



HAL
open science

Quasi-boundary method for a fractional ill-posed problem

Claire Joseph, Maryse Moutamal

► **To cite this version:**

Claire Joseph, Maryse Moutamal. Quasi-boundary method for a fractional ill-posed problem. Fractional Differential Calculus, In press. hal-03654123

HAL Id: hal-03654123

<https://hal.univ-antilles.fr/hal-03654123>

Submitted on 28 Apr 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Quasi-boundary method for a fractional ill-posed problem

Claire JOSEPH *

Maryse MOUTAMAL[†]

April 28, 2022

Abstract

A quasi-boundary method is used to study an ill-posed, time-fractional diffusion equation involving the fractional Riemann-Liouville derivative. In particular, we consider an ill-posed problem for a family of well-posed problems, and prove, by means of eigenfunction expansions, that the solutions of the latter problems converge to the solutions associated with the former problem. The analysis presented includes providing conditions for the rate of the convergence.

Mathematics Subject Classification. 35K57, 35R25, 26A33

Key-words :Riemann-Liouville derivative, quasi-boundary method, fractional diffusion equation, ill-posed problem, inverse-problem

1 Introduction

Let $d \in \mathbb{N}^*$ and Ω be a bounded open subset of \mathbb{R}^d , for a boundary $\partial\Omega$ of class C^2 . For $T > 0$, we set $Q = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$ and consider the fractional diffusion equation:

$$\begin{cases} D_{RL}^\alpha y(x, t) - \Delta y(x, t) = f(x, t) & \text{in } Q, \\ y(\sigma, t) = 0 & \text{on } \Sigma, \\ I^{1-\alpha} y(x, T) = y^1(x) & \text{on } \Omega, \end{cases} \quad (1.1) \quad \boxed{\text{FVP}}$$

where $0 < \alpha < 1$, $f \in L^2(Q)$, $y^1 \in H_0^1(\Omega)$. The operators $I^{1-\alpha} y$ and $D_{RL}^\alpha y$ are, respectively, the Riemann-Liouville fractional integral of order $1 - \alpha$, and the left Riemann-Liouville fractional derivative of order α of y .

The fractional diffusion equation has been of significant interest for many decades. The equation (specifically the time-fractional diffusion equation) is obtained by replacing the first order time derivative of the classical diffusion equation with a time fractional derivative. In comparison with the conventional first order derivative, Left Riemann-Liouville fractional derivatives are characterised by a convolution integral (see Definition 2.6). This shows that the (fractional) derivative depends on the behavior of function y over the interval $[0, t]$. This is the reason why researchers speak about the memory effect associated with a fractional derivative, and why, in this context fractional derivatives are used in other fields such as Physics, Biology or Economics, where the memory association of a field is mandatory.

*Laboratoire LAMIA, Université des Antilles , Campus Fouillole, 97159 Pointe-à-Pitre Guadeloupe (FWI) (email :claire.joseph@univ-antilles.fr).

[†]University of Buea, Department of mathematics, Buea, Cameroon (email :maryse.moutamal@aims-cameroon.org).

The model compounded in Equation (1.1), can be used to investigate environmental phenomenon. In such cases, one might not have all the necessary information on complete the model. In the case considered here, the initial condition is missing so that Equation (1.1) appears as an ill-posed backward time-fractional diffusion equation. The problem compounded in Equation (1.1) does not satisfies the Hadamard conditions. This is because we cannot prove that (1.1) admits a unique solution which depends continuously of y^1 . Nowadays, there are many methods such as inverse methods that may be used to approach such ill-posed problems. In this paper, we use a Quasi-boundary method which was originally introduced by the quasi-reversibility method developed in [8].

The quasi-boundary method is based on perturbing the final condition. Some researches on the topic have shown that this method gives better numerical results than the quasi-reversibility method. For example, in [20], Yang et *al.* apply the quasi-boundary method to approximate an inverse problem for identifying the initial data for a time-fractional diffusion equation on a pherically symmetric domain. Jayakumar [5] use a modified quasi-boundary method to solve a non-homogeneous time fractional diffusion problem involving the left fractional Caputo derivative. More recently, Huynh et *al.* [3] applied a modified quasi-boundary method for a fractional diffusion equation involving the Caputo-Fabrizio fractional derivative.

We refer to [6, 9, 16, 18, 1, 10, 17, 19] and references therein for more information in regard to the quasi-reversibility method and quasi-boundary method. In this context, the Riemann-Liouville and Caputo based fractional derivatives are closely related. The best of the authors' knowledge, and, judging from the open literature available, there are no studies on the quasi-boundary method for fractional diffusion equations involving the Riemann-Liouville fractional derivative. In this paper, we approach the ill-posed problem compounded in Equation (1.1) through a family of well-posed problems. More precisely, we consider, for any $\beta > 0$, the following quasi-boundary value problem:

$$\begin{cases} D_{RL}^\alpha y_\beta(x, t) - \Delta y_\beta(x, t) = f(x, t) & (x, t) \in Q, \\ y_\beta(\sigma, t) = 0 & (\sigma, t) \in \Sigma, \\ I^{1-\alpha} y_\beta(x, T) + \beta I^{1-\alpha} y_\beta(x, 0^+) = y^1(x) & x \in \Omega, \end{cases}$$

and prove that the family of solutions y_β converge to the solution of Equation (1.1) in an appropriate Hilbert space, specifying the rate of convergence.

This paper is structured as follows: In Section 2, we provide some definitions on fractional operators, examples of their properties and some preliminary results. In Section 3, we use the spectral method to prove the existence and uniqueness of the solution of the problem. The convergence results are provided in Section 4.

2 Preliminaries

prelim

In this section, we recall some basic definitions and results on fractional integration and derivative.

gamma

Definition 2.1 [7, 13] *Let z be a complex number such that $Re(z) > 0$. Then the Gamma function is given by*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Remark 2.1 *It follows from the definition above that*

$$\Gamma(z + 1) = z\Gamma(z).$$

def2

Definition 2.2 [7, 13] Let x and y be two complex numbers such that $\operatorname{Re}(x) > 0$ and $\operatorname{Re}(y) > 0$. The Beta function is given by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

Remark 2.2 One can prove [7, 13] that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad (2.1) \quad \text{beta_formule}$$

Definition 2.3 [7, 13] Let $\alpha > 0$ and $\beta > 0$. Then, the two-parameter Mittag-Leffler function is given by

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}. \quad (2.2) \quad \text{mittag}$$

Thus, we have

$$E_{\alpha, \alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \alpha)}, \quad z \in \mathbb{C}. \quad (2.3) \quad \text{Eaa}$$

In what follows, we set

$$E_{\alpha, 1}(t) = E_{\alpha}(t).$$

majora

Theorem 2.1 [13, 15] Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$. We consider that μ satisfies

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}.$$

In this case, there exists a constant $C = C(\alpha, \beta, \mu) > 0$, such that

$$|E_{\alpha, \beta}(z)| \leq \frac{C}{1 + |z|}, \quad \mu \leq |\arg(z)| \leq \pi.$$

Definition 2.4 [2, 7] Let $\alpha, \beta, \rho \in \mathbb{C}$ such that $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\beta) > 0$. Then, the generalized Mittag-Leffler function is defined by

$$\mathcal{E}_{\alpha, \beta}^{\rho}(t) = \sum_{n=0}^{+\infty} \frac{(\rho)_n t^n}{\Gamma(\alpha n + \beta) n!}, \quad \text{for all } t \in \mathbb{C},$$

where $(\rho)_n = \rho(\rho+1)\dots(\rho+n-1)$.

Remark 2.3 Note that, when $\rho = 1$ we obtain

$$\mathcal{E}_{\alpha, \beta}^1(t) = E_{\alpha, \beta}(t),$$

where $E_{\alpha, \beta}$ is the classical Mittag-Leffler function defined in (2.2).

The following result gives the Laplace transform of the generalized Mittag-Leffler function.

