\lambda^{25}-8652816000\,\lambda^{26}\\+8156911894790\,\lambda^{23}-1050941832560\,\lambda^{24}\\+291624916827179\,\lambda^{21}-53593644332126\,\lambda^{22}\\-6465363544810366\,2\lambda^8+1627625218824\,\lambda\\-19744908073848\,\lambda^2-831277990913364\,\lambda^4\\+151845798562668\,\lambda^3+3448576501074018\,\lambda^5\\-11268232559588502\,\lambda^6+125657219231572896\,\lambda^{15}\\-186150918901235109\,\lambda^{14}+234488280310373037\,\lambda^{13}\Big)\end{array}\right)}{\left(\begin{array}{c}-864\left(3\,\lambda^2-7\,\lambda+3\right)\left(13\,\lambda^2-18\,\lambda+6\right)\left(2\,\lambda^2-4\,\lambda+3\right)(-3+4\,\lambda)^2\\(-6+5\,\lambda)\,(-3+\lambda)\,(-4+3\,\lambda)\left(9\,\lambda^2-16\,\lambda+6\right)\left(5\,\lambda^2-10\,\lambda+6\right)\\(-3+5\,\lambda)^2\,(\lambda-2)\,(\lambda-1)^3\,(-3+2\,\lambda)\left(\lambda^2-2\,\lambda+3\right)\left(3-5\,\lambda+\lambda^2\right)\end{array}\right)} & \mathit{if}\ \ 0\le\lambda\le\frac{1}{2}\\[2ex]
\dfrac{\left(\begin{array}{c}2099520-18230665471\,\lambda^7-43292910605\,\lambda^{11}\\+55648570181\,\lambda^{10}-53946322373\,\lambda^9+624000\,\lambda^{18}\\-15121600\,\lambda^{17}+155807560\,\lambda^{16}+25686340889\,\lambda^{12}\\+38189483083\,\lambda^8-14475024\,\lambda+5218992\,\lambda^2\\-863638434\,\lambda^4+234422424\,\lambda^3+614152332\,\lambda^5\\+4456679443\,\lambda^6-949706470\,\lambda^{15}+3906190562\,\lambda^{14}\\-11586749657\,\lambda^{13}\end{array}\right)}{\left(\begin{array}{c}4320\,(3-2\,\lambda)\left(\lambda^2-2\,\lambda+3\right)(\lambda-2)\,\lambda^3\,(-4+3\,\lambda)\left(5\,\lambda^2-10\,\lambda+6\right)\\(-6+5\,\lambda)\,(\lambda-1)\,(-3+\lambda)\,(1+4\,\lambda)\left(2\,\lambda^2-4\,\lambda+3\right)\end{array}\right)} & \mathit{if}\ \ \frac{1}{2}\le\lambda\le\frac{2}{3}\\[2ex]
\dfrac{\left(\begin{array}{c}18720000\,\lambda^{18}-264992000\,\lambda^{17}+1517810000\,\lambda^{16}\\-4612688020\,\lambda^{15}+7219079938\,\lambda^{14}-701040792\,\lambda^{13}\\-23566887867\,\lambda^{12}+58714236393\,\lambda^{11}-80453992152\,\lambda^{10}\\+71388488778\,\lambda^9-41360020859\,\lambda^8+13989995429\,\lambda^7\\-1120219921\,\lambda^6-1234798607\,\lambda^5+524869929\,\lambda^4\\-42508179\,\lambda^3-24266682\,\lambda^2+6888564\,\lambda-524880\end{array}\right)}{\left(\begin{array}{c}4320\left(2\,\lambda^2-4\,\lambda+3\right)(-2\,\lambda+1)\,(-5+9\,\lambda)\,(5\,\lambda-1)\\(-3+5\,\lambda)^2\,(1+4\,\lambda)\,(4+\lambda)\,(-3+2\,\lambda)\left(\lambda^2-2\,\lambda+3\right)(\lambda-2)\,\lambda^3\end{array}\right)} & \mathit{if}\ \ \frac{2}{3}\le\lambda\le 1
\end{cases}
$$

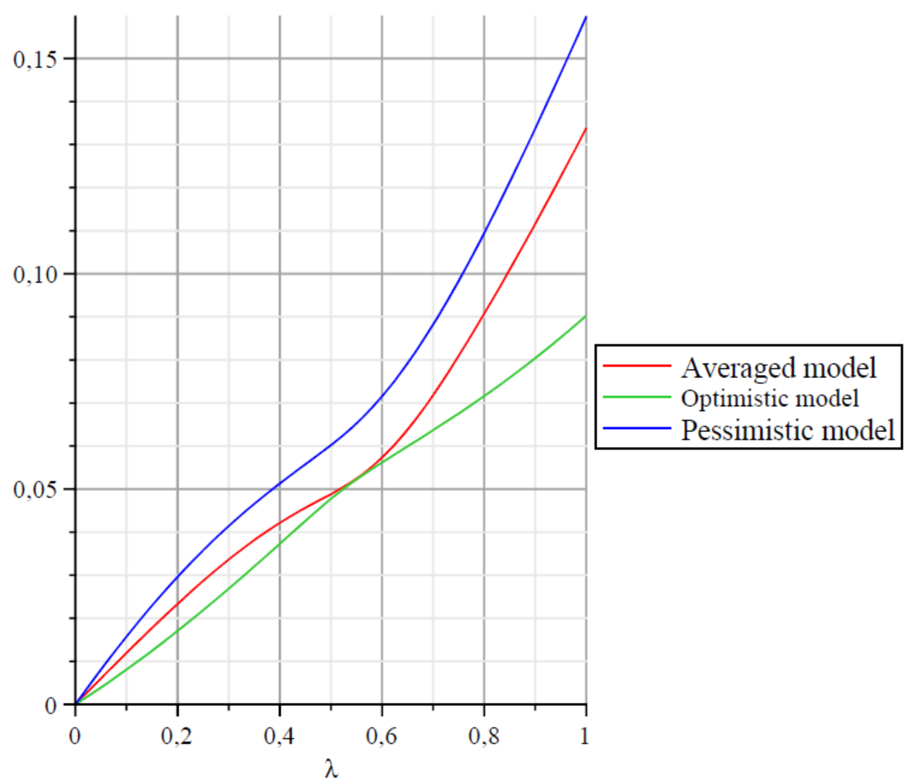
For reasons of length, we omit the calculation details of Proposition 6, but they are available upon request. In Table 3 we report some numerical evaluations of $P_{\mathrm{Pe}}\left(F'_\lambda\right)$, $P_{\mathrm{Op}}\left(F'_\lambda\right)$ and $P_{\mathrm{Av}}\left(F'_\lambda\right)$, while Figure 2 gives an overview of their behavior.

