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***Abstract*— This paper deals with the stabilization of a linear hyperbolic system subject to a boundary disturbance. Our feedback design relies on a super-twisting control algorithm, which leads to a feedback that is continuous with respect to the state, in contrast with the classical sliding mode design. Our first result is the existence of solutions of the closed-loop system. Moreover, the global asymptotic stability, that is our second result, is proved together with the guarantee that the disturbance is rejected.**

I. INTRODUCTION

This paper is concerned with the stabilization of a linear hyperbolic system with a boundary control and subject to a disturbance (see e.g, [1] for a review on this class of system). To be more precise, we aim at designing a super-twisting algorithm [13] for that purpose.

Systems of linear transport equations have been the subject of much attention for many years because of the many physical phenomena they model: e.g pressure drilling [10], aeronomy [19]. A good overview of the actual research lines concerning this topic is provided in [1]. For more details, the reader can also refer to [3]–[5].

Sliding mode control (SMC) strategy has been proved to be efficient for robust control of nonlinear systems of ordinary differential equations (ODEs) [6], [21], [23], [25]. Such controllers allow to force, thanks to discontinuous terms, the system trajectories to reach in a finite time a manifold, called sliding surface, and to evolve on it. This manifold being defined from control objectives [21]. Roughly speaking, the control design is decomposed into two steps: firstly, a sliding variable is selected such that, once this variable equals zero, global asymptotic stability is ensured; secondly, a discontinuous feedback-law is designed such that the trajectory reaches the sliding surface, that is defined thanks to the sliding variable. On this sliding surface, the disturbance is rejected. The generalization of the SMC procedure to the partial differential equations (PDE) case is not new. In [16], [17], a definition of equivalent control (which is the control applied to the system after reaching the sliding surface, to ensure that the trajectories stays on the surface thereafter) for systems governed by semilinear differential equations in Banach spaces has been proposed. One can refer also to [11], [12] where differential inclusions and viability theory are combined to design sliding mode controllers for semilinear differential equations in Banach spaces. In the last decade, a backstepping strategy has been used to select a sliding variable [8], [18], [24]. Note also that the SMC feedback-law is discontinuous, which creates chattering phenomena when implementing the control numerically. Recall that chattering phenomena are characterized by oscillations having finite amplitude. Therefore, in practical control cases, it is important to reduce this phenomena by providing continuous or smooth controller.

Based on second-order sliding mode techniques (see e.g, [21, Chapter 4]), the super twisting (ST) algorithm has been developed for systems whose the sliding variable admits a relative degree (see [21, Definition 1.6]) equal to 1. The essential feature of the ST control is to require that only the measurement of the sliding variable to guarantee the convergence in finite time to zero of the sliding variable and its derivative. Moreover, the ST feedback-law is continuous with respect to the state and this drastically attenuates the chattering phenomenon. The contribution of this paper is to use a super-twisting strategy to design a continuous feedback-law which allows to reject the disturbance in finite time and to ensure that the resulting closed-loop system is globally asymptotically stable. The sliding variable comes from the gradient of a Lyapunov functional that is used for hyperbolic systems [1, Section 2.3]. This imposes to measure the integral of the state with a certain weight and a boundary of the state. Such a sliding variable allows to directly use well-known results on the stabilization of hyperbolic systems.

This paper is organized as follows. Section II presents a system of linear hyperbolic equations, the super-twisting based control law and the main results of the paper. Section III contains the proofs of the main results. Finally, Section IV collects some remarks and introduces some future research lines to be followed.

Notation. The set of non-negative real numbers is denoted in this paper by $\mathbb{R}_+$. When a function f only depends on the time variable t (resp. on the space variable x), its derivative is denoted by $\dot{f}$ (resp. f'). Given any subset of $\mathbb{R}$ denoted by Ω ($\mathbb{R}_+$ or an interval, for instance), $L^2(\Omega;\mathbb{R}^n)$ denotes the set of (Lebesgue) measurable functions $f_1, \ldots, f_n$ such that, for $i = \{1,\ldots,n\}$, $\int_\Omega |f_i(x)|^2 dx < +\infty$. The associated norm is $\|(f_1,\ldots,f_n)\|^2_{L^2(\Omega;\mathbb{R}^n)} := \int_\Omega |f_1(x)|^2 dx + \ldots + \int_\Omega |f_n(x)|^2 dx$. Given two vector spaces E and F, $\mathcal{L}(E,F)$ denotes the space of linear applications from E into F. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$, if it is continuous, strictly increasing and satisfies $\alpha(0) = 0$. A function α : $\mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}_\infty$, if it is of class $\mathcal{K}$ and unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$, if for each

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fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and, for each fixed $r \geq 0$, $\beta(r, \cdot)$ is decreasing and satisfies $\lim_{t\to\infty}\beta(r,t) = 0$.