Lemma 2.1 [2] Let α, β, ρ be complexes such that $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\rho) > 0$ and $\operatorname{Re}(\beta) > 0$. Then, we have

$$\mathcal{L}^{-1} \left\{ \frac{s^{\rho-1}}{s^{\alpha} + as^{\beta} + b}; t \right\} = t^{\alpha-\rho} \sum_{k=0}^{+\infty} (-a)^k t^{(\alpha-\beta)k} \mathcal{E}_{\alpha, \alpha+(\alpha-\beta)k-\rho+1}^{k+1}(-bt^{\alpha}), \quad (2.4) \quad \text{Lpmittag}$$

where $|as^{\beta}/(s^{\alpha} + b)| < 1$ and \mathcal{L}^{-1} is the inverse Laplace transform.

IRL

Definition 2.5 [7, 13] Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a continuous function, and $\alpha > 0$. Then, the expression

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0,$$

is called the Riemann-Liouville integral of order α of the function f .

def26

Definition 2.6 [7, 13] Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$. The left Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ of f is defined as

$$D_{RL}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds, \quad t > 0,$$

provided that the integral exists.

Remark 2.4 From Definition 2.5, we see that :

$$D_{RL}^\alpha f(t) = \frac{d}{dt} I^{1-\alpha} f(t). \quad (2.5)$$

DandI

We also with the right Caputo fractional derivative given by:

Definition 2.7 [7, 13] Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, $0 < \alpha < 1$. The right Caputo fractional derivative of order α of f is defined by

$$\mathcal{D}_C^\alpha f(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f'(s) ds, \quad 0 < t < T, \quad (2.6)$$

provided that the integral exists.

We consider a result obtained through integration by parts, which involves the left Riemann-Liouville fractional derivative and the right Caputo fractional derivative.

integrale1

Lemma 2.2 [11] Let $0 < \alpha < 1$, $y \in \mathcal{C}^\infty(\bar{Q})$ and $\varphi \in \mathcal{C}^\infty(\bar{Q})$. Then, we have,

$$\begin{aligned} & \int_0^T \int_\Omega (D_{RL}^\alpha y(x, t) - \Delta y(x, t)) \varphi(x, t) dx dt = \\ & \int_\Omega \varphi(x, T) I^{1-\alpha} y(x, T) dx - \int_\Omega \varphi(x, 0) I^{1-\alpha} y(x, 0) dx \\ & + \int_0^T \int_{\partial\Omega} y(\sigma, t) \frac{\partial \varphi}{\partial \nu}(\sigma, t) d\sigma dt - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu}(\sigma, t) \varphi(\sigma, t) d\sigma dt \\ & + \int_\Omega \int_0^t y(x, t) (-\mathcal{D}_C^\alpha \varphi(x, t) - \Delta \varphi(x, t)) dx dt, \end{aligned} \quad (2.7)$$

where \mathcal{D}_C^α is the right Caputo fractional of order $0 < \alpha < 1$.

integrale0T

Corollary 2.1 [11] Let $\mathbb{D}(0, T)$ be the set of C^∞ functions on $(0, T)$ with compact support. Then for all $\varphi \in \mathbb{D}(0, T)$,

$$\int_0^T D_{RL}^\alpha y(t) \varphi(t) dt = - \int_0^T y(t) \mathcal{D}_C^\alpha \varphi(t) dt,$$

where \mathcal{D}_C^α is the right fractional Caputo derivative.

article1

Theorem 2.2 [12] Let $1/2 < \alpha < 1$, $y^0 \in H_0^1(\Omega)$ and $f \in L^2(Q)$. Then, the problem

$$\begin{cases} D_{RL}^\alpha y(x, t) - \Delta y(x, t) = f(x, t) & \text{in } Q, \\ y(\sigma, t) = 0 & \text{on } \Sigma, \\ I^{1-\alpha} y(x, 0) = y^0 & \text{in } \Omega. \end{cases}$$

has a unique solution $y \in L^2((0, T), H_0^1(\Omega))$. Moreover, $I^{1-\alpha} y \in C([0, T], H_0^1(\Omega))$, and, there exists a constant $C > 0$, such that the following estimations hold:

$$\|y\|_{L^2((0, T); H_0^1(\Omega))} \leq \Delta \left(\|y^0\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q)} \right), \quad (2.8)$$

$$\|I^{1-\alpha} y\|_{C([0, T]; H_0^1(\Omega))} \leq \Pi \left(\|y^0\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q)} \right), \quad (2.9)$$

with

$$\Delta = \max \left(C \sqrt{\frac{2T^{2\alpha-1}}{2\alpha-1}}, \frac{C}{\alpha} \sqrt{\frac{2T}{\lambda_1}} \right),$$

and

$$\Pi = \sup \left(C\sqrt{2}, C\sqrt{\frac{2T^{1-\alpha}}{(1-\alpha)}} \right).$$

On other hand, it is well-known that $(-\Delta)$ is a symmetric uniform elliptic operator. Thus, it admits real eigenvalues, $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ with $\lambda_k \rightarrow \infty$ when $k \rightarrow \infty$. Moreover, there exists an orthonormal basis $\{w_k\}_{k=1}^\infty$ of $L^2(\Omega)$, where $w_k \in H_0^1(\Omega)$ is an eigenfunction corresponding to λ_k : $-\Delta w_k = \lambda_k w_k$. Further, we have,

$$\int_{\Omega} \nabla \varphi(x) \cdot \nabla \psi(x) dx = \lambda_k \int_{\Omega} \varphi(x) \psi(x) dx, \quad \forall \varphi, \psi \in H_0^1(\Omega). \quad (2.10) \quad \text{valprop1}$$

In what follows, for all $\varphi, \psi \in L^2(\Omega)$, we denote

$$(\varphi, \psi)_{L^2(\Omega)} = \int_{\Omega} \varphi(x) \psi(x) dx,$$

as the inner product in $L^2(\Omega)$ and $\|\varphi\|_{L^2(\Omega)}$ as the associated norm.

We set

$$a(\varphi, \psi) = \int_{\Omega} \nabla \varphi(x) \cdot \nabla \psi(x) dx, \quad \forall \varphi, \psi \in H_0^1(\Omega). \quad (2.11) \quad \text{formebi1}$$

Then, the bilinear functional $a(\cdot, \cdot)$ defines an inner product on $H_0^1(\Omega)$, and we have

$$\|\varphi\|_{H_0^1(\Omega)}^2 = a(\varphi, \varphi), \quad (2.12) \quad \text{equivalence1}$$

which is a norm on $H_0^1(\Omega)$. Since $\left\{ \frac{w_k}{\sqrt{\lambda_k}} \right\}_{k=1}^\infty$ is an orthonormal basis of $H_0^1(\Omega)$ for the inner product $a(\cdot, \cdot)$, we can write

$$\|\phi\|_{H_0^1(\Omega)}^2 = \sum_{i=1}^{+\infty} \lambda_i (\phi, w_i)_{L^2(\Omega)}^2, \quad \forall \phi \in H_0^1(\Omega). \quad (2.13) \quad \text{H101}$$

3 Approximate problem

approx

In this section, using eigenfunctions expansions of the Laplace operator, we prove the existence and uniqueness of solution to the approximate problem given by

$$\begin{cases} D_{RL}^\alpha y_\beta(x, t) - \Delta y_\beta(x, t) = f(x, t) & (x, t) \in Q, \\ y_\beta(\sigma, t) = 0 & (\sigma, t) \in \Sigma, \\ I^{1-\alpha} y_\beta(x, T) + \beta I^{1-\alpha} y_\beta(x, 0^+) = y^1(x) & x \in \Omega, \end{cases} \quad (3.1) \quad \text{eqA1}$$

where $1/2 < \alpha < 1$, $f \in L^2(Q)$, $y^1 \in H_0^1(\Omega)$ and $I^{1-\alpha} y_\beta(x, 0^+) = \lim_{t \downarrow 0} I^{1-\alpha} y_\beta(x, t)$.