Table 3: Likelihood of the truncation paradox for runoff scoring rules with one-by-one elimination

	λ										
$Models$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Pessimistic	–	0.01575	0.02963	0.04142	0.05127	0.06018	0.07151	0.08817	0.10942	0.13374	0.15972
Averaged	–	0.01195	0.02335	0.03361	0.04216	0.04884	0.05728	0.07176	0.09071	0.11162	0.13389
Optimistic	–	0.00806	0.01706	0.02686	0.03728	0.04774	0.05615	0.06367	0.07161	0.08042	0.09028

We notice that with a large number of voters, and under the IAC assumption, the limit probability of observing a voting situation in which the truncation paradox may occur given a runoff scoring rule F'_λ increases from 0 to 15.97%, from 0 to 13.38%, and from 0 to 9.03% under the pessimistic, the averaged, and the optimistic models respectively, as the weight λ increases from 0 (the Plurality runoff rule) to 1 (the Antiplurality runoff rule). For any $\lambda \in]0\ 1]$, the paradox is more likely to occur with the pessimistic model than under the averaged and the optimistic models. Also, for $0.52 < \lambda < 0.57$ the probabilities $P_{\text{Op}}(F'_\lambda)$ and $P_{\text{Av}}(F'_\lambda)$ are quite close (but not equal) while still having $P_{\text{Op}}(F'_\lambda) < P_{\text{Av}}(F'_\lambda)$.

Figure 2: Vulnerability of runoff scoring rules to the truncation paradox



Our results tell us that the model under which we operate does indeed have a significant impact on the probability of the truncation paradox for scoring runoff rules with three candidates. Although no model completely blocks strategic truncation behavior, the optimistic model, by comparison with the other two models, makes it possible to reduce the probability of manipulation by truncation almost by half.

4.2.3 The case of iterative scoring rules with elimination according to the average

Using the sets of linear constraints derived from the conditions in Proposition 4 and by applying the same technique as in Cervone et al. (2005), we are able to compute the likelihood of the truncation paradox under a rule $\tilde{F}'_\lambda$ under the optimistic, the pessimistic, and the averaged models as displayed in Proposition 7.[9]

Proposition 7. *Consider the scoring runoff rule associated with the scoring vector* $w_\lambda = (1, \lambda, 0)$ *with* $0 < \lambda \leq 1$ *and where eliminations proceed according to the average. As the total number* n *of voters tends to infinity, the limiting probability of observing a voting situation in which the truncation paradox may occur is given by :*

$$P_{Av}\left(\tilde{F}'_\lambda\right) = \begin{cases} \dfrac{\left(\begin{array}{c} (2880\,\lambda^{12} - 16512\,\lambda^{11} + 35217\,\lambda^{10} - 164910\,\lambda^{9} \\ +980311\,\lambda^{8} - 3247752\,\lambda^{7} + 6419912\,\lambda^{6} - 8127852\,\lambda^{5} \\ +6766612\,\lambda^{4} - 3684952\,\lambda^{3} + 1263440\,\lambda^{2} - 247520\,\lambda + 21120)\lambda \end{array}\right)}{5184(-3\,\lambda+2)(\lambda^2+2-2\,\lambda)(\lambda^2-4\,\lambda+2)(\lambda-2)(-1+\lambda)^4(5\,\lambda-2)} & \textit{if} \;\; 0 \leq \lambda \leq \frac{2}{7} \\[2ex] \dfrac{\left(\begin{array}{c} 192\,\lambda^{11} - 512\,\lambda^{10} - 4165\,\lambda^{9} + 24304\,\lambda^{8} \\ -59575\,\lambda^{7} + 85782\,\lambda^{6} - 79784\,\lambda^{5} + 50732\,\lambda^{4} \\ -22860\,\lambda^{3} + 7040\,\lambda^{2} - 1216\,\lambda + 64 \end{array}\right)}{5184\lambda\,(1-\lambda)^4(\lambda^2-4\,\lambda+2)(\lambda^2+2-2\,\lambda)} & \textit{if} \;\; \frac{2}{7} \leq \lambda \leq \frac{1}{2} \\[2ex] \dfrac{\left(\begin{array}{c} 640\,\lambda^{12} - 3616\,\lambda^{11} + 3664\,\lambda^{10} + 12871\,\lambda^{9} \\ -44588\,\lambda^{8} + 62770\,\lambda^{7} - 52659\,\lambda^{6} + 36661\,\lambda^{5} \\ -27394\,\lambda^{4} + 17106\,\lambda^{3} - 6780\,\lambda^{2} + 1456\,\lambda - 128 \end{array}\right)}{5184(-5\,\lambda+2)(\lambda^2+2-2\,\lambda)(-1+\lambda)^2\lambda^4} & \textit{if} \;\; \frac{1}{2} \leq \lambda \leq \frac{2}{3} \\[2ex] \dfrac{\left(\begin{array}{c} 384\,\lambda^{16} - 5600\,\lambda^{15} + 30573\,\lambda^{14} - 72250\,\lambda^{13} \\ +30315\,\lambda^{12} + 217574\,\lambda^{11} - 492963\,\lambda^{10} + 325431\,\lambda^{9} \\ +352601\,\lambda^{8} - 916801\,\lambda^{7} + 888872\,\lambda^{6} - 502368\,\lambda^{5} \\ +181240\,\lambda^{4} - 43476\,\lambda^{3} + 7304\,\lambda^{2} - 896\,\lambda + 64 \end{array}\right)}{5184(\lambda^2-4\,\lambda+2)(-3\,\lambda+1)(\lambda^2+2-2\,\lambda)(-5\,\lambda+\lambda^2+2)(-1+\lambda)^2\lambda^4} & \textit{if} \;\; \frac{2}{3} \leq \lambda \leq -1+\sqrt{3} \\[2ex] \dfrac{\left(\begin{array}{c} 384\,\lambda^{12} - 4064\,\lambda^{11} + 11632\,\lambda^{10} + 5044\,\lambda^{9} \\ -56144\,\lambda^{8} + 35238\,\lambda^{7} + 47947\,\lambda^{6} - 74683\,\lambda^{5} \\ +42124\,\lambda^{4} - 12446\,\lambda^{3} + 2244\,\lambda^{2} - 320\,\lambda + 32 \end{array}\right)}{5184(\lambda^2-4\,\lambda+2)(-3\,\lambda+1)(-5\,\lambda+\lambda^2+2)\lambda^4} & \textit{if} \;\; -1+\sqrt{3} \leq \lambda \leq 1 \end{cases}$$