II. Main results

A. Problem statement

Let $\ell > 0$ and λ_1, $\lambda_2 \in C^1([0,\ell])$ such that, for all $x \in [0,\ell]$, $\lambda_1(x) > 0$, $\lambda_2(x) > 0$ and $k_2 \in \mathbb{R}\backslash\{0\}$. Consider the following linear hyperbolic system

$$\begin{cases} \partial_t R_1(t,x) + \lambda_1(x)\partial_x R_1(t,x) = 0, \\ \partial_t R_2(t,x) - \lambda_2(x)\partial_x R_2(t,x) = 0, \\ R_1(t,0) = u(t) + d(t), \\ R_2(t,\ell) = k_2 R_1(t,\ell), \\ R_1(0,x) = R_1^0(x), R_2(0,x) = R_2^0(x), \end{cases} \tag{1}$$

where $R_1^0, R_2^0 \in L^2(0,\ell)$, u denotes the control and $d(\cdot)$ is an unknown disturbance. Assume that $d(\cdot)$ is bounded and globally Lipschitz over $\mathbb{R}_+$. Furthermore, there exists a known positive constant C such that, for a.e $t \in \mathbb{R}_+$,

$$|\dot{d}(t)| \leq C. \tag{2}$$

When the system (1) is undisturbed ($d = 0$), it is known that the feedback law

$$u(t) := k_1 R_2(t,0),$$

allows to stabilize the system if $|k_1 k_2| < 1$, see [1, Theorem 2.11.]. The proof, relies on the following Lyapunov functional

$$V(t) = \int_0^\ell \left(q_1(x)R_1^2(t,x) + q_2(x)R_2^2(t,x)\right) dx, \tag{3}$$

$$\begin{aligned} q_1(x) =& \frac{p_1}{\lambda_1(x)} \exp\left(-\int_0^x \frac{\nu}{\lambda_1(\sigma)} d\sigma\right), \\ q_2(x) =& \frac{p_2}{\lambda_2(x)} \exp\left(\int_0^x \frac{\nu}{\lambda_2(\sigma)} d\sigma\right), \end{aligned} \tag{4}$$

for any $\nu, p_1, p_2 > 0$ selected as in [1, Theorem 2.11.].

In this paper, the goal is to develop a super-twisting based controller. To do so, we will consider a continuous robust feedback-law u which allows to reject the disturbance in finite-time and to globally asymptotically stabilize the system around the equilibrium point $(0,0)$ in the functional space

$$X := L^2(0,\ell;\mathbb{R}^2).$$

More precisely, the aim is to find a sliding surface Σ on which (1) becomes in finite-time the following system

$$\begin{cases} \partial_t R_1(t,x) + \lambda_1(x)\partial_x R_1(t,x) = 0, \\ \partial_t R_2(t,x) - \lambda_2(x)\partial_x R_2(t,x) = 0, \\ R_1(t,0) = k_1 R_2(t,0), \\ R_2(t,\ell) = k_2 R_1(t,\ell), \end{cases} \tag{5}$$

with k_1 chosen such that $|k_1 k_2| < 1$. From [1, Theorem 2.11], it is known that $(0,0)$ is exponentially stable for (5). The next section will provide a definition of this sliding surface Σ (and its related sliding variable) and the associated super-twisting sliding mode controller.

B. Control Design

Introduce the sliding surface Σ defined as follows,

$$\Sigma := \left\{(f,g) \in X \mid \int_0^\ell \left(q_1(x)f(x) + q_2(x)g(x)\right) dx = 0\right\} \tag{6}$$

with q_1 and q_2 defined in in (4). Then, for any solution (R_1, R_2) of (1), the sliding variable $S : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$S(t) := \int_0^\ell \left(q_1(x)R_1(t,x) + q_2(x)R_2(t,x)\right) dx \tag{7}$$

for all $t \geq 0$, with $\nu > 0$ and

$$\begin{cases} p_1 = 1, \\ p_2 = k_1 = \dfrac{\exp\left(-\nu \int_0^\ell \left[\frac{1}{\lambda_1(\sigma)} + \frac{1}{\lambda_2(\sigma)}\right] d\sigma\right)}{k_2}. \end{cases} \tag{8}$$

Consider the super-twisting based controller u defined by, for a.e $t \geq 0$,

$$\begin{cases} u(t) = \nu S(t) + k_1 R_2(t,0) - \alpha|S(t)|^{\frac{1}{2}}\operatorname{sign}(S(t)) + v(t), \\ \dot{v}(t) \in -\beta \operatorname{sign}(S(t)) \end{cases} \tag{9}$$

where k_1 is defined in (8), S is introduced in (7), β and α are positive constants which will be chosen later and the set-valued function sign is defined by

$$\operatorname{sign}(z) := \begin{cases} -1 & \text{if } z < 0, \\ [-1,1] & \text{if } z = 0, \\ 1 & \text{if } z > 0. \end{cases}$$

From (9), only the measurements of $R_2(t,0)$ and $S(t)$ are required. Furthermore, the full-state is not needed to measure $S(t)$. Indeed, it is just needed to measure the integral of the state with the weight functions q_1 and q_2.