Let us assume that (3.1) has a solution $y_\beta \in C^\infty(\bar{Q})$. If we multiply the first equation in (3.1) by a function $v \in H_0^1(\Omega)$ and integrate by parts over Ω , we obtain

$$\int_\Omega D_{RL}^\alpha y_\beta(x, t) v(x) dx + \int_\Omega \nabla y_\beta(x, t) \cdot \nabla v(x) dx = \int_\Omega f(x, t) v(x) dx. \quad (3.2) \quad \text{chin}$$

Observing that $(D_{RL}^\alpha y_\beta(t), v) = D_{RL}^\alpha (y_\beta(t), v)$ and using (2.11), problem (3.1) becomes for all $t \in (0, T)$:

$$\begin{cases} D_{RL}^\alpha (y_\beta(t), v)_{L^2(\Omega)} + a(y_\beta(t), v) = (f(t), v)_{L^2(\Omega)} & \text{in } \Omega, \quad \forall v \in H_0^1(\Omega), \\ y_\beta(t) = 0 & \text{on } \partial\Omega, \\ I^{1-\alpha} y_\beta(x, T) + \beta I^{1-\alpha} y_\beta(x, 0^+) = y^1 & \text{in } \Omega. \end{cases} \quad (3.3) \quad \text{eqB1}$$

eq11

We can then consider the following problem : Given $1/2 < \alpha < 1$, $y^1 \in H_0^1(\Omega)$ and $f \in L^2(Q)$, find

$$y_\beta \in L^2((0, T), H_0^1(\Omega)), \quad (3.4a)$$

$$I^{1-\alpha} y_\beta \in C([0, T]; H_0^1(\Omega)), \quad (3.4b)$$

diff1

such that

$$D_{RL}^\alpha (y_\beta(t), v)_{L^2(\Omega)} + a(y_\beta(t), v) = (f(t), v)_{L^2(\Omega)} \quad \forall t \in (0, T), \quad \forall v \in H_0^1(\Omega), \quad (3.5a) \quad \text{eq21}$$

$$I^{1-\alpha} y_\beta(T) + \beta I^{1-\alpha} y_\beta(0^+) = y^1 \quad \text{in } \Omega. \quad (3.5b) \quad \text{eq22}$$

In this context, the following existence and uniqueness theorems hold.

existeyb

Theorem 3.1 *Let $1/2 < \alpha < 1$ and $a(.,.)$ be the bilinear form defined by (2.11). Then, the approximate problem (3.4)-(3.5) has a unique solution $y_\beta \in L^2((0, T), H_0^1(\Omega))$ given by*

$$\begin{aligned} y_\beta(t) = & \sum_{i=1}^{+\infty} \left\{ \frac{y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du}{\beta + E_\alpha(-\lambda_i T^\alpha)} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) \right. \\ & \left. + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha) f_i(s) ds \right\} w_i. \end{aligned} \quad (3.6) \quad \text{solution11}$$

where λ_i is the eigenvalue of the operator $-\Delta$ corresponding to the eigenfunction w_i . $E_{\alpha,\alpha}$ as given in (2.3), $y_i^1 = (y^1, w_i)$ and $f_i(t) = (f(t), w_i)$ are respectively, the i -th component of y^1 and $f(t)$ in the

orthonormal basis $\{w_i\}_{i=1}^\infty$ of $L^2(\Omega)$. Moreover, $I^{1-\alpha}y_\beta \in C([0, T], H_0^1(\Omega))$ and there exists a constant $C > 0$ such that,

$$\|y_\beta\|_{L^2((0, T), H_0^1(\Omega))} \leq \Pi \left(\|y^1\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q)} \right), \quad (3.7) \quad \text{estimation_yb}$$

and

$$\|I^{1-\alpha}y_\beta\|_{C([0, T], H_0^1(\Omega))} \leq \Theta \left(\|y^1\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q)} \right), \quad (3.8) \quad \text{estimation_Iyb}$$

where

$$\Pi = \max \left(\frac{2C}{\beta} \sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}, \sqrt{\frac{2C^2T^\alpha}{\beta^2(1-\alpha)(2\alpha-1)} + \frac{4C^2T^\alpha}{\alpha - \frac{1}{2}}} \right)$$

and

$$\Theta = \sup \left(\frac{2C}{\beta}, \sqrt{\frac{2C^4T^{1-\alpha}}{\beta^2(1-\alpha)} + \frac{2CT^{1-\alpha}}{1-\alpha}} \right).$$

Proof. If we replace v by w_i in (3.5a) and use the fact that

$$a(y_\beta(t), w_i) = \lambda_i(y_\beta(t), w_i)_{L^2(\Omega)} = \lambda_i y_{\beta i},$$

we deduce from (3.5) that $y_{\beta i}$ is a solution of the ordinary differential equation

$$\begin{cases} D_{RL}^\alpha y_{\beta i}(t) + \lambda_i y_{\beta i}(t) = f_i(t), & t \in (0, T), \\ I^{1-\alpha} y_{\beta i}(T) + \beta I^{1-\alpha} y_{\beta i}(0^+) = y_i^1, \end{cases} \quad (3.9) \quad \text{edo11}$$

where $y_i^1 = (y^1, w_i)$.

Now, using the Laplace transform, we obtain from the first equation of (3.9) that,

$$\hat{D}_{RL}^\alpha y_{\beta i}(s) + \lambda_i \hat{y}_{\beta i}(s) = \hat{f}_i(s), \quad (3.10) \quad \text{eqlaplace1}$$

where

$$\begin{aligned} \hat{D}_{RL}^\alpha y_{\beta i}(s) &= \mathcal{L}(D_{RL}^\alpha y_{\beta i}(t))(s), \\ \hat{y}_{\beta i}(s) &= \mathcal{L}(y_{\beta i}(t))(s), \\ \hat{f}_i(s) &= \mathcal{L}(f_i(t))(s) \end{aligned}$$

and \mathcal{L} denotes the Laplace transform operator.

From (2.5), we have

$$\hat{D}_{RL}^\alpha y_{\beta i}(s) = -I^{1-\alpha} y_{\beta i}(0^+) + s^\alpha \hat{y}_{\beta i}(s),$$

which, combining with (3.10), gives

$$-I^{1-\alpha} y_{\beta i}(0^+) + s^\alpha \hat{y}_{\beta i}(s) + \lambda_i \hat{y}_{\beta i}(s) = \hat{f}_i(s).$$

Hence,

$$\hat{y}_{\beta i}(s) = I^{1-\alpha} y_{\beta i}(0^+) \times \frac{1}{s^\alpha + \lambda_i} + \hat{f}_i(s) \times \frac{1}{s^\alpha + \lambda_i},$$

and it follows from (2.4) that

$$y_{\beta i}(t) = I^{1-\alpha} y_{\beta i}(0^+) t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds.$$

Therefore,

$$\begin{aligned}
I^{1-\alpha}y_{\beta i}(t) &= I^{1-\alpha}(I^{1-\alpha}y_{\beta i}(0)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)) \\
&+ I^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)f_i(s)ds\right), \\
&= A+B
\end{aligned}$$

where

$$\begin{aligned}
A &= I^{1-\alpha}(I^{1-\alpha}y_{\beta i}(0)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)), \\
B &= I^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)f_i(s)ds\right).
\end{aligned}$$

Let us now compute A and B . We have,

$$\begin{aligned}
A &= I^{1-\alpha}(I^{1-\alpha}y_{\beta i}(0)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)) \\
&= I^{1-\alpha}y_{\beta i}(0) \times I^{1-\alpha}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)) \\
&= I^{1-\alpha}y_{\beta i}(0) \times \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i s^\alpha) ds\right) \\
&= \frac{I^{1-\alpha}y_{\beta i}(0)}{\Gamma(1-\alpha)} \sum_{k=0}^{+\infty} \frac{(-\lambda_i)^k}{\Gamma(\alpha k + \alpha)} \int_0^t (t-s)^{-\alpha} s^{\alpha-1} s^{\alpha k} ds \\
&= \frac{I^{1-\alpha}y_{\beta i}(0)}{\Gamma(1-\alpha)} \sum_{k=0}^{+\infty} \frac{(-\lambda_i)^k t^{\alpha k-1}}{\Gamma(\alpha k + \alpha)} \int_0^1 (1-u)^{-\alpha} u^{\alpha-1+\alpha k} t du \\
&= \frac{I^{1-\alpha}y_{\beta i}(0)}{\Gamma(1-\alpha)} \sum_{k=0}^{+\infty} \frac{(-\lambda_i)^k t^{\alpha k}}{\Gamma(\alpha k + \alpha)} B(1-\alpha, \alpha k + \alpha),
\end{aligned}$$

which in view of (2.1) gives

$$A = I^{1-\alpha}y_{\beta i}(0) \sum_{k=0}^{+\infty} \frac{(-\lambda_i t^\alpha)^k}{\Gamma(\alpha k + 1)} = I^{1-\alpha}y_{\beta i}(0) E_\alpha(-\lambda_i t^\alpha). \quad (3.11) \quad \boxed{\text{Ca1A}}$$