[9] Computation details are available upon request.

$$
P_{Op}\left(\tilde{F}'_\lambda\right) = \begin{cases}
\dfrac{\begin{pmatrix} (2007040\,\lambda^{16} - 31567968\,\lambda^{15} + 254487636\,\lambda^{14} \\ -1673042594\,\lambda^{13} + 7891593436\,\lambda^{12} - 25242771985\,\lambda^{11} \\ +56041630779\,\lambda^{10} - 89218569797\,\lambda^{9} + 104324126150\,\lambda^{8} \\ -90845074086\,\lambda^{7} + 59183341011\,\lambda^{6} - 28718089245\,\lambda^{5} \\ +10225905993\,\lambda^{4} - 2593961334\,\lambda^{3} + 443687625\,\lambda^{2} \\ -45867951\,\lambda + 2165130)\lambda \end{pmatrix}}{\begin{pmatrix} 432\left(-7\,\lambda + 3 + \lambda^2\right)(7\,\lambda - 3)^2 (8\,\lambda - 3)(-1 + \lambda)^4 \\ (-3 + \lambda)\left(3 + 2\,\lambda^2 - 4\,\lambda\right)(2\,\lambda - 3)(4\,\lambda - 3)^2 (-3 + 5\,\lambda) \end{pmatrix}} & \textit{if}\ \ 0 \le \lambda \le \frac{1}{4} \\[2ex]
\dfrac{\begin{pmatrix} 25088\,\lambda^{15} + 209394\,\lambda^{14} - 6426216\,\lambda^{13} + 40984961\,\lambda^{12} \\ -135428896\,\lambda^{11} + 282711391\,\lambda^{10} - 405851583\,\lambda^{9} + 418674743\,\lambda^{8} \\ -317925857\,\lambda^{7} + 179789199\,\lambda^{6} - 75771612\,\lambda^{5} + 23495913\,\lambda^{4} \\ -5184162\,\lambda^{3} + 759618\,\lambda^{2} - 64152\,\lambda + 2187 \end{pmatrix}}{\begin{pmatrix} 432\lambda\left(-7\,\lambda + 3 + \lambda^2\right)(7\,\lambda - 3)^2 (-1 + \lambda)^4 \\ \left(3 + 2\,\lambda^2 - 4\,\lambda\right)(-3 + \lambda)(4\,\lambda - 3)^2 \end{pmatrix}} & \textit{if}\ \ \frac{1}{4} \le \lambda \le \frac{1}{3} \\[2ex]
\dfrac{\begin{pmatrix} 32\,\lambda^{12} - 552\,\lambda^{11} + 3636\,\lambda^{10} - 13918\,\lambda^{9} \\ +33134\,\lambda^{8} - 51899\,\lambda^{7} + 55638\,\lambda^{6} - 42130\,\lambda^{5} \\ +23152\,\lambda^{4} - 9234\,\lambda^{3} + 2520\,\lambda^{2} - 405\,\lambda + 27 \end{pmatrix}}{432(-7\,\lambda+3+\lambda^2)(-3+\lambda)(3+2\,\lambda^2-4\,\lambda)\lambda^2(-1+\lambda)^4} & \textit{if}\ \ \frac{1}{3} \le \lambda \le -1 + \sqrt{2} \\[2ex]
\dfrac{\begin{pmatrix} 64\,\lambda^{12} - 606\,\lambda^{11} + 2412\,\lambda^{10} - 5345\,\lambda^{9} \\ +7132\,\lambda^{8} - 5470\,\lambda^{7} + 1662\,\lambda^{6} + 664\,\lambda^{5} \\ -538\,\lambda^{4} - 138\,\lambda^{3} + 234\,\lambda^{2} - 81\,\lambda + 9 \end{pmatrix}}{432(1+2\,\lambda)(-3+\lambda)(3+2\,\lambda^2-4\,\lambda)\lambda^3(-1+\lambda)^4} & \textit{if}\ \ -1 + \sqrt{2} \le \lambda \le \frac{1}{2} \\[2ex]
\dfrac{\begin{pmatrix} 301056\,\lambda^{16} - 3833088\,\lambda^{15} + 13750656\,\lambda^{14} - 6734912\,\lambda^{13} \\ -32548680\,\lambda^{12} + 4100776\,\lambda^{11} + 77490304\,\lambda^{10} - 39137456\,\lambda^{9} \\ -54511130\,\lambda^{8} + 47314071\,\lambda^{7} + 7194988\,\lambda^{6} - 20050104\,\lambda^{5} \\ +6805017\,\lambda^{4} + 521721\,\lambda^{3} - 819072\,\lambda^{2} + 184680\,\lambda - 13851 \end{pmatrix}}{\begin{pmatrix} 31104\left(-7\,\lambda + 3 + \lambda^2\right)(7\,\lambda - 3)^2 (8\,\lambda - 3) \\ (1 + 2\,\lambda)(3 - \lambda)\lambda^4\left(-1 + \lambda^2\right) \end{pmatrix}} & \textit{if}\ \ \frac{1}{2} \le \lambda \le \frac{3}{5} \\[2ex]
\dfrac{\begin{pmatrix} 1505280\,\lambda^{15} - 9230592\,\lambda^{14} + 7289472\,\lambda^{13} + 36742208\,\lambda^{12} \\ -45946536\,\lambda^{11} - 19131136\,\lambda^{10} + 37859624\,\lambda^{9} - 4031512\,\lambda^{8} \\ +1437614\,\lambda^{7} - 15432807\,\lambda^{6} + 10976459\,\lambda^{5} - 956646\,\lambda^{4} \\ -1752966\,\lambda^{3} + 804843\,\lambda^{2} - 142155\,\lambda + 9234 \end{pmatrix}}{\begin{pmatrix} 31104(1 + 2\,\lambda)(5\,\lambda - 2)(8\,\lambda - 3) \\ (7\,\lambda - 3)^2 (3 - \lambda)\lambda^4\left(-1 + \lambda^2\right) \end{pmatrix}} & \textit{if}\ \ \frac{3}{5} \le \lambda \le \frac{3}{4} \\[2ex]
\dfrac{\begin{pmatrix} 1505280\,\lambda^{16} - 13558272\,\lambda^{15} + 31075584\,\lambda^{14} + 23913152\,\lambda^{13} \\ -107966000\,\lambda^{12} - 66112144\,\lambda^{11} + 241896312\,\lambda^{10} - 86654760\,\lambda^{9} \\ -115938132\,\lambda^{8} + 116580298\,\lambda^{7} - 34211530\,\lambda^{6} - 7077125\,\lambda^{5} \\ +10409724\,\lambda^{4} - 4974300\,\lambda^{3} + 1430676\,\lambda^{2} - 236925\,\lambda + 17010 \end{pmatrix}}{\begin{pmatrix} 31104(4\,\lambda - 1)(5\,\lambda - 2)(1 + 2\,\lambda)(7\,\lambda - 3)^2 \\ \left(-8\,\lambda + 3 + 2\,\lambda^2\right)(3 - \lambda)(1 + \lambda)\lambda^4 \end{pmatrix}} & \textit{if}\ \ \frac{3}{4} \le \lambda \le 1
\end{cases}
$$