For a.e $t \geq 0$, after some formal integration by parts, one gets

$$\begin{cases} \dot{S}(t) = -\alpha|S(t)|^{\frac{1}{2}}\operatorname{sign}(S(t)) + v(t) + d(t), \\ \dot{v}(t) \in -\beta \operatorname{sign}(S(t)). \end{cases} \tag{10}$$

along the trajectories of (1)–(9). Then, according to the following transformation

$$W(t) = d(t) + v(t), \tag{11}$$

the system (10) is rewritten as

$$\begin{cases} \dot{S}(t) = -\alpha|S(t)|^{\frac{1}{2}}\operatorname{sign}(S(t)) + W(t), \\ \dot{W}(t) \in \dot{d}(t) - \beta \operatorname{sign}(S(t)). \end{cases} \tag{12}$$

The system (12) is understood in the sense of Filippov [7] and the existence of solutions is given in Lemma 1. From [20, Theorem 1], all trajectories of (12) converge to zero in finite time.

Proposition 1: ([20, Theorem 1]) Assuming that

$$\beta > C \quad \text{and } \alpha > \sqrt{\beta + C}, \tag{13}$$

there exists a finite time $t_r > 0$ such that $S(t) = 0$ and $W(t) = 0$ for any $t > t_r$.

Then, the closed−loop system (1)−(9) can be seen as follows:

$$\begin{cases}\partial_t R_1(t,x)+\lambda_1(x)\partial_x R_1(t,x)=0,\\ \partial_t R_2(t,x)-\lambda_2(x)\partial_x R_2(t,x)=0,\\ R_1(t,0)=k_1R_2(t,0)+\nu S(t)-\alpha|S(t)|^{\frac{1}{2}}\mathrm{sign}(S(t))+W(t),\\ \dot W(t)\in \dot d(t)-\beta \mathrm{sign}(S(t))\\ R_2(t,\ell)=k_2R_1(t,\ell),\\ R_1(0,x)=R_1^0(x), R_2(0,x)=R_2^0(x)\\ W(0)=W_0.\end{cases} \tag{14}$$

Remark 1: According to Proposition 1 and the first line of (12), $\dot S(t)=0$ for any $t>t_r$. Then, the solution (R_1,R_2) of (14) reaches the sliding surface Σ in finite time t_r and remains on it. Since $W(t)=0$ for any $t>t_r$, then according to (11), one has $v(t)+d(t)=0$ for any $t>t_r$. As a consequence, the system (14) can be rewritten as (5) on the sliding surface, which is exponentially stable from [1, Theorem 2.11].

Our proof strategy is based on semigroup theory. For this, it is interesting to mention the scalar product which will be used in this paper.

Since $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ are positive functions, define a scalar product on X as follows: for all $\begin{pmatrix} z_1\\ w_1\end{pmatrix}, \begin{pmatrix} z_2\\ w_2\end{pmatrix}\in X$,

$$\left\langle \begin{pmatrix} z_1\\ w_1\end{pmatrix}, \begin{pmatrix} z_2\\ w_2\end{pmatrix}\right\rangle := \int_0^\ell \frac{1}{\lambda_1(x)} z_1(x)z_2(x)dx + \int_0^\ell \frac{1}{\lambda_2(x)} w_1(x)w_2(x)dx. \tag{15}$$

Now, consider the following system

$$\frac{d}{dt}Y(\cdot,t)=\mathcal{A}Y(\cdot,t)+\mathcal{B}\hat d(t), \tag{16}$$

where $Y=(Y_1,Y_2)$, and $\hat d\in L^2(0,T;\mathbb{R})$. The operator $\mathcal{A}$ is defined as

$$\begin{cases}\mathcal{A}Y=(-\lambda_1 Y_1', \lambda_2 Y_2'),\\ D(\mathcal{A})=\{Y\in (H^1(0,\ell))^2 \mid Y_1(0)=k_1Y_2(0),\\ \qquad Y_2(\ell)=k_2Y_1(\ell)\},\end{cases} \tag{17}$$

