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During the last decades, much progress has been made concerning the prediction of hurricane tracks, but the dynamics near the hurricane's eye is still not very well understood. This motivates our study of non-smooth solutions to equations of fluid dynamics (derived from Navier-Stokes equations) with vortex or shock-wave type irregularities. We have in mind the wind field near the eye of a hurricane where it drops from its maximal value to (nearly) zero within a very small region of space.
In the first part of this paper, we discuss a numerical and analytic study of a two-dimensional model which, in spite of its simplicity, predicts remarkably correctly the "wall of the eye" and allows us to get analytic expressions for the asymptotic behaviour of radial and tangential wind field near this wall.
In the second part, we prove a new theorem concerning solutions with weak singularities (specifically : a jump discontinuity on an arbitrary hypersurface), to such nonlinear PDE in *n* dimensions. To do so, we briefly introduce a framework of *algebras* of generalized function, required in order to deal with *nonlinear* differential equations with non-smooth functions, which in spite of real-world applications can't be formulated with ordinary distribution theory.

Asymptotics of the wind field near the eye of a hurricane

In this part, we consider the two-dimensional differential equation

$$\partial_t\,(u,v) \;+\; (u,v)\cdot \mathtt{grad}\,(u,v) = \sigma\,(u,v) + \omega\,(v, -u),$$

which follows from the general Navier-Stokes equations in the limit of a nonviscid fluid and after coordinate transformation to a frame having its origin on the surface of the rotating earth, which induces the "Coriolis force" terms proportional to $\omega = (2\square\,/\,24h)\cos(latitude)$. The parameter σ accounts for friction (negative contribution) but also for energy supply from the hot water surface. Both are proportional to (u,v) only for small to moderately large winds: the energy supply can be considered to saturate at some limiting value $\sigma_*\,u_*$, while the friction is known to acquire quadratic terms for larger wind speed. Concerning the energy supply, many more more or less complicated piecewise defined heat transfer functions are considered in the literature, we also considered one of the form $\sigma\,\boldsymbol{u} = \sigma_*\,\boldsymbol{u}\,/\,(1+|\boldsymbol{u}|/u_*)$.
We assume radial symmetry around the center of the hurricane assumed located in $(x,y) = (0,0)$, *i.e.*, that the wind field (u, v) only depends on $\square = \sqrt{\square^2 + \square^2}$, and we reparametrize it using radial and tangential components (a, b), viz: $(\square,\square) = \square(\square)\,(\square,\square) + \square(\square)\,(-\square,\square)$. This yields the system of equations

$$a' = [(\,\sigma - a + (\,\omega + b\,)\,b\,/\,a\,]\,/\,r\,, \tag{1}$$

$$b' = [(\,\sigma\,/\,a - 2\,)\,b - \omega\,]\,/\,r\,. \tag{2}$$

We can integrate these equations numerically, starting at a given point r_i, if we know the initial conditions $a(r_i)$, $b(r_i)$, and of course the parameters σ and ω. We can also make a qualitative study of this system of ODE.
Both approaches lead to the following results: the tangent component $b(r)$ has a very small positive value for large r, which increases monotonically as r tends to some finite value R_{eye}. (This corresponds to the counter-clockwise rotating winds.) The radial component $a(r)$ has a small negative value for large r, which decreases to a minimum at $R_{\min} \sim k\,R_{\mathrm{eye}}$ with $2 < k < 9$, and then tends to zero as $r \to R_{\mathrm{eye}}$.
Both $a'(r)$ and $b'(r)$ tend to $-\infty$ as $r \to R_{\mathrm{eye}}$. When we first noticed this behaviour, we thought $b(r)$ might be unbounded, which would not be physical but could be explained by the right hand side $\sigma\,u$ which can allow an unbounded energy supply. At this point we introduced the more sophisticated form explained above, which saturates at a value $\sigma_*\,u_*$. However, a more thorough analysis shows that even with constant σ, we get a finite value b_0 as $r \to R_{\mathrm{eye}}$.

Indeed, we can solve a simplified version of the system which is valid asymptotically, as $r \to R_{\mathrm{eye}}$ and $a \to 0$.
In this limit, the equation for b becomes $b'\,/\,b \sim \sigma\,/\,(r\,a(r))$, which we can integrate to get $b(r) = b_0\,J(r)$ with $\square(\square) \sim \int_{\square_0}^{\square} \frac{\square}{\square\,\square(\square)}\,\square\square$. We find that a solution is given using the ansatz $a(r) = -A\,(r/R_{\mathrm{eye}} - 1)^{\alpha}$, which gives $b(r) \sim b_0\,\exp(-\sigma\,(r/R_{\mathrm{eye}} - 1)^{1-\alpha}/\,((1-\alpha)\,A\,r/R_{\mathrm{eye}}))$ and back in (1), $\alpha = 1/2$ and $A^2 = (\omega + b_0)\,b_0$.
So we have found that $a(r) \sim -\sqrt{\{(\omega + b_0)\,b_0\,(r/R_{\mathrm{eye}} - 1)\}}$ and $b(r) \sim b_0\,\exp(-2\,\sigma\,\sqrt{\{(\omega + b_0)\,b_0\,(r/R_{\mathrm{eye}} - 1)\}}\,R_{\mathrm{eye}}/r)$.