$$
P_{Pe}\left(\tilde{F}'_{\lambda}\right)=\begin{cases}
\dfrac{\left(\begin{array}{c}
(2903040\,\lambda^{23}-118612992\,\lambda^{22}+1712646880\,\lambda^{21}\\
-13313615448\,\lambda^{20}+65434297542\,\lambda^{19}-220412439853\,\lambda^{18}\\
+536605932138\,\lambda^{17}-1018649209293\,\lambda^{16}+1827119013493\,\lambda^{15}\\
-4031565194601\,\lambda^{14}+10291301918478\,\lambda^{13}-23690326437459\,\lambda^{12}\\
+44052063146307\,\lambda^{11}-64932299077275\,\lambda^{10}+76123434660435\,\lambda^{9}\\
-71268763376898\,\lambda^{8}+53265310271586\,\lambda^{7}-31585102212528\,\lambda^{6}\\
+14666810997429\,\lambda^{5}-5216338389618\,\lambda^{4}+1370646805176\,\lambda^{3}\\
-250425805167\,\lambda^{2}+28370446344\,\lambda-1498663620)\lambda
\end{array}\right)}{\left(\begin{array}{c}
11664\left(-9\,\lambda+5\,\lambda^{2}+3\right)\left(-5\,\lambda+3\,\lambda^{2}+3\right)(7\,\lambda-3)\\
(4\,\lambda-3)\left(3+\lambda^{2}-5\,\lambda\right)\left(2\,\lambda^{2}-6\,\lambda+3\right)(2\,\lambda-3)^{2}\\
\left(2\,\lambda^{2}-3\,\lambda+3\right)(-1+\lambda)^{4}\left(3-2\,\lambda+\lambda^{2}\right)\left(-9+\lambda^{2}\right)
\end{array}\right)} & \text{if } 0\le\lambda\le\frac{1}{3}\\[2ex]
\dfrac{\left(\begin{array}{c}
4782969-95659380\,\lambda+10553855654\,\lambda^{23}\\
-44491583279\,\lambda^{22}+130813670335\,\lambda^{21}-1043958364246\,\lambda^{17}\\
+2582857536531\,\lambda^{16}-3962768871240\,\lambda^{15}+4390631916813\,\lambda^{14}\\
-3684538242360\,\lambda^{13}+2423339276040\,\lambda^{12}-1345108504311\,\lambda^{11}\\
+740308915035\,\lambda^{10}-461744433036\,\lambda^{9}+298059010713\,\lambda^{8}\\
-262810400222\,\lambda^{20}+306887928296\,\lambda^{19}+36733020908\,\lambda^{18}\\
-172175599080\,\lambda^{7}+85658336163\,\lambda^{6}-37248404445\,\lambda^{5}\\
+13841380845\,\lambda^{4}-4026374163\,\lambda^{3}+809916084\,\lambda^{2}\\
-1695481480\,\lambda^{24}+171671768\,\lambda^{25}-9517488\,\lambda^{26}+207360\,\lambda^{27}
\end{array}\right)}{\left(\begin{array}{c}
11664\left(-9\,\lambda+5\,\lambda^{2}+3\right)\left(-5\,\lambda+3\,\lambda^{2}+3\right)\left(-7\,\lambda+3+\lambda^{2}\right)\\
\lambda^{2}(1+2\,\lambda)\left(3+\lambda^{2}-5\,\lambda\right)\left(2\,\lambda^{2}-6\,\lambda+3\right)(2\,\lambda-3)^{2}\\
\left(2\,\lambda^{2}-3\,\lambda+3\right)(-1+\lambda)^{4}\left(3-2\,\lambda+\lambda^{2}\right)\left(-9+\lambda^{2}\right)
\end{array}\right)} & \text{if } \frac{1}{3}\le\lambda\le 1-\frac{\sqrt{10}}{5}\\[2ex]
\dfrac{\left(\begin{array}{c}
-1594323+43578162\,\lambda+30747544\,\lambda^{23}-273630908\,\lambda^{22}\\
+1461029470\,\lambda^{21}-90972856343\,\lambda^{17}+321380482003\,\lambda^{16}\\
-748479200948\,\lambda^{15}+1311198713673\,\lambda^{14}-1809827919078\,\lambda^{13}\\