and according to the proof of [1, Theorem A.1], it generates a C_0−semigroup $(\mathbb{T}(t))_{t\geq 0}$ of contractions in X. The operator $\mathcal{B}$ is the delta function at $x=0$ in $\mathcal{L}(\mathbb{R}^2, D(\mathcal{A}^*)')$ i.e $\langle \varphi, \mathcal{B}y\rangle_{D(\mathcal{A}^*),D(\mathcal{A}^*)'}=\varphi(0)y$ for all $y\in\mathbb{R}^2$ and $\varphi\in D(\mathcal{A}^*)$ where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$, $D(\mathcal{A}^*)$ its domain and $\langle\cdot,\cdot\rangle_{D(\mathcal{A}^*),D(\mathcal{A}^*)'}$ is the dual product.

In this paper, we consider the mild solution of (16) in the sense of the next definition.

Definition 1: Let $T>0$, $\hat d\in L^2(0,T;\mathbb{R})$. Then for every (R_1^0,R_2^0), we say that the map $(Y_1,Y_2):[0,T]\times(0,\ell)\to\mathbb{R}^2$ is a mild solution of (16), if $(Y_1,Y_2)\in C(0,T;X)\cap H^1(0,T;D(\mathcal{A}^*)')$ such that for all $t\in[0,T]$

$$\begin{pmatrix} Y_1(t,\cdot)\\ Y_2(t,\cdot)\end{pmatrix}=\mathbb{T}(t)\begin{pmatrix} Y_1^0\\ Y_2^0\end{pmatrix}+\int_0^t \mathbb{T}(t-s)\mathcal{B}\hat d(s)ds \tag{18}$$

where $(\mathbb{T}(t))_{t\geq 0}$ is the C_0−semigroup generated by the operator $\mathcal{A}$ define in (17).

Next, the solutions of (14) are understood in the sense of the following definition.

Definition 2: Let $T>0$ and $(R_1^0,R_2^0,W_0)\in X\times\mathbb{R}$. We say that the map $(R_1,R_2):[0,T]\times(0,\ell)\to\mathbb{R}^2$ and $W:[0,T]\to\mathbb{R}$ is a mild solution of the Cauchy problem (14) if $(R_1,R_2)\in C(0,T;X)$ such that for all $t\in[0,T]$, $(R_1(t,\cdot),R_2(t,\cdot))$ satisfies (18) with

$$\hat d(t)=\begin{pmatrix} -\alpha|S(t)|^{\frac{1}{2}}\mathrm{sign}(S(t))+W(t)+\nu S(t)\\ 0\end{pmatrix} \tag{19}$$

and W is absolutely continuous and satisfies

$$\dot W(t)\in \dot d(t)-\beta\mathrm{sign}(S(t)) \tag{20}$$

for a.e $t\in[0,T]$ where S is given in (7).

The main results of this paper can be formulated as follows:

Theorem 1 (Well-posedness): Assume that (13) holds. Then, for all $T>0$ and for all $(R_1^0,R_2^0,W_0)\in X\times\mathbb{R}$, the closed-loop system (14) admits a mild solution (R_1,R_2,W).

Theorem 2 (Global asymptotic stability): Assume that (13) holds. Then, for any $(R_1^0,R_2^0,W_0)\in X\times\mathbb{R}$, 0 is globally asymptotically stable for (14). In other words, there exists a $\mathcal{KL}$-function τ such that for any $(R_1^0,R_2^0,W_0)\in X\times\mathbb{R}$ and for any $t\geq 0$:

$$\|(R_1(t,\cdot),R_2(t,\cdot)\|_X+|W(t)|\leq \tau(\|(R_1^0,R_2^0)\|_X+|W_0|,t). \tag{21}$$

III. Proof of Theorem 1 and Theorem 2

A. *Proof of Theorem 1*

This section provides a proof of Theorem 1. More precisely, the aim consists in proving the well-posedness of the closed−loop system (14) and the regularity of the function S defined by (7). Let $(R_1^0,R_2^0,W_0)\in X\times\mathbb{R}$ and consider the following ODE

$$\begin{cases}\dot\gamma(t)=-\alpha|\gamma(t)|^{\frac{1}{2}}\mathrm{sign}(\gamma(t))+\eta(t), \quad t\in\mathbb{R}_+,\\ \dot\eta(t)\in\dot d(t)-\beta\mathrm{sign}(\gamma(t)),\\ \gamma(0)=S_0, \eta(0)=W_0.\end{cases} \tag{22}$$

where

$$S_0=\int_0^\ell \left(q_1(x)R_1^0(x)+q_2(x)R_2^0(x)\right)dx.$$

The system (22) is understood in the sense of Filippov [7]. In the next lemma, we state that there exists a solution to (22).