On the other hand,

$$\begin{aligned}
B &= I^{1-\alpha} \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha) f_i(s) ds \right) \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left(\int_0^s (s-u)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(s-u)^\alpha) f_i(u) du \right) ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t f_i(u) \left(\int_u^t (t-s)^{-\alpha} (s-u)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(s-u)^\alpha) ds \right) du \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t f_i(u) \sum_{k=0}^{+\infty} \frac{(-\lambda_i)^k}{\Gamma(\alpha k + \alpha)} \left(\int_u^t (t-s)^{-\alpha} (s-u)^{\alpha-1+\alpha k} ds \right) du \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t f_i(u) \sum_{k=0}^{+\infty} \frac{(-\lambda_i)^k (t-u)^{\alpha k}}{\Gamma(\alpha k + \alpha)} \left(\int_0^1 (1-z)^{-\alpha} z^{\alpha-1+\alpha k} dz \right) du \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t f_i(u) \left(\sum_{k=0}^{+\infty} \frac{(-\lambda_i)^k (t-u)^{\alpha k}}{\Gamma(\alpha k + \alpha)} B(1-\alpha, \alpha k + \alpha) \right) du \\
&= \int_0^t f_i(u) \left(\sum_{k=0}^{+\infty} \frac{(-\lambda_i(t-u)^\alpha)^k}{\Gamma(\alpha k + 1)} \right) du = \int_0^t f_i(u) E_\alpha(-\lambda_i(t-u)^\alpha) du.
\end{aligned}$$

Thus,

$$B = \int_0^t f_i(u) E_\alpha(-\lambda_i(t-u)^\alpha) du \quad (3.12) \quad \boxed{\text{CalB}}$$

Finally, adding (3.11) to (3.12), we obtain

$$I^{1-\alpha} y_{\beta i}(t) = I^{1-\alpha} y_{\beta i}(0) E_\alpha(-\lambda_i t^\alpha) + \int_0^t f_i(u) E_\alpha(-\lambda_i(t-u)^\alpha) du. \quad (3.13) \quad \boxed{\text{I1-a}}$$

Hence,

$$I^{1-\alpha} y_{\beta i}(T) = I^{1-\alpha} y_{\beta i}(0) E_\alpha(-\lambda_i T^\alpha) + \int_0^T f_i(u) E_\alpha(-\lambda_i(T-u)^\alpha) du.$$

From (3.9)₂, we have that,

$$I^{1-\alpha} y_{\beta i}(0) E_\alpha(-\lambda_i T^\alpha) + \int_0^T f_i(u) E_\alpha(-\lambda_i(T-u)^\alpha) du + \beta I^{1-\alpha} y_{\beta i}(0) = y_i^1,$$

from which, we deduce that

$$I^{1-\alpha} y_{\beta i}(0) = \frac{y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du}{\beta + E_\alpha(-\lambda_i T^\alpha)}. \quad (3.14) \quad \boxed{\text{Iybo}}$$

Finally, we obtain

$$y_{\beta i}(t) = \left[\frac{y_i^1 - \int_0^T E_{\alpha}(-\lambda_i(T-u)^{\alpha})f_i(u)du}{\beta + E_{\alpha}(-\lambda_i T^{\alpha})} \right] t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^{\alpha})f_i(s)ds.$$

The rest of proof can be done in three steps.

Step 1: We give the formulation of a solution to an approximate problem associated to Equations (3.4)-(3.5).

Let V_m be a subspace of $H_0^1(\Omega)$ generated by the w_1, w_2, \dots, w_m .

Consider the following approximate problem associated to Equations (3.4) – (3.5):

Find $y_{\beta m} : t \in (0, T] \rightarrow y_{\beta m}(t) \in V_m$, the solution of

$$D_{RL}^{\alpha}(y_{\beta m}(t), v)_{L^2(\Omega)} + a(y_{\beta m}(t), v) = (f(t), v)_{L^2(\Omega)}, \forall v \in V_m, \quad (3.15) \quad \boxed{\text{eq111}}$$

$$I^{1-\alpha}y_{\beta m}(T) + \beta I^{1-\alpha}y_{\beta m}(0) = y_m^1 = \sum_{i=1}^m y_{\beta i}^1 w_i. \quad (3.16) \quad \boxed{\text{eq121}}$$

As $y_{\beta m}(t) \in V_m$, we have

$$y_{\beta m}(t) = \sum_{i=1}^m (y(t), w_i)_{L^2(\Omega)} w_i = \sum_{i=1}^m y_{\beta i}(t) w_i.$$

Proceeding as per the computation of y_{β} , we show that $y_{\beta m}$ is a solution of the problem given by Equations (3.15) – (3.16) and obtain,

$$y_{\beta m}(t) = \sum_{i=1}^m \left\{ \frac{y_i^1 - \int_0^T E_{\alpha}(-\lambda_i(T-u)^{\alpha})f_i(u)du}{\beta + E_{\alpha}(-\lambda_i T^{\alpha})} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^{\alpha}) \right\} w_i + \sum_{i=1}^m \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^{\alpha})f_i(s)ds \right\} w_i. \quad (3.17) \quad \boxed{\text{solm1}}$$

Step 2: We show that the sequences $(y_{\beta m})$ and $(I^{1-\alpha}y_{\beta m})$ are respectively, Cauchy sequences in $L^2((0, T); H_0^1(\Omega))$ and $C([0, T]; H_0^1(\Omega))$.

Let m and p be two integers such that $p > m \geq 1$. Then, from (3.17)

$$y_{\beta p}(t) - y_{\beta m}(t) = \sum_{i=m+1}^p y_{\beta i}(t) w_i.$$

Set $b_i = \frac{y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du}{\beta + E_\alpha(-\lambda_i T^\alpha)}$. Then, we have that,

$$\begin{aligned} a(y_{\beta p}(t) - y_{\beta m}(t), y_{\beta p}(t) - y_{\beta m}(t)) &= \sum_{i=m+1}^p \lambda_i [y_{\beta i}(t)]^2 \\ &\leq 2 \sum_{i=m+1}^p \lambda_i t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha) |b_i|^2 \\ &\quad + 2 \sum_{i=m+1}^p \lambda_i \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha) f_i(s) ds \right\}^2. \end{aligned}$$

Hence,

$$\begin{aligned} \|y_{\beta p}(t) - y_{\beta m}(t)\|_{L^2((0,T);H_0^1(\Omega))}^2 &= \int_0^T a(y_{\beta p}(t) - y_{\beta m}(t), y_{\beta p}(t) - y_{\beta m}(t)) dt \\ &\leq A_p + B_p, \end{aligned}$$

with

$$\begin{aligned} A_p &= 2 \sum_{i=m+1}^p \lambda_i |b_i|^2 \int_0^T t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha) dt, \\ B_p &= 2 \sum_{i=m+1}^p \int_0^T \lambda_i \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha) f_i(s) ds \right\}^2 dt. \end{aligned}$$

Note that from Theorem 2.1, we know that there exists a generic constant $C > 0$ such that

$$\begin{aligned} A_p &= 2 \sum_{i=m+1}^p \lambda_i |b_i|^2 \int_0^T t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha) dt \\ &\leq 2C^2 \sum_{i=m+1}^p \lambda_i |b_i|^2 \int_0^T t^{2\alpha-2} dt \\ &\leq 2C^2 \sum_{i=m+1}^p \lambda_i |b_i|^2 \left[\frac{t^{2\alpha-1}}{2\alpha-1} \right]_0^T \leq \frac{2C^2 T^{2\alpha-1}}{2\alpha-1} \sum_{i=m+1}^p \lambda_i |b_i|^2. \end{aligned}$$

Remark 3.1 From the latter estimation, we see that we have to take $1/2 < \alpha < 1$ to give a sense to our computation.