+2003050907451\,\lambda^{12}-1776882280482\,\lambda^{11}+1237384433214\,\lambda^{10}\\
-636074088768\,\lambda^{9}+194952392460\,\lambda^{8}-4789458190\,\lambda^{20}\\
+8004589643\,\lambda^{19}+7080421727\,\lambda^{18}+15115611609\,\lambda^{7}\\
-58967400843\,\lambda^{6}+37959485625\,\lambda^{5}-14345265645\,\lambda^{4}\\
+3474797454\,\lambda^{3}-521697915\,\lambda^{2}-1828848\,\lambda^{24}+41472\,\lambda^{25}
\end{array}\right)}{\left(\begin{array}{c}
11664\left(-5\,\lambda+3\,\lambda^{2}+3\right)\left(-7\,\lambda+3+\lambda^{2}\right)\lambda^{2}(1+2\,\lambda)\\
\left(3+\lambda^{2}-5\,\lambda\right)\left(2\,\lambda^{2}-6\,\lambda+3\right)\left(2\,\lambda^{2}-3\,\lambda+3\right)\\
(-1+\lambda)^{4}\left(3-2\,\lambda+\lambda^{2}\right)(2\,\lambda-3)^{2}\left(-9+\lambda^{2}\right)
\end{array}\right)} & \text{if } 1-\frac{\sqrt{10}}{5}\le\lambda\le -1+\sqrt{2}\\[2ex]
\dfrac{\left(\begin{array}{c}
20736\,\lambda^{21}-723264\,\lambda^{20}+8133080\,\lambda^{19}-45725100\,\lambda^{18}\\
+143092260\,\lambda^{17}-194940914\,\lambda^{16}-385308330\,\lambda^{15}\\
+2938277649\,\lambda^{14}-9068598631\,\lambda^{13}+19127158605\,\lambda^{12}\\
-30791986851\,\lambda^{11}+39552168654\,\lambda^{10}-41375080647\,\lambda^{9}\\
+35491017510\,\lambda^{8}-24867614709\,\lambda^{7}+14036981994\,\lambda^{6}\\
-6231535011\,\lambda^{5}+2099474073\,\lambda^{4}-510039018\,\lambda^{3}\\
+82845747\,\lambda^{2}-7971615\,\lambda+354294
\end{array}\right)}{\left(\begin{array}{c}
11664\left(-5\,\lambda+3\,\lambda^{2}+3\right)\lambda^{2}\left(3+\lambda^{2}-5\,\lambda\right)\left(2\,\lambda^{2}-6\,\lambda+3\right)(\lambda+3)\\
\left(2\,\lambda^{2}-3\,\lambda+3\right)(-1+\lambda)^{4}\left(3-2\,\lambda+\lambda^{2}\right)(2\,\lambda-3)^{2}
\end{array}\right)} & \text{if } -1+\sqrt{2}\le\lambda\le\frac{1}{2}\\[2ex]
\dfrac{\left(\begin{array}{c}
2903040\,\lambda^{19}-20017152\,\lambda^{18}+14120448\,\lambda^{17}+247169392\,\lambda^{16}\\
-964092424\,\lambda^{15}+1100979688\,\lambda^{14}+2472535432\,\lambda^{13}-12202032664\,\lambda^{12}\\
+24718668784\,\lambda^{11}-31754061672\,\lambda^{10}+28895976381\,\lambda^{9}-20243366055\,\lambda^{8}\\
+12068956404\,\lambda^{7}-6282102879\,\lambda^{6}+2359841202\,\lambda^{5}-304402455\,\lambda^{4}\\
-209007216\,\lambda^{3}+122229243\,\lambda^{2}-26473635\,\lambda+2165130
\end{array}\right)}{\left(\begin{array}{c}
46656\left(-9\,\lambda+5\,\lambda^{2}+3\right)(1+3\,\lambda)(-7\,\lambda+3)(2\,\lambda-3)^{2}\\
\lambda^{4}(-1+\lambda)\left(3-2\,\lambda+\lambda^{2}\right)\left(2\,\lambda^{2}-3\,\lambda+3\right)(\lambda+3)
\end{array}\right)} & \text{if } \frac{1}{2}\le\lambda\le\frac{3}{5}
\end{cases}
$$