Lemma 1: Assume that (13) holds. There exists an absolutely continuous map (γ,η) that satisfies (22) for almost $t\geq 0$.

Proof: We consider the function $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by

$$f(\gamma,\eta)=\left\{\begin{array}{ll} f^+(\gamma,\eta)=(-\alpha\sqrt{\gamma}+\eta,-\beta) & \text{if } \gamma>0,\\ f^-(\gamma,\eta)=(\alpha\sqrt{-\gamma}+\eta,\beta) & \text{if } \gamma<0\end{array}\right. \tag{23}$$

and let $F_d:(\gamma,\eta)\in\mathbb{R}^2\mapsto F_d(\gamma,\eta)$ be the set-valued map defined by

$$F_d(\gamma,\eta)=\bar{B}(0,C)+\left\{\begin{array}{ll} \{f(\gamma,\eta)\} & \text{if } \gamma\neq 0,\\ \overline{\text{conv}}\{f^+(\gamma,\eta),f^-(\gamma,\eta)\} & \text{if } \gamma=0\end{array}\right. \tag{24}$$

where $\bar{B}(0,C)$ is a closed ball of $\mathbb{R}^2$ centered at 0 and of radius C. Since f is continuous on $\mathbb{R}\setminus\{0\}\times\mathbb{R}$, then the function F_d is non−empty, compact, convex and upper semi−continuous. Then according to [2, Theorem 3.6], there exists at least one solution of the differential inclusion

$$\dot{\zeta}\in F_d(\zeta) \tag{25}$$

where $\zeta=(\gamma,\eta)$. Since F_d is the Filippov's construction associated to (22), then, there exists an absolutely continuous map that satisfies (22) for almost $t\geq 0$, concluding therefore the proof. ∎

Since γ and η are continuous then, according to the first line of (22), we deduce that $\dot{\gamma}$ is also continuous.

Next, we show the following well-posedness result for the system (16).

Lemma 2: Let (γ,η) be a solution of (22). Then, for all $(Y_1^0,Y_2^0)\in X$ and for all $T>0$, the system (16) with

$$\hat{d}(t)=\begin{pmatrix}\nu\gamma(t)+\dot{\gamma}(t)\\ 0\end{pmatrix} \tag{26}$$

admits a mild solution $(Y_1,Y_2)\in C([0,T];X)$.

Proof: Let $T>0$ and (γ,η) be a solution of (22). Then according to Lemma 1, γ and $\dot{\gamma}$ are continuous. Therefore, $\hat{d}\in L^2(0,T)$. As a consequence, since $\mathcal{A}$ generates a C_0−semigroup $(\mathbb{T}(t))_{t\geq 0}$ of contractions in X and if one proves that the operator $\mathcal{B}$ is an admissible operator (see e.g [22, Chapter 4]) for the C_0−semigroup $(\mathbb{T}(t))_{t\geq 0}$, one can apply the result provided by [9, Theorem 2.2], and conclude that the statement of Lemma 2 holds.

Since X is a Hilbert space, proving that the operator $\mathcal{B}$ is an admissible operator for the C_0−semigroup $(\mathbb{T}(t))_{t\geq 0}$ is equivalent to prove that the adjoint operator $\mathcal{B}^*$ of $\mathcal{B}$ is an admissible observation operator for the adjoint of the C_0−semigroup $(\mathbb{T}(t))_{t\geq 0}$. Then, we consider the dual system of (16):

$$\begin{cases}\dfrac{d}{dt}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\mathcal{A}^*\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\\ y^*=\mathcal{B}^*\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\end{cases} \tag{27}$$

where $\mathcal{A}^*$ and B^* are given by

$$\begin{cases}\mathcal{A}^*\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\begin{pmatrix}\lambda_1\varphi_1'\\ -\lambda_2\varphi_1'\end{pmatrix},\\ D(\mathcal{A}^*)=\left\{\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\in(H^1(0,\ell))^2\mid\varphi_2(0)=k_1\varphi_1(0),\right.\\ \left.\varphi_1(\ell)=k_2\varphi_2(\ell)\right\},\\ \mathcal{B}^*:\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\in D(\mathcal{A}^*)\mapsto\begin{pmatrix}\varphi_1(0)\\ 0\end{pmatrix}.\end{cases} \tag{28}$$