Using again Theorem 2.1 and noting that $1 - \alpha \neq 0$, we obtain

$$\begin{aligned}
\sum_{i=m+1}^p \lambda_i |b_i|^2 &= \sum_{i=m+1}^p \lambda_i \left| \frac{y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du}{\beta + E_\alpha(-\lambda_i T^\alpha)} \right|^2 \\
&\leq \frac{1}{\beta^2} \sum_{i=m+1}^p \lambda_i \left| y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du \right|^2 \\
&\leq \frac{2}{\beta^2} \sum_{i=m+1}^p \lambda_i |y_i^1|^2 + \frac{2}{\beta^2} \sum_{i=m+1}^p \lambda_i \left| \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du \right|^2 \\
&\leq \frac{2}{\beta^2} \sum_{i=m+1}^p \lambda_i |y_i^1|^2 + \frac{2C^2}{\beta^2} \left[\frac{-(T-u)^{1-\alpha}}{1-\alpha} \right]_0^T \sum_{i=m+1}^p \int_0^T |f_i(u)|^2 du.
\end{aligned}$$

Consequently,

$$\sum_{i=m+1}^p \lambda_i |b_i|^2 \leq \frac{2}{\beta^2} \sum_{i=m+1}^p \lambda_i |y_i^1|^2 + \frac{2C^2 T^{1-\alpha}}{\beta^2(1-\alpha)} \sum_{i=m+1}^p \int_0^T |f_i(u)|^2 du. \quad (3.18) \quad \boxed{\text{sum_bi}}$$

and we have that

$$A_p \leq \frac{4C^2 T^{2\alpha-1}}{\beta^2(2\alpha-1)} \sum_{i=m+1}^p \lambda_i |y_i^1|^2 + \frac{4C^4 T^\alpha}{\beta^2(1-\alpha)(2\alpha-1)} \left[\sum_{i=m+1}^p \int_0^T |f_i(u)|^2 du \right]. \quad (3.19) \quad \boxed{\text{CaIAP}}$$

On the other hand, using the Cauchy-Schwartz inequality,

$$\begin{aligned}
B_p &= 2 \sum_{i=m+1}^p \int_0^T \lambda_i \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha) f_i(s) ds \right\}^2 dt \\
&= 2 \sum_{i=m+1}^p \int_0^T \lambda_i \left\{ \int_0^t \left[(t-s)^{\frac{\alpha}{2}-\frac{1}{4}} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha) \right] \left[(t-s)^{\frac{\alpha}{2}-\frac{3}{4}} f_i(s) \right] ds \right\}^2 dt \\
&\leq 2 \sum_{i=m+1}^p \int_0^T \lambda_i \left[\int_0^t (t-s)^{\alpha-\frac{1}{2}} E_{\alpha,\alpha}^2(-\lambda_i(t-s)^\alpha) ds \right] \left[\int_0^t (t-s)^{\alpha-\frac{3}{2}} |f_i(s)|^2 ds \right] dt,
\end{aligned}$$

which in view of Theorem 2.1 gives

$$\begin{aligned}
B_p &\leq 2C^2 \sum_{i=m+1}^p \int_0^T \left[\int_0^t (t-s)^{-1/2} ds \right] \left[\int_0^t (t-s)^{\alpha-\frac{3}{2}} |f_i(s)|^2 ds \right] dt \\
&\leq 4C^2 T^{1/2} \sum_{i=m+1}^p \int_0^T \int_0^t (t-s)^{\alpha-\frac{3}{2}} |f_i(s)|^2 ds dt \\
&= 4C^2 T^{1/2} \sum_{i=m+1}^p \int_0^T |f_i(s)|^2 \int_s^T (t-s)^{\alpha-\frac{3}{2}} dt ds \\
&\leq \frac{4C^2 T^\alpha}{\alpha - \frac{1}{2}} \sum_{i=m+1}^p \int_0^T |f_i(s)|^2 ds.
\end{aligned}$$

Thus,

$$B_p \leq \frac{4C^2 T^\alpha}{\alpha - \frac{1}{2}} \sum_{i=m+1}^p \int_0^T |f_i(s)|^2 ds \quad (3.20) \quad \boxed{\text{Ca1BP}}$$

Adding (3.19) to (3.20), we obtain

$$\begin{aligned}
\|y_{\beta p}(t) - y_{\beta m}(t)\|_{L^2((0,T);H_0^1(\Omega))}^2 &\leq A_p + B_p \\
&\leq \frac{4C^2 T^{2\alpha-1}}{\beta^2(2\alpha-1)} \sum_{i=m+1}^p \lambda_i |y_i^1|^2 \\
&\quad + \left[\frac{4C^2 T^\alpha}{\beta^2(1-\alpha)(2\alpha-1)} + \frac{4C^2 T^\alpha}{\alpha - \frac{1}{2}} \right] \left[\sum_{i=m+1}^p \int_0^T |f_i(s)|^2 ds \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|y_{\beta p}(t) - y_{\beta m}(t)\|_{L^2((0,T);H_0^1(\Omega))} &\leq \frac{2C}{\beta} \sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}} \left(\sum_{i=m+1}^p \lambda_i |y_i^1|^2 \right)^{1/2} \\
&\quad + \sqrt{\frac{4C^2 T^\alpha}{\beta^2(1-\alpha)(2\alpha-1)} + \frac{4C^2 T^\alpha}{\alpha - \frac{1}{2}}} \left(\sum_{i=m+1}^p \int_0^T |f_i(s)|^2 ds \right)^{1/2}.
\end{aligned} \quad (3.21) \quad \boxed{\text{normyb}}$$

In view of Equation (3.13), we have

$$I^{1-\alpha}(y_{\beta p}(t) - y_{\beta m}(t)) = \sum_{i=m+1}^p |b_i| E_{\alpha,1}(-\lambda_i t^\alpha) w_i + \sum_{i=m+1}^p \left\{ \int_0^t f_i(u) E_{\alpha,1}(-\lambda_i(t-u)^\alpha) du \right\} w_i,$$

from which we deduce that,

$$\begin{aligned}
\|I^{1-\alpha}(y_{\beta p}(t) - y_{\beta m}(t))\|_{H_0^1(\Omega)}^2 &\leq a(I^{1-\alpha}(y_{\beta p}(t) - y_{\beta m}(t)), I^{1-\alpha}(y_{\beta p}(t) - y_{\beta m}(t))) \\
&\leq 2 \sum_{i=m+1}^p \lambda_i |b_i|^2 E_{\alpha,1}^2(-\lambda_i t^\alpha) \\
&\quad + 2 \sum_{i=m+1}^p \lambda_i \left\{ \int_0^t f_i(u) E_{\alpha,1}(-\lambda_i(t-u)^\alpha) du \right\}^2.
\end{aligned}$$

If we set

$$\begin{aligned}
C_p &= 2 \sum_{i=m+1}^p \lambda_i |b_i|^2 E_{\alpha,1}^2(-\lambda_i t^\alpha), \\
Z_p &= 2 \sum_{i=m+1}^p \lambda_i \left\{ \int_0^t f_i(u) E_{\alpha,1}(-\lambda_i(t-u)^\alpha) du \right\}^2,
\end{aligned}$$

we have from Theorem 2.1, (3.18) and the Cauchy-Schwartz inequality that,

$$\begin{aligned}
C_p &\leq 2C^2 \sum_{i=m+1}^p \lambda_i |b_i|^2 \\
&\leq 2C^2 \left(\frac{2}{\beta^2} \sum_{i=m+1}^p \lambda_i |y_i^1|^2 + \frac{2C^2 T^{1-\alpha}}{\beta^2(1-\alpha)} \sum_{i=m+1}^p \int_0^T |f_i(u)|^2 du \right) \\
&\leq \frac{4C^2}{\beta^2} \sum_{i=m+1}^p \lambda_i |y_i^1|^2 + \frac{4C^4 T^{1-\alpha}}{\beta^2(1-\alpha)} \left(\sum_{i=m+1}^p \int_0^T |f_i(u)|^2 du \right)
\end{aligned}$$

and

$$\begin{aligned}
Z_p &\leq 2 \sum_{i=m+1}^p \lambda_i \left(\int_0^t E_{\alpha,1}^2(-\lambda_i(t-u)^\alpha) du \right) \left(\int_0^t |f_i(u)|^2 du \right) \\
&\leq 2C \sum_{i=m+1}^p \left(\int_0^t (t-u)^{-\alpha} du \right) \left(\int_0^t |f_i(u)|^2 du \right) \\
&\leq \frac{2Ct^{1-\alpha}}{1-\alpha} \sum_{i=m+1}^p \left(\int_0^t |f_i(u)|^2 du \right).
\end{aligned}$$