$$
= \begin{cases}
\dfrac{\begin{pmatrix} 2903040\,\lambda^{22} - 18855936\,\lambda^{21} - 114304512\,\lambda^{20} + 1027669360\,\lambda^{19} \\ -1668644520\,\lambda^{18} - 5247625496\,\lambda^{17} + 25533624832\,\lambda^{16} - 34561876784\,\lambda^{15} \\ -22186963496\,\lambda^{14} + 135514412736\,\lambda^{13} - 162221029091\,\lambda^{12} - 20704466061\,\lambda^{11} \\ +292946881953\,\lambda^{10} - 391906525824\,\lambda^{9} + 254201545656\,\lambda^{8} - 66047183730\,\lambda^{7} \\ -20154910536\,\lambda^{6} + 21140360424\,\lambda^{5} - 5777591085\,\lambda^{4} - 78406137\,\lambda^{3} \\ +403941087\,\lambda^{2} - 89597016\,\lambda + 6613488 \end{pmatrix}}{\begin{pmatrix} 46656\,(-7\,\lambda+3)\,(1+3\,\lambda)\,(\lambda^2+6\,\lambda-3)\,(2\,\lambda^2-3\,\lambda+3)\,(\lambda+3) \\ (5\,\lambda-2)\,(-7\,\lambda+3+\lambda^2)\,(2\,\lambda-3)^2\,\lambda^4\,(-1+\lambda)\,(3-2\,\lambda+\lambda^2) \end{pmatrix}} & \textit{if } \frac{3}{5} \leq \lambda \leq \frac{3}{4} \\[2ex]
\dfrac{\begin{pmatrix} -116917020 + 2277756126\,\lambda - 39907126160\,\lambda^{21} + 280564712\,\lambda^{22} \\ +1803452272\,\lambda^{23} - 182539776\,\lambda^{24} - 21758976\,\lambda^{25} \\ +2903040\,\lambda^{26} - 9659697906082\,\lambda^{15} + 28445222256624\,\lambda^{14} \\ -1247686892712\,\lambda^{18} + 1557739367010\,\lambda^{6} - 13516718508\,\lambda^{5} - 190867848714\,\lambda^{4} \\ +84897699750\,\lambda^{3} - 18983958255\,\lambda^{2} - 45543320881656\,\lambda^{13} \\ -43238256934734\,\lambda^{11} + 31600325845197\,\lambda^{10} - 21156802536186\,\lambda^{9} \\ +12497155380657\,\lambda^{8} - 5675008697694\,\lambda^{7} + 50515926752016\,\lambda^{12} \\ +2813377375616\,\lambda^{17} - 948197496131\,\lambda^{16} + 98257359624\,\lambda^{19} + 115301721144\,\lambda^{20} \end{pmatrix}}{\begin{pmatrix} 46656\,(-7\,\lambda+3)\,(1+3\,\lambda)\,(\lambda^2+6\,\lambda-3)\,\lambda^4\,(\lambda+3) \\ (\lambda^2+5\,\lambda-3)\,(5\,\lambda-2)\,(-7\,\lambda+3+\lambda^2)\,(3-2\,\lambda+\lambda^2) \\ (-6\,\lambda+3+\lambda^2)\,(-1+\lambda)\,(2\,\lambda-3)^2\,(2\,\lambda^2-3\,\lambda+3) \end{pmatrix}} & \textit{if } \frac{3}{4} \leq \lambda \leq \frac{4}{5} \\[2ex]
\dfrac{\begin{pmatrix} -4204761192\,\lambda - 35054449844\,\lambda^{21} + 4968684392\,\lambda^{22} \\ +1215015136\,\lambda^{23} - 239107584\,\lambda^{24} - 13049856\,\lambda^{25} + 2903040\,\lambda^{26} \\ -8513414733355\,\lambda^{15} + 10491474428697\,\lambda^{14} - 849266140932\,\lambda^{18} \\ -3082786513965\,\lambda^{6} + 623393679627\,\lambda^{5} + 117502635177\,\lambda^{4} \\ -111062331333\,\lambda^{3} + 31001669784\,\lambda^{2} - 8116380522411\,\lambda^{13} \\ -5551814983041\,\lambda^{11} + 3416549079384\,\lambda^{10} + 2539013699232\,\lambda^{9} \\ -7152149276739\,\lambda^{8} + 6509528169381\,\lambda^{7} + 5739980627925\,\lambda^{12} \\ +166625511404\,\lambda^{17} + 3438941055241\,\lambda^{16} + 323165732892\,\lambda^{19} \\ +9573958980\,\lambda^{20} + 233834040 \end{pmatrix}}{\begin{pmatrix} 46656\,(-7\,\lambda+3)\,(1+3\,\lambda)\,(\lambda^2+6\,\lambda-3)\,(\lambda^2+5\,\lambda-3) \\ (-7\,\lambda+3+\lambda^2)\,(-6\,\lambda+3+\lambda^2)\,(2+\lambda)\,(2\,\lambda-3)^2 \\ (2\,\lambda^2-3\,\lambda+3)\,\lambda^4\,(\lambda+3)\,(3-2\,\lambda+\lambda^2)\,(5\,\lambda-2) \end{pmatrix}} & \textit{if } \frac{4}{5} \leq \lambda \leq 1
\end{cases}
$$