For all $(\varphi_1^0,\varphi_2^0)\in D(\mathcal{A}^*)$, the function

$$\begin{pmatrix}\varphi_1(t)\\ \varphi_2(t)\end{pmatrix}=\mathbb{T}^*(t)\begin{pmatrix}\varphi_1^0\\ \varphi_2^0\end{pmatrix} \tag{29}$$

defines the unique classical solution of (27) where $\mathbb{T}^*(t)$ is a C_0−semigroup with infinitesimal generator $\mathcal{A}^*$ on X. Now, consider the following function

$$E(t)=\int_0^\ell\left(Q_1(x)\varphi_1^2(t,x)+Q_2(x)\varphi_2^2(t,x)\right)dx, \tag{30}$$

$$\begin{aligned}Q_2(x)=&\frac{a_2}{\lambda_2(x)}\exp\left(-\int_0^x\frac{\nu}{\lambda_2(\sigma)}d\sigma\right),\\ Q_1(x)=&\frac{a_1}{\lambda_1(x)}\exp\left(\int_0^x\frac{\nu}{\lambda_1(\sigma)}d\sigma\right),\end{aligned} \tag{31}$$

where $\nu,a_1,a_2>0$ will be chosen later. If we select ν, a_1 and a_2 as in [1, Proof of Theorem 2.11], then one deduces that for all $t\geq 0$

$$|\varphi_1(t,0)|^2\leq\frac{-1}{a_1-a_2k_1^2}\dot{E}(\varphi_1(t,\cdot),\varphi_2(t,\cdot)). \tag{32}$$

Therefore, for all $T>0$ and for all $\begin{pmatrix}\varphi_1(0,\cdot)\\ \varphi_2(0,\cdot)\end{pmatrix}\in D(\mathcal{A}^*)$

$$\begin{aligned}\int_0^T|y^*(t)|^2dt\leq&-\frac{1}{a_1-a_2k_1^2}\int_0^T\dot{E}(\varphi_1(t,\cdot),\varphi_2(t,\cdot))dt\\ \leq&\frac{1}{a_1-a_2k_1^2}E(\varphi_1(0,\cdot),\varphi_2(0,\cdot))\\ &-\frac{1}{a_1-a_2k_1^2}E(\varphi_1(T,\cdot),\varphi_2(T,\cdot))\\ \leq&\frac{1}{a_1-a_2k_1^2}E(\varphi_1(0,\cdot),\varphi_2(0,\cdot)).\end{aligned} \tag{33}$$

Since E is equivalent to the usual norm, there exists a positive constant C such that, for all $T>0$ and for all $\begin{pmatrix}\varphi_1(0,\cdot)\\ \varphi_2(0,\cdot)\end{pmatrix}\in D(\mathcal{A}^*)$

$$\int_0^T|y^*(t)|^2dt\leq C\|(\varphi_1(0,\cdot),\varphi_2(0,\cdot))\|_X^2. \tag{34}$$

This proves that $\mathcal{B}$ is admissible for the C_0-semigroup $(\mathbb{T}(t))_{t\geq 0}$ and concludes the proof of Lemma 2.

■

The aim is now to prove that, for any Filippov solution (γ, η) of (22) with initial condition (S_0, W_0), the solution $(Y_1(\cdot), Y_2(\cdot), \eta)$ of (16)-(26)-(22) is a mild solution of (14). To that end, we will show that the following function

$$\sigma(t) = S(Y_1(t,\cdot), Y_2(t,\cdot)), \tag{35}$$

with S defined in (7) and (Y_1, Y_2) the solution of (16), is equal to γ for any $t > 0$.

Lemma 3: For any $T > 0$, σ is a Carathéodory[1] solution of

$$\begin{cases} \dot{\sigma}(t) = -\nu\sigma(t) + \dot{\gamma}(t) + \nu\gamma(t), \quad t \in [0, T], \\ \sigma(0) = S_0. \end{cases} \tag{36}$$

Proof: Let $T > 0$. According to (4) and (8), we have $\begin{pmatrix}\lambda_1 q_1 \\ \lambda_2 q_2\end{pmatrix} \in D(\mathcal{A}^*)$. Then, taking the inner product with $\begin{pmatrix}\lambda_1 q_1 \\ \lambda_2 q_2\end{pmatrix}$ on both sides of (16), we obtain for almost every $t \in [0, T]$

$$\frac{d}{dt}\left\langle \begin{pmatrix} Y_1(t,\cdot) \\ Y_2(t,\cdot)\end{pmatrix}, \begin{pmatrix}\lambda_1 q_1 \\ \lambda_2 q_2\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix} Y_1(t,\cdot) \\ Y_2(t,\cdot)\end{pmatrix}, \mathcal{A}^* \begin{pmatrix}\lambda_1 q_1 \\ \lambda_2 q_2\end{pmatrix}\right\rangle + \mathcal{B}^* \begin{pmatrix}\lambda_1 q_1 \\ \lambda_2 q_2\end{pmatrix} \hat{d}(t) \tag{37}$$

where $\langle\cdot,\cdot\rangle$ is defined in (15). This implies that for a.e $t \in [0, T]$

$$\begin{aligned} &\frac{d}{dt}\left(\int_0^\ell (q_1(x)Y_1(t,x) + q_2(x)Y_2(t,x))\,dx\right) \\ &= -\nu \int_0^\ell (q_1(x)Y_1(t,x) + q_2(x)Y_2(t,x))\,dx \\ &+ \begin{pmatrix}1 & 0\end{pmatrix}\hat{d}(t) \end{aligned} \tag{38}$$