Our next goal is the analysis of this simple system is to obtain further analytical results for $r \to R_{\mathrm{eye}}$ and $R_{\min}$ in terms of the initial data. But we are also aware that this model has its limits. In particular, we think that an extension to 3D is required to capture more of the dynamics of the real hurricane's eye. The vertical airflows near the eye and between the rainbands are physically important, and taking them into account in our model should improve its potential for predicting realistic values.

Jump condition at the eye's wall

In this second part we prove a new theorem for solutions to Navier-Stokes type equations with presence of a jump discontinuity. We first found this result for the case of the two-dimensional model analysed earlier, but could generalize it to an arbitrary number of n dimensions. Indeed, we will consider the system of partial differential equations $\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \mathrm{grad}\, \boldsymbol{u} = A(\boldsymbol{u} - \boldsymbol{u}_*)$, where $\boldsymbol{u} : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is an n-dimensional vector field depending on $(t,x) \in \mathbb{R}^{1+n}$, and A is a continuous linear operator, or simply a matrix acting on the n-component vector $\boldsymbol{u}$, while $\boldsymbol{u}_*$ is a given wind field (of the trade wind in our application to hurricanes).

We wish to consider a solution of the form $\boldsymbol{u}(t,x) = \boldsymbol{u}_*(t,x) + H(S(t,x))\, \hat{\boldsymbol{u}}(t,x)$, where H is the Heaviside step function, equal to 1 for positive arguments and zero else. Thus, we have $\boldsymbol{u} = \boldsymbol{u}_*$ inside the region $D = \{ (t,x) \mid S(t,x) < 0\}$ which represents the hurricane's eye, where we have nearly no wind (or more precisely, just the roughly constant trade wind), and $\boldsymbol{u} = \boldsymbol{u}_* + \hat{\boldsymbol{u}}$ outside this domain, where we have very strong winds right next to the eye's wall.
In classical distribution theory, this nonlinear differential problem does not make sense, because one cannot consistently multiply distributions. (Classically, $H = H^2 = H^3$, and using Leibniz' rule we get for the derivative $H' = 2\, H\, H' = 3\, H\, H'$, whence $H\, H' = o$, but we also have $H' = \delta \neq o$.)
To formulate, study and solve nonlinear differential equations with irregular solutions, we need a theory of **algebras** of generalized function, as are given by the Colombeau type algebras of sequence spaces, in which differentiation and multiplication is always well-defined.
More precisely, we consider the space $\mathcal{E}$ of smooth functions equipped with seminorms $\mathcal{P} = \{ p_{\alpha,K} : \Box \mapsto \|\partial^\alpha \Box\|_{K,\infty} \}$, and the scale $M = \{ (\varepsilon^m)_{\varepsilon \in \Lambda = (0,1]} ; m \in \mathbb{N} \}$. Then the space of "moderate" sequences is $\mathcal{E}_M = \{ f \in \mathcal{E}^\Lambda \mid \forall p \in \mathrm{P}\ \exists a \in \mathrm{M}: p(f_\varepsilon) = \mathrm{O}(1/a_\varepsilon)\}$, and the Colombeau-type algebra is the associated Hausdorff space $\mathcal{G}_M = \mathcal{E}_M / \mathcal{N}$ where $\mathcal{N} = \{ f \in \mathcal{E}^\Lambda \mid \forall p \in \mathrm{P},\ a \in M: p(f_\varepsilon) = \mathrm{O}(a_\varepsilon)\}$ is the closure (intersection of all neighborhoods) of zero for the naturally associated so-called "sharp" topology (for which multiplication is continuous). The construction is functorial, the spaces are (pre)sheaves of topological algebras, and the point values of generalized functions are generalized numbers $^*\Box$ which can be infinitely small or large and still mathematically well-defined, as for example $\delta(0)$.
Schwarz distributions $\mathcal{D}'(\Omega)$ and L^1_{loc} are injected into $\mathcal{G}_M(\Omega)$ via $\mathrm{i}_\varphi : \mathrm{T} \mapsto [\varphi_\varepsilon \Box \mathrm{T}]$, *i.e.,* convolution with mollifier $\varphi_\varepsilon = \varepsilon^{-n}\, \varphi(\cdot/\varepsilon)$, where φ has Fourier transform equal to 1 near the origin, as to have $\int_{\mathrm{R}^n} \varphi = 1$ and vanishing higher moments. Consequently, $\boldsymbol{\delta} = i_\varphi \delta = [\varphi_\varepsilon]$ is a Dirac delta function, which is for $n = 1$ the derivative of $\mathbf{H}(\mathrm{x}) = i_\varphi H(\mathbf{x}) = [\int_{-\infty}^{\mathrm{x}/\varepsilon} \varphi(y)\, \mathrm{d}y]$; $\mathbf{H}(0) = ½$. The composition of two generalized functions is well defined if the first one is c-bounded [1, Def. 1.2.7], which is the case for smooth functions. Thus, $\mathbf{H}\circ S(x) = [H_\varepsilon(S(x))] \in \mathcal{G}_M(\mathbb{R}^n)$ is well defined.