In Table 4 we report some values of $P_{\text{Pe}}\left(\tilde{F}'_{\lambda}\right)$, $P_{\text{Op}}\left(\tilde{F}'_{\lambda}\right)$, and $P_{\text{Av}}\left(\tilde{F}'_{\lambda}\right)$. Figure 3 provides a complete overview of the evolution of these probabilities over [0 1].

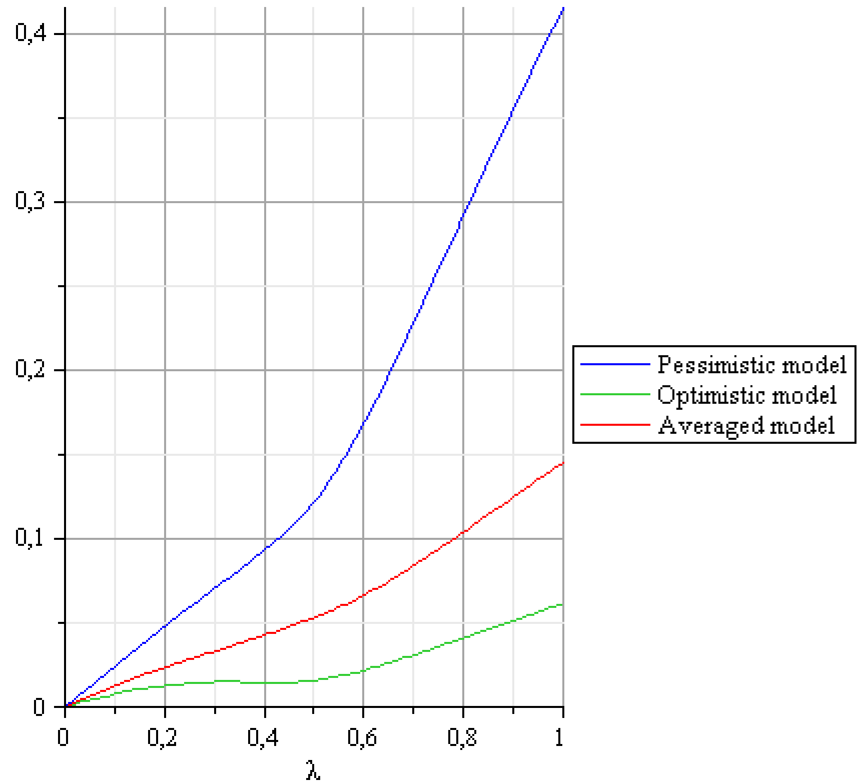
Table 4: Likelihood of the truncation paradox for runoff scoring rules with elimination by the average

	λ										
Models	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Pessimistic	–	0.02419	0.04783	0.06985	0.09285	0.12164	0.16868	0.22858	0.29247	0.35552	0.41542
Averaged	–	0.01233	0.02337	0.03302	0.04272	0.05312	0.06636	0.08402	0.10402	0.12479	0.14525
Optimistic	–	0.00758	0.01266	0.01485	0.01451	0.01562	0.02181	0.03086	0.04095	0.05126	0.06141

Under each of the models, the probability of the truncation paradox is maximized by the Kim-Roush rule ($\lambda = 1$) with 41.54% under the pessimistic model, 14.52% under the averaged model, and 6.14% for the optimistic model. For the Nanson rule, we get 12.16%, 5.31%, and 1.56% for the pessimistic, averaged, and the optimistic models respectively.

We also note that the probabilities of the truncation paradox are between 2 and 3 times higher under the pessimistic model than under the averaged model and between 4 and 5 times higher than under the optimistic model. The optimistic model is thus the one which offers the least possibility of manipulation.

Figure 3: Vulnerability of runoff scoring rules (with elimination by the average) to the truncation paradox



It has been established that the average-elimination rules behave rather well in the face of voting paradoxes compared to one-to-one elimination scoring rules and one-shot scoring rules(Favardin and Lepelley, 2006, Kamwa, 2019, Kim and Roush, 1996, Lepelley and Valognes, 2003). A comparison of Tables 3 and 4 shows that apart from the optimistic model, the elimination rules at the mean tend to exhibit the paradox of truncation more than the one-to-one elimination scoring rules. Our results thus lead us to conclude that, to our knowledge, the truncation paradox is one of the rare cases where the rules of elimination at the mean perform less well than scoring runoff rules (one-to-one elimination). A comparison of the probabilities leads to the conclusion that iterative scoring rules perform better than one-shot scoring rules.

5 Concluding remarks

The objective of this paper was to contribute to the small pool of works on the evaluation of the probability of occurrence of the truncation paradox. For three-candidate elections, we have characterized all the voting situations under which the truncation paradox is likely to occur for one-shot scoring rules and for scoring runoff rules, under the three models that can be adopted when dealing with incomplete preferences: the pessimistic, the optimistic, and the averaged models. Then we computed the limiting probability of the truncation paradox for each of the models. It emerged that for any one-shot scoring rule such that $\lambda \in]0\ 1]$, the truncation paradox never occurs under the optimistic scoring model, while it is almost twice as likely to occur under the pessimistic scoring model as under the averaged scoring model. For scoring runoff rules (with elimination one by one and elimination by the average), we found that for any $\lambda \in]0\ 1]$, the truncation paradox is slightly more likely to occur under the pessimistic scoring model than under the average scoring model, and more likely to occur than under the optimistic scoring model. The lesson we draw from our analysis is that the likelihood of occurrence of the truncation paradox is highly dependent on the model that is applied to truncated preferences. Thus, for ballots which use scoring rules or which employ scoring in the second-round runoff, and where voters must provide a (complete) ranking of the candidates, care must be taken to choose the "best" model for dealing with truncated preferences—although this does not mean, of course, that we necessarily want to strip voters of all possibility of strategic action. Compared to the other models, however, the pessimistic model is less to be recommended, since it leaves the door open for a considerable degree of manipulation by truncation, while the optimistic model limits the scope for such manipulation.'

Appendix

Appendix A: Proof of Proposition 1

Consider a voting situation π where $n \geq 2$ voters have sincere strict rankings on $m \geq 3$ candidates in a set A. Assume for a given scoring rule that candidate x is the winner; this means that for all $y \in A \setminus \{x\}$, we have $S(\pi, w, x) > S(\pi, w, y)$. Let consider a group of $\bar{n}$ voters who rank candidate x at position $p > 1$ and who want to favor by sincere truncation a more preferred candidate say y ranked at one of the positions $k < p$. If they truncate after position l ($k \leq l < p$) and that the optimistic model is in effect, this implies that the score of all the candidates they rank from the top to position $l + 1$ are not affected (including that of the candidate they wish to favor) while the new score of candidate x is equal to $S(\pi, w, x) + (-w_p + w_{l+1})\bar{n}$. Since by definition $w_{l+1} \geq w_p$, candidate x records an increase of his score and he is still get more than y the candidate to favor. Thus, when the optimistic model is used to deal with truncated preferences, no one-shot scoring rule is manipulable by sincere truncation.