Then, after integration by parts, one gets for a.e $t \in [0, T]$

$$\begin{aligned} &\int_0^\ell (q_1(x)Y_1(T,x) + q_2(x)Y_2(T,x))\,dx \\ &- \int_0^\ell (q_1(x)Y_1(0,x) + q_2(x)Y_2(0,x))\,dx \\ &= -\nu \int_0^T \int_0^\ell (q_1(x)Y_1(t,x) + q_2(x)Y_2(t,x))\,dxdt \\ &+ \int_0^T \begin{pmatrix}1 & 0\end{pmatrix}\hat{d}(t)dt \end{aligned} \tag{39}$$

Using (7), (26), (35) one has for all $T > 0$

$$\sigma(T) - \sigma(0) = -\nu \int_0^T \sigma(t)dt + \int_0^T \nu\gamma(t) + \dot{\gamma}(t)dt. \tag{40}$$

This concludes the proof of Lemma 3.

■

[1]A Carathéodory solution of (36) is an absolutely continuous map that satisfies (36) for almost every t.

Lemma 4: For all $(R_1^0, R_2^0) \in X$, for all $W_0 \in \mathbb{R}$ and for all $t \in \mathbb{R}_+$, $\sigma(t) = \gamma(t)$ and $W(t) = \eta(t)$.

Proof: Note that it is enough to prove $\sigma(t) = \gamma(t)$ to be able to conclude $W(t) = \eta(t)$. For this proof, we refer to [14, Section 4.1]. This concludes the proof of Lemma 4.

■

Then, according to Lemma 2, Lemma 3 and Lemma 4, one concludes that, for any Filippov solution (γ, η) of (22) with initial condition $\gamma(0) = S_0$, $\eta(0) = W_0$, $\hat{d}$ satisfies (19) and the associated solutions (Y_1, Y_2, η) is a mild solution of (14) in the sense of Definition 2. This concludes the proof of Theorem 1.

B. Proof of Theorem 2

Let us start proving (12). Let $T > 0$ and $(R_1^0, R_2^0, W_0) \in X \times \mathbb{R}$. We consider (R_1, R_2, W) a mild solution of (14) with initial condition (R_1^0, R_2^0, W_0). Then, according to Definition 2, there exists $w \in L^1(0, T)$ with $w(t) \in \text{sign}(S(t))$ such that for a.e $t \in [0, T]$, $\dot{W}(t) = \dot{d}(t) - \beta w(t)$ and $(R_1(t,\cdot), R_2(t,\cdot))$ satisfies (18) with $\hat{d}$ which satisfies (19). As a consequence, (R_1, R_2) satisfies (16). Then, by replacing (Y_1, Y_2) with (R_1, R_2) in (39) and using (7), (19) we obtain that, for all $T > 0$

$$S(T) - S(0) = \int_0^T (-\alpha|S(t)|^{\frac{1}{2}}\text{sign}(S(t)) + W(t))dt. \tag{41}$$

Then, according to (41), we obtain for a.e $t \in [0, T]$

$$\begin{cases} \dot{S}(t) = -\alpha|S(t)|^{\frac{1}{2}}\text{sign}(S(t)) + W(t), \\ \dot{W}(t) = \dot{d}(t) - \beta w(t). \end{cases} \tag{42}$$

Since $w(t) \in \text{sign}(S(t))$, then (S, W) is a Filippov solution of (12) with initial condition $(S(R_1^0, R_2^0), W_0)$.

Now, we are going to prove the Theorem 2. Let $(R_1^0, R_2^0, W_0) \in X \times \mathbb{R}$. Then, according to Proposition 1 and Remark 1, there exists a finite time t_r such that, for all $t > t_r$, the system (14) is equivalent to the system (5) and hence is exponentially in X from [1, Theorem 2.11]. Therefore, to conclude the proof of Theorem 2, it is just necessary to prove the Lyapunov stability of the system (14) over the time interval $[0, t_r]$. It is stated in the following Lemma.