Using the estimations of C_p and Z_p , we obtain

$$\begin{aligned} \|I^{1-\alpha}(y_{\beta p}(t) - y_{\beta m}(t))\|_{H_0^1(\Omega)}^2 &\leq \frac{4C^2}{\beta^2} \sum_{i=m+1}^p \lambda_i |y_i^1|^2 \\ &+ \left(\frac{4C^4 T^{1-\alpha}}{\beta^2(1-\alpha)} + \frac{2Ct^{1-\alpha}}{1-\alpha} \right) \left(\sum_{i=m+1}^p \int_0^t |f_i(u)|^2 du \right). \end{aligned}$$

Thus,

$$\begin{aligned} \sup_{t \in [0, T]} \|I^{1-\alpha}(y_{\beta p}(t) - y_{\beta m}(t))\|_{H_0^1(\Omega)} &\leq \frac{2C}{\beta} \left(\sum_{i=m+1}^p \lambda_i |y_i^1|^2 \right)^{1/2} \\ &+ \sqrt{\frac{4C^4 T^{1-\alpha}}{\beta^2(1-\alpha)} + \frac{2CT^{1-\alpha}}{1-\alpha}} \left(\sum_{i=m+1}^p \int_0^t |f_i(u)|^2 du \right)^{1/2}. \end{aligned} \quad (3.22) \quad \boxed{\text{supI}}$$

As $y^1 \in H_0^1(\Omega)$ and $f \in L^2(Q)$,

$$\lim_{m, p \rightarrow +\infty} \left(\sum_{i=m+1}^p \lambda_i |y_i^1|^2 \right)^{1/2} = \lim_{m, p \rightarrow +\infty} \left(\sum_{i=m+1}^p \int_0^t |f_i(u)|^2 du \right)^{1/2} = 0.$$

Then, from Equation (3.21) and Equation (3.22), we obtain

$$\lim_{m, p \rightarrow +\infty} \int_0^T \|y_{\beta p}(t) - y_{\beta m}(t)\|_{H_0^1(\Omega)}^2 dt = 0$$

and

$$\sup_{t \in [0, T]} \|I^{1-\alpha}(y_{\beta p}(t) - y_{\beta m}(t))\|_{H_0^1(\Omega)} = 0.$$

Consequently, $(y_{\beta m})$ and $(I^{1-\alpha}y_{\beta m})$ are Cauchy sequences in $L^2((0, T); H_0^1(\Omega))$ and $C([0, T], H_0^1(\Omega))$ respectively. This implies that

$$y_{\beta m} \rightarrow y_\beta \quad \text{in } L^2((0, T); H_0^1(\Omega)), \quad (3.23) \quad \boxed{\text{limy1}}$$

and

$$I^{1-\alpha}y_{\beta m} \rightarrow \xi \quad \text{in } C([0, T]; H_0^1(\Omega)).$$

Since $y_\beta \in L^2((0, T), H_0^1(\Omega))$ and $I^{1-\alpha}y_\beta$ are continuous, we have $\xi = I^{1-\alpha}y_\beta$ and

$$I^{1-\alpha}y_{\beta m} \rightarrow I^{1-\alpha}y_\beta \quad \text{in } C([0, T]; H_0^1(\Omega)). \quad (3.24) \quad \boxed{\text{limI1}}$$

Step 3: We show that y_β satisfies (3.4) – (3.5).

Let $\varphi \in \mathbb{D}(0, T)$ and $\mu \geq 1$ an integer. Then, from (3.15), we have for all $m \geq \mu$,

$$\begin{aligned} \int_0^T (f(t), v)_{L^2(\Omega)} \varphi(t) dt &= \int_0^T D_{RL}^\alpha(y_{\beta m}(t), v)_{L^2(\Omega)} \varphi(t) dt \\ &+ \int_0^T a(y_{\beta m}(t), v) \varphi(t) dt, \quad \forall v \in V_\mu, \end{aligned}$$

which according to Corollary 2.1 implies that,

$$\begin{aligned} \int_0^T (f(t), v)_{L^2(\Omega)} \varphi(t) dt &= - \int_0^T (y_{\beta m}(t), v)_{L^2(\Omega)} \mathcal{D}_C^\alpha \varphi(t) dt \\ &+ \int_0^T a(y_{\beta m}(t), v) \varphi(t) dt, \quad \forall v \in V_\mu. \end{aligned}$$

Therefore, passing to the limit and using (3.23), we obtain

$$\begin{aligned} \int_0^T (f(t), v)_{L^2(\Omega)} \varphi(t) dt &= - \int_0^T (y_\beta(t), v)_{L^2(\Omega)} \mathcal{D}_C^\alpha \varphi(t) dt \\ &+ \int_0^T a(y_\beta(t), v) \varphi(t) dt, \quad \forall v \in V_\mu. \end{aligned}$$

Since $\cup_{\mu \geq 1} V_\mu$ is dense in $H_0^1(\Omega)$ because (w_i) is a base of $H_0^1(\Omega)$, we have for all $v \in H_0^1(\Omega)$ that

$$\begin{aligned} \int_0^T (f(t), v)_{L^2(\Omega)} \varphi(t) dt &= - \int_0^T (y_\beta(t), v)_{L^2(\Omega)} \mathcal{D}_C^\alpha \varphi(t) dt \\ &+ \int_0^T a(y_\beta(t), v) \varphi(t) dt, \quad \forall v \in H_0^1(\Omega). \end{aligned}$$

Using, once again, Corollary 2.1, we can write

$$\begin{aligned} \int_0^T (f(t), v)_{L^2(\Omega)} \varphi(t) dt &= \int_0^T D_{RL}^\alpha (y_\beta(t), v)_{L^2(\Omega)} \varphi(t) dt \\ &+ \int_0^T a(y_\beta(t), v) \varphi(t) dt, \quad \forall v \in H_0^1(\Omega). \end{aligned}$$

This implies that for all $v \in H_0^1(\Omega)$,

$$(f(t), v)_{L^2(\Omega)} \varphi(t) = D_{RL}^\alpha (y_\beta(t), v)_{L^2(\Omega)} \varphi(t) + a(y_\beta(t), v) \varphi(t), \quad \forall t \in (0, T).$$

From (3.24), we have

$$I^{1-\alpha} y_{\beta m}(0) \rightarrow I^{1-\alpha} y_\beta(0) \quad \text{in } H_0^1(\Omega),$$

and

$$I^{1-\alpha} y_{\beta m}(T) \rightarrow I^{1-\alpha} y_\beta(T) \quad \text{in } H_0^1(\Omega).$$

But

$$I^{1-\alpha} y_{\beta m}(T) + \beta I^{1-\alpha} y_{\beta m}(0) = \sum_{i=1}^m y_i^1 w_i \rightarrow \sum_{i=1}^{+\infty} y_i^1 w_i = y^1.$$

Thus,

$$I^{1-\alpha} y_\beta(T) + \beta I^{1-\alpha} y_\beta(0) = y^1.$$

To complete the proof of Theorem 3.1, we need to prove Equation (3.7) and Equation (3.8). Since y_β is the solution of (3.4)-(3.5), we have

$$\begin{aligned} y_\beta(t) &= \sum_{i=1}^{+\infty} \left(\frac{y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du}{\beta + E_\alpha(-\lambda_i T^\alpha)} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) \right. \\ &\quad \left. + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha) f_i(s) ds \right) w_i. \end{aligned}$$