Now, let us assume the following profile on a the set $\{x, y, z\}$ with $5r + t - 2$ voters ($r, t \in \mathbb{R}_+$ and $r > 2$, $t > 2$).

$r+1$	$r-2$	r	t	r	$r-1$
x	x	y	y	z	z
y	z	x	z	x	y
z	y	z	x	y	x

With this profile, let us set $t = r$ and consider iterative scoring rules with one-by-one eliminations ($\forall\lambda \in]0\ 1]$). One can check that z is eliminated at the first round and x wins the majority duel against y. Assume that the r voters with the ranking yxz truncate. If the optimistic model or the averaged model, x is now rushed out at the first round and y wins the majority duel against z. So, it always possible to build a voting situation where the truncation paradox occurs for a given iterative scoring rule with one-by-one eliminations under the optimistic model or the averaged model. Instead, if we consider all the iterative rules with eliminations on the average ($\forall\lambda \in]0\ 1]$), x and y are qualified for the second round since they score above the average and x wins this round. When the r voters with the ranking yxz truncate, if the optimistic model is assumed, x is now the only one scoring below the average and y defeats z in the last round. Thus, all iterative rules with eliminations on the average are vulnerable to the truncation paradox when the optimistic model is assumed.

By following the same approach on our profile with $t = r - 1$, one can easily show that when r voters with the ranking yxz truncate, the truncation paradox occurs for all the one-shot scoring rules ($\forall\lambda \in]0\ 1]$) when the averaged model is assumed and for all the iterative rules with eliminations on the average both under the pessimistic and the averaged models.

Appendix B: Proof of Proposition 3

The proof for the pessimistic model provided in appendix C of Kamwa and Moyouwou (2020) applies here. We choose to skip the proof for the averaged model as it follows the same scheme as that of the pessimistic model. So, we only need to focus on the proof for the optimistic model. For a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on A and a runoff rule associated with $0 < \lambda \leq 1$, where x is the winner, y is the challenger, and z is the first-round loser, let us assume the optimistic model.

- *The truncation paradox is liable to occur at π only in favor of y if and only if x is the first-round loser at $\pi([yxz])$* and y wins the majority duel against z.

  By truncating their true preferences, voters who strictly prefer y to x (those with the ranking yxz) increase the score of z while those of x and y remain unchanged in such a way that x is ruled out in the first round at $\pi([yxz])$ and y wins the second round against z. Hence, the truncation paradox occurs in favor of y. This is not possible if voters with the ranking yzx wanted to favor y since in the first round of $\pi([yzx])$, the score of x increases while those of y and z remain unchanged and x still defeats y in the second round.

- *The truncation paradox cannot occur at π in favor of z.*

Let us assume that voters who strictly prefer z to x want to engage in manipulation. If those with the ranking zxy truncate, this increases the score of y while those of x and z are unchanged: z is ruled out in the first round at $\pi([zxy])$. If those with the ranking zyx truncate, it does not affect the scores of y and z while that of x increases: z is ruled out in the first round at $\pi([zyx])$. Thus, there is no way to favor candidate z.

Appendix C: Proof of Proposition 4

Here, we only focus on the proof of the first part of the proposition; that of the second part follows an almost similar approach. Let us consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{x, y, z\}$ and a scoring rule $\widetilde{F}'_\lambda$ associated with the scoring vector $w_\lambda = (1, \lambda, 0)$.

Assume that x wins at the first round. By definition, we get: $S(\pi, \lambda, x) \geq T(\pi)$, $S(\pi, \lambda, y) < T(\pi)$ and $S(\pi, \lambda, z) < T(\pi)$ with $T(\pi) = \frac{S(\pi,\lambda,x)+S(\pi,\lambda,y)+S(\pi,\lambda,z)}{3}$.

Assume that voters with the ranking yxz truncate in order to favor y; if the the optimistic model is assumed, it follows that the scores of x and y are still unchanged while that of z increases by λn_3. The average of the scores at $\pi([yxz])$ is now equal to $T(\pi) + \frac{\lambda}{3}n_3$. Since $S(\pi, \lambda, y) < T(\pi)$ it follows that $S(\pi, \lambda, y) < T(\pi) + \frac{\lambda}{3}n_3$: candidate y is certainly rushed out at the first run. Also in order to favor candidate y, if voters with yzx truncates, it is easy to see that this is unfruitful at $\pi([yzx])$. We reach the same conclusion if one to favor candidate z at $\pi([zxy])$ or $\pi([zyx])$. So, the truncation paradox is not liable under the optimistic model.

If the averaged model is assumed when voters with the ranking yxz (resp. yzx) truncate, it follows at $\pi([yxz])$ (resp. at $\pi([yzx])$) that the score of y is still unchanged while that of z increases (resp. decreases) by $\frac{\lambda}{2}n_3$ (resp. by $\frac{\lambda}{2}n_4$) and that of x decreases (resp. increases) by $\frac{\lambda}{2}n_3$ (resp. by $\frac{\lambda}{2}n_4$). The average of the scores at $\pi([yxz])$ (resp. at $\pi([yzx])$) remains the same as at π: in fact, x remains the winner. We get the same conclusion if one tries to favor z at $\pi([zxy])$ or at $\pi([zxy])$. So, the truncation paradox is not liable under the averaged model.

Let us assume the pessimistic model and that voters with the ranking yxz truncate. At $\pi([yxz])$, it follows that one the score of x decreases by λn_3 and the new average of the scores is T' such that $T' = T(\pi) - \frac{\lambda n_3}{3}$. If the set of voters who truncate is enough, candidate y wins at $\pi([yxz])$ if he scores above T' : either he is the only one in this case or he wins the majority duel against one of his competitors meeting this condition. At $\pi([zxy])$, we get the same conclusion in favor of z.

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