Writing $\mathrm{D}_t = \partial_t + \boldsymbol{u} \cdot \nabla$, we have $\mathrm{D}_t(\boldsymbol{u}_* + (H \circ S)\, \hat{\boldsymbol{u}}) = \mathrm{D}_t \boldsymbol{u}_* + (H \circ S)\, \mathrm{D}_t \hat{\boldsymbol{u}} + (\delta \circ S)\, (\mathrm{D}_t S)\, \hat{\boldsymbol{u}}$. To prove the result, it is sufficient to take the scalar product of the differential equation with $\hat{\boldsymbol{u}}$. Putting to the right hand side all terms without δ, we have: $\hat{\boldsymbol{u}}^2 (\delta \circ \mathrm{S})\, \mathrm{D}_t \mathrm{S} = \hat{\boldsymbol{u}} \cdot ((\mathrm{A}\, \hat{\boldsymbol{u}} - \mathrm{D}_t \hat{\boldsymbol{u}})\, \mathrm{H} \circ \mathrm{S} - \mathrm{D}_t \boldsymbol{u}_*)$. For $x \in \mathrm{S}_0 := \partial D$, $\delta \circ S(x) = \delta(0)$ is an infinitely large, invertible generalized number, by which we can divide the equation. Using $\mathrm{H}(0) = ½$, we have $\forall \mathbf{x} \in \mathrm{S}_0 : \hat{\boldsymbol{u}}^2 (\partial_t + (\boldsymbol{u}_* + ½\, \hat{\boldsymbol{u}})\cdot\nabla)S = (1/\delta(0))\, \hat{\boldsymbol{u}} \cdot (½\, (A.\hat{\boldsymbol{u}} - \mathrm{D}_t \hat{\boldsymbol{u}}) - D_t \boldsymbol{u}^*)$. All expressions except $1/\delta(0)$ take finite real values in any $x \in S_0$. Therefore, the left hand side must be zero, i.e., $\forall\ x \in S_0$: $\hat{\boldsymbol{u}} \cdot \nabla S = -2\, (\partial_t + \boldsymbol{u}_* \cdot \nabla)\, S(x)$. So we have proved the following

Theorem: If the PDE $(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{u} = A(\boldsymbol{u} - \boldsymbol{u}_*)$ has a solution of the form $\boldsymbol{u} = \boldsymbol{u}_* + (H \circ S)\, \hat{\boldsymbol{u}}$, where H is the Heaviside step function and $S \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$, then $\hat{\boldsymbol{u}} \cdot \nabla S = -2\, (\partial_t + \boldsymbol{u}_* \cdot \nabla)S$ on ∂D, $D = \{ x \mid S(x) < 0 \}$.

Corollary: If $(\partial_t + \boldsymbol{u}_* \cdot \nabla)S = 0$ on $\partial D \Leftrightarrow$ the domain D moves with speed $\boldsymbol{u}_*$ (consider, *e.g.*, $S(x,t) = \|\boldsymbol{x} - t\, \boldsymbol{u}_*\|^2 - r_0^2$, as in the case of our hurricane), then the jump $\hat{\boldsymbol{u}}$ is tangent (orthogonal to the normal vector) to the boundary of D in each point of this boundary.

This is (fortunately) in agreement with real-world observations and also with the previous study of the 2-dimensional case: As predicted by the Corollary, the radial wind component goes to zero when approaching the wall of the hurricane's eye.

Summary & Conclusion

The very simple equation $(\partial_t + u\cdot\nabla)\, u = \sigma\, u + \Omega \times u$ predicts qualitatively correctly the existence of the eye of the hurricane and finite limiting speed of wind at the border, in spite of the right hand side possibly violating laws of conservation. We found an analytic expression for the asymptotic form of the solution near the eye's wall. We also prove, in arbitrary dimension, that if there is a “shock wave” hypersurface where the wind field makes jump, then the wind must be tangent to that surface. Non-linearity is essential for producing singularities (shock waves etc.) but nonlinear differential problems with singularities cannot be handled in the classical mathematical framework of Schwartz distributions which is a linear theory. Our theorem was straightforwardly proved using Colombeau-type differential algebras, and the result is confirmed by numerical and algebraic analysis of the 2-dimensional case and real-world observations. We plan to continue our work on one hand to try to get explicit non-smooth solution to Euler/Navier-Stokes equations in 2D, and to extend the analytic and numerical study to 3 dimensions.

References

[1] M Grosser, M Kunzinger, M Oberguggenberger, R Steinbauer: Geometric Theory of Generalized Functions with Applications to General Relativity. Kluwer Academic Publ., Dordrecht, 2001.