Lemma 5: There exists a $\mathcal{K}$-function ψ such that for all $(R_1^0, R_2^0, W_0) \in X \times \mathbb{R}$, for all $t \in [0, t_r]$,

$$\begin{aligned} &\|(R_1(t,\cdot), R_2(t,\cdot))^\top\|_X + |W(t)| \leq \\ &\psi\left(\|(R_1^0(\cdot), R_2^0(\cdot))^\top\|_X + |W_0|\right). \end{aligned} \tag{43}$$

for all mild solutions $(R_1(t,\cdot), R_2(t,\cdot), W(\cdot))$ of (14).

Proof: Let $(R_1^0, R_2^0, W_0) \in X \times \mathbb{R}$ and we consider (R_1, R_2, W) a mild solution of (14) associated (R_1^0, R_2^0, W_0). Then, using the Definition 2, there exists

$C > 0$ such that, for all $t \in [0, t_r]$, we have

$$\|(R_1(t,\cdot), R_2(t,\cdot))^T\|_X \le C\|(R_1^0(\cdot), R_2^0(\cdot))^T\|_X + \left\| \int_0^t \mathbb{T}(t-s)\mathcal{B} \begin{pmatrix} -\alpha|S(s)|^{\frac{1}{2}}\operatorname{sign}(S(s)) + W(s) + \nu S(s) \\ 0 \end{pmatrix} ds \right\|_X. \tag{44}$$

As a consequence, since $(\mathbb{T}(t))_{t\ge0}$ is exponentially stable and $\mathcal{B}$ is admissible operator for $(\mathbb{T}(t))_{t\ge0}$, then according to [22, Proposition 4.3.3], there exists $K_1 > 0$ independent of t_r such that

$$\|(R_1(t,\cdot), R_2(t,\cdot))^T\|_X \le K_1\Big(\|(R_1^0(\cdot), R_2^0(\cdot))^T\|_X + \| -\alpha|S(\cdot)|^{\frac{1}{2}}\operatorname{sign}(S(\cdot)) + W(\cdot) + \nu S(\cdot)\|_{L^2((0,t_r),\mathbb{R})}\Big). \tag{45}$$

Since the couple (R_1, R_2) is continuous on $[0, t_r]$, then, according to (7), S is also continuous. Therefore, S is bounded on $[0, t_r]$. Moreover, d is bounded on $[0, t_r]$. As a consequence, W is bounded on $[0, t_r]$ according to (10) and (11). Then, the function $-\alpha|S(\cdot)|^{\frac{1}{2}}\operatorname{sign}(S(\cdot)) + W(\cdot) + \nu S$ is also bounded on $[0, t_r]$. Therefore, there exists $K_2 > 0$ such that

$$\| -\alpha|S(\cdot)|^{\frac{1}{2}}\operatorname{sign}(S(\cdot)) + W(\cdot) + \nu S(\cdot)\|_{L^2((0,t_r),\mathbb{R})} \le K_2 t_r^{\frac{1}{2}}. \tag{46}$$

Now, according to [15, Theorem 2], there are positive constants K_3, K_4 (dependent on the bound of $\dot{d}$) such that

$$\begin{cases} t_r < K_3\left(|S(0)| + |W(0)|\right), \\ |W(t)| \le K_4|W(0)|. \end{cases} \tag{47}$$

Using Holder's inequality there exists $C > 0$ such that

$$\begin{aligned} |S(0)| &\le \|q_1(\cdot), q_2(\cdot)\|_{L^\infty((0,L),\mathbb{R}^2)}\|R_1^0(\cdot), R_2^0(\cdot)\|_{L^1((0,L),\mathbb{R}^2)} \\ &\le C\|R_1^0(\cdot), R_2^0(\cdot)\|_X. \end{aligned} \tag{48}$$

As a consequence, according to (45), (46), (47) and (48), exists $C_1 > 0$ (independent of t_r) such that for all $t \in [0, t_r]$,

$$\begin{aligned} &\|(R_1(t,\cdot), R_2(t,\cdot))^T\|_X + |W(t)| \le \\ &\psi\left(\|(R_1^0(\cdot), R_2^0(\cdot))^T\|_X + |W_0|\right). \end{aligned} \tag{49}$$

where is given by $\psi : x \in \mathbb{R}_+ \mapsto C_1(x + \sqrt{x})$.

■

This concludes the proof of Theorem 2.

IV. Conclusion

A new approach for sliding mode control (precisely super-twisting control) has been proposed for a class of PDEs, namely a system of two transport equations. It is a Lyapunov approach, since the sliding variable is based on the gradient of the classical Lyapunov function given in [1]. The existence of solutions of the closed-loop system has been proved as well as the disturbance rejection and the asymptotic stability of the closed-loop control system.

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