Proceeding as above for estimations on $y_{\beta m}$, we can prove that there exists a constant $C > 0$ such that

$$\begin{aligned} \|y_{\beta}(t)\|_{L^2((0,T);H_0^1(\Omega))} &\leq \frac{2C}{\beta} \sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}} \left(\sum_{i=1}^{+\infty} \lambda_i |y_i^1|^2 \right)^{1/2} \\ &+ \sqrt{\frac{4C^2 T^\alpha}{\beta^2(1-\alpha)(2\alpha-1)} + \frac{4C^2 T^\alpha}{\alpha - \frac{1}{2}}} \left(\sum_{i=1}^{+\infty} \int_0^T |f_i(s)|^2 ds \right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} \sup_{t \in [0,T]} \|I^{1-\alpha} y_{\beta}(t)\|_{H_0^1(\Omega)} &\leq \frac{2C}{\beta} \left(\sum_{i=1}^{+\infty} \lambda_i |y_i^1|^2 \right)^{1/2} \\ &+ \sqrt{\frac{4C^4 T^{1-\alpha}}{\beta^2(1-\alpha)} + \frac{2C T^{1-\alpha}}{1-\alpha}} \left(\sum_{i=1}^{+\infty} \int_0^T |f_i(s)|^2 ds \right)^{1/2}, \end{aligned}$$

from which we deduce, respectively, Equation (3.7) and Equation (3.8). ■

4 Convergence results

convergence

In this section we provide some convergence results.

conv_I_y1

Theorem 4.1 *For all $y^1 \in H_0^1(\Omega)$, we have*

$$\lim_{\beta \rightarrow 0} \|I^{1-\alpha} y_{\beta}(T) - y^1\| = 0.$$

That is $I^{1-\alpha} y_{\beta}(T)$ converges to y^1 in $H_0^1(\Omega)$.

Proof. Since $y^1 \in H_0^1(\Omega)$, we know that

$$\forall \epsilon > 0, \exists N_{\epsilon} \in \mathbb{N} \text{ such that } \sum_{i=N_{\epsilon}+1}^{+\infty} \lambda_i |y_i^1|^2 < \frac{\epsilon}{2}.$$

Also, since $f \in L^2(Q)$, we know that

$$\forall \epsilon > 0, \exists N_{\epsilon} \in \mathbb{N} \text{ such that } \sum_{i=N_{\epsilon}+1}^{+\infty} \int_0^T |f_i(s)|^2 ds < \frac{\epsilon}{2}.$$

Let $\epsilon > 0$ and choose $N > 0$ such that

$$\sum_{i=N+1}^{+\infty} \lambda_i |y_i^1|^2 < \frac{\epsilon}{2} \quad \text{and} \quad \sum_{i=N+1}^{+\infty} \int_0^T |f_i(s)|^2 ds < \frac{\epsilon}{2}.$$

Then, we have

$$\begin{aligned}
\|I^{1-\alpha}y_\beta(T) - y^1\|_{H_0^1(\Omega)}^2 &= a(-\beta I^{1-\alpha}y_\beta(0), -\beta I^{1-\alpha}y_\beta(0)) \\
&= \beta^2 \sum_{i=1}^{+\infty} \lambda_i |b_i|^2 \\
&\leq A + B,
\end{aligned}$$

where

$$\begin{aligned}
A &= 2\beta^2 \sum_{i=1}^{+\infty} \frac{\lambda_i |y_i^1|^2}{(\beta + E_\alpha(-\lambda_i T^\alpha))^2}, \\
B &= 2\beta^2 \sum_{i=1}^{+\infty} \frac{\lambda_i}{(\beta + E_\alpha(-\lambda_i T^\alpha))^2} \left(\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du \right)^2.
\end{aligned}$$

We firstly have,

$$\begin{aligned}
A &= 2\beta^2 \sum_{i=1}^N \frac{\lambda_i |y_i^1|^2}{(\beta + E_\alpha(-\lambda_i T^\alpha))^2} + 2\beta^2 \sum_{i=N+1}^{+\infty} \frac{\lambda_i |y_i^1|^2}{(\beta + E_\alpha(-\lambda_i T^\alpha))^2} \\
&\leq \beta^2 \sum_{i=1}^N \frac{2\lambda_i |y_i^1|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + 2 \sum_{i=N+1}^{+\infty} \lambda_i |y_i^1|^2 \\
&\leq \beta^2 \sum_{i=1}^N \frac{2\lambda_i |y_i^1|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + \epsilon.
\end{aligned}$$

Secondly, using the Cauchy-Schwartz inequality, we can write

$$\begin{aligned}
B &= 2\beta^2 \sum_{i=1}^{+\infty} \frac{\lambda_i}{(\beta + E_\alpha(-\lambda_i T^\alpha))^2} \left(\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du \right)^2 \\
&\leq 2\beta^2 \sum_{i=1}^{+\infty} \frac{C^2 T^{1-\alpha}}{(1-\alpha)(\beta + E_\alpha(-\lambda_i T^\alpha))^2} \left(\int_0^T |f_i(u)|^2 du \right) \\
&\leq \beta^2 \sum_{i=1}^N \frac{2C^2 T^{1-\alpha}}{(1-\alpha)E_\alpha^2(-\lambda_i T^\alpha)} \left(\int_0^T |f_i(u)|^2 du \right) + \frac{C^2 T^{1-\alpha}}{(1-\alpha)} \epsilon.
\end{aligned}$$

Finally, using the estimations of A and B , we have

$$\begin{aligned}
\|I^{1-\alpha}y_\beta(T) - y^1\|_{H_0^1(\Omega)}^2 &\leq \beta^2 \left[\sum_{i=1}^N \frac{2\lambda_i |y_i^1|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + \sum_{i=1}^N \frac{2C^2 T^{1-\alpha}}{(1-\alpha)E_\alpha^2(-\lambda_i T^\alpha)} \left(\int_0^T |f_i(u)|^2 du \right) \right] \\
&\quad + \left(1 + \frac{C^2 T^{1-\alpha}}{(1-\alpha)} \right) \epsilon.
\end{aligned}$$

Since

$$\sum_{i=1}^N \frac{2\lambda_i |y_i^1|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + \sum_{i=1}^N \frac{2C^2 T^{1-\alpha}}{(1-\alpha)E_\alpha^2(-\lambda_i T^\alpha)} \left(\int_0^T |f_i(u)|^2 du \right) < \infty,$$

we choose β such that

$$\beta^2 < \epsilon \left[\sum_{i=1}^N \frac{2\lambda_i |y_i^1|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + \sum_{i=1}^N \frac{2C^2 T^{1-\alpha}}{(1-\alpha)E_\alpha^2(-\lambda_i T^\alpha)} \left(\int_0^T |f_i(u)|^2 du \right) \right]^{-1}.$$

■

Theorem 4.2 *Supposing there exists $\epsilon \in (0, 2)$ such that*

$$D = 2 \sum_{i=1}^{+\infty} \frac{\lambda_i |y_i^1|^2}{E_\alpha^\epsilon(-\lambda_i T^\alpha)} + \frac{2C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^{+\infty} \frac{\int_0^T |f_i(s)|^2 ds}{E_\alpha^\epsilon(-\lambda_i T^\alpha)}$$

converges, then $\|I^{1-\alpha} y_\beta - y^1\|_{H_0^1(\Omega)}$ converges to zero with order $\epsilon^{-2}\beta^\epsilon$.

Proof. Let $\epsilon \in (0, 2)$ such that D converges and $k \in (0, 2)$. We fix a natural integer i , and define

$$g_i(\beta) = \frac{\beta^k}{[\beta + E_\alpha(-\lambda_i T^\alpha)]^2}.$$

Differentiating g_i with respect to β , we obtain

$$\begin{aligned} g_i'(\beta) &= \frac{(k-2)\beta^k + k\beta^{k-1}E_\alpha(-\lambda_i T^\alpha)}{[\beta + E_\alpha(-\lambda_i T^\alpha)]^3} \\ &= \beta^{k-1} \times \frac{(k-2)\beta + kE_\alpha(-\lambda_i T^\alpha)}{[\beta + E_\alpha(-\lambda_i T^\alpha)]^3}. \end{aligned}$$

Observing that $g_i'(\beta) = 0$ if $\beta = 0$ or $(k-2)\beta + kE_\alpha(-\lambda_i T^\alpha) = 0$. We have

$$(k-2)\beta + kE_\alpha(-\lambda_i T^\alpha) = 0 \Leftrightarrow \beta = \frac{k}{2-k} E_\alpha(-\lambda_i T^\alpha).$$

As $g_i(\beta) > 0$, $g_i(0) = 0$ and $\lim_{\beta \rightarrow +\infty} g_i(\beta) = 0$. Indeed,

$$\lim_{\beta \rightarrow +\infty} g_i(\beta) = \lim_{\beta \rightarrow +\infty} \frac{\beta^k}{\beta^2} = \lim_{\beta \rightarrow +\infty} \frac{1}{\beta^{2-k}} = 0.$$

We know that g_i achieves its maximum at $\beta_0 = \frac{k}{2-k} E_\alpha(-\lambda_i T^\alpha)$. Hence, we have,

$$\begin{aligned}
g_i(\beta) \leq g_i(\beta_0) &\Leftrightarrow g_i(\beta) \leq \frac{(\beta_0)^k}{[\beta_0 + E_\alpha(-\lambda_i T^\alpha)]^2} \\
&\Leftrightarrow g_i(\beta) \leq \frac{\left(\frac{k}{2-k}\right)^k E_\alpha^k(-\lambda_i T^\alpha)}{[\beta_0 + E_\alpha(-\lambda_i T^\alpha)]^2} \\
&\Leftrightarrow g_i(\beta) \leq \left(\frac{k}{2-k}\right)^k E_\alpha^{k-2}(-\lambda_i T^\alpha).
\end{aligned}$$

Since we can write

$$\begin{aligned}
\|I^{1-\alpha} y_\beta - y^1\|_{H_0^1(\Omega)}^2 &\leq 2\beta^2 \sum_{i=1}^{+\infty} \frac{\lambda_i |y_i^1|^2}{(\beta + E_\alpha(-\lambda_i T^\alpha))^2} \\
&+ 2\beta^2 \sum_{i=1}^{+\infty} \frac{\lambda_i}{(\beta + E_\alpha(-\lambda_i T^\alpha))^2} \left(\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du \right)^2 \\
&= 2\beta^{2-k} \sum_{i=1}^{+\infty} \lambda_i |y_i^1|^2 g_i(\beta) + 2\beta^{2-k} \sum_{i=1}^{+\infty} \lambda_i \left(\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du \right)^2 g_i(\beta),
\end{aligned}$$

it follows that

$$\begin{aligned}
\|I^{1-\alpha} y_\beta - y^1\|_{H_0^1(\Omega)}^2 &\leq 2\beta^{2-k} \sum_{i=1}^{+\infty} \lambda_i |y_i^1|^2 g_i(\beta) + 2\beta^{2-k} \frac{C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^{+\infty} \left(\int_0^T |f_i(u)|^2 du \right) g_i(\beta) \\
&\leq \beta^{2-k} \left(\frac{k}{2-k}\right)^k \left[2 \sum_{i=1}^{+\infty} \lambda_i |y_i^1|^2 E_\alpha^{k-2}(-\lambda_i T^\alpha) \right. \\
&\quad \left. + \frac{2C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^{+\infty} \left(\int_0^T |f_i(u)|^2 du \right) E_\alpha^{k-2}(-\lambda_i T^\alpha) \right].
\end{aligned}$$

If we choose, $k = 2 - \epsilon$ (then $\epsilon = 2 - k$), we then obtain

$$\begin{aligned}
\|I^{1-\alpha}y_\beta - y^1\|_{H_0^1(\Omega)}^2 &\leq \beta^\epsilon \left(\frac{2-\epsilon}{\epsilon}\right)^{2-\epsilon} \left[2 \sum_{i=1}^{+\infty} \lambda_i |y_i^1|^2 E_\alpha^{-\epsilon}(-\lambda_i T^\alpha) \right. \\
&\quad \left. + \frac{2C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^{+\infty} \left(\int_0^T |f_i(u)|^2 du \right) E_\alpha^{-\epsilon}(-\lambda_i T^\alpha) \right] \\
&\leq \beta^\epsilon \left(\frac{2}{\epsilon}\right)^2 \left[2 \sum_{i=1}^{+\infty} \lambda_i |y_i^1|^2 E_\alpha^{-\epsilon}(-\lambda_i T^\alpha) \right. \\
&\quad \left. + \frac{2C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^{+\infty} \left(\int_0^T |f_i(u)|^2 du \right) E_\alpha^{-\epsilon}(-\lambda_i T^\alpha) \right].
\end{aligned}$$

Since D converges, there exists a constant $K > 0$ such that

$$2 \sum_{i=1}^{+\infty} \lambda_i |y_i^1|^2 E_\alpha^{-\epsilon}(-\lambda_i T^\alpha) + \frac{2C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^{+\infty} \left(\int_0^T |f_i(u)|^2 du \right) E_\alpha^{-\epsilon}(-\lambda_i T^\alpha) < K.$$

which implies that

$$\begin{aligned}
\|I^{1-\alpha}y_\beta(T) - y^1\|_{H_0^1(\Omega)}^2 &\leq \beta^\epsilon \left(\frac{2}{\epsilon}\right)^2 K \\
&= \epsilon^{-2} \beta^\epsilon (4K) \\
&= \epsilon^{-2} \beta^\epsilon K'.
\end{aligned}$$

It then suffices to take $K' = 4K$ to achieve the proof. ■

Theorem 4.3 *For all $y^1 \in H_0^1(\Omega)$, the problem compounded in Equation (1.1) has a solution y if and only if the sequence $I^{1-\alpha}y_\beta(0^+)$ converges in $H_0^1(\Omega)$. Furthermore, we have that y_β converges to y as β tends to zero in $L^2((0, T); H_0^1(\Omega))$.*

Proof. We proceed in two steps.

Step 1: We show that if $I^{1-\alpha}y_\beta(0)$ converges in $H_0^1(\Omega)$, then the problem (1.1) admits a solution.

Assume that $\lim_{\beta \rightarrow 0} I^{1-\alpha}y_\beta(0) = y^0$ exists. Since $y^0 \in H_0^1(\Omega)$, we can write

$$y^0 = \sum_{i=1}^{+\infty} y_i^0 w_i \quad \text{where} \quad y_i^0 = (y^0, w_i).$$

Let y the solution of the following equation

$$\begin{cases} D_{RL}^\alpha y(x, t) - \Delta y(x, t) = f(x, t) & \text{in } Q, \\ y(\sigma, t) = 0 & \text{on } \Sigma, \\ I^{1-\alpha} y(x, 0) = y^0 & \text{in } \Omega. \end{cases}$$

where $1/2 < \alpha < 1$. Then, from Theorem 2.2, we know that $y \in L^2((0, T); H_0^1(\Omega))$ is given by

$$y(t) = \sum_{i=1}^{+\infty} \left\{ t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i t^\alpha) y_i^0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds \right\} w_i.$$

Moreover, $I^{1-\alpha} y \in C([0, T], H_0^1(\Omega))$. Thus, $I^{1-\alpha} y(T) \in H_0^1(\Omega)$ exists.

Now, let $t \in [0, T]$, we have

$$\begin{aligned} y_\beta(t) - y(t) &= \sum_{i=1}^{+\infty} \left[I^{1-\alpha} y_{\beta i}(0) t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds \right] w_i \\ &\quad - \sum_{i=1}^{+\infty} \left[y_i^0 t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds \right] w_i \\ &= \sum_{i=1}^{+\infty} \left(I^{1-\alpha} y_{\beta i}(0) - y_i^0 \right) t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i t^\alpha) w_i. \end{aligned}$$

consequently,

$$\begin{aligned} \|y_\beta - y\|_{L^2((0, T), H_0^1(\Omega))}^2 &= \int_0^T a(y_\beta(t) - y(t), y_\beta(t) - y(t)) dt \\ &= \int_0^T \left(\sum_{i=1}^{+\infty} \lambda_i \left(I^{1-\alpha} y_{\beta i}(0) - y_i^0 \right)^2 \left(t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i t^\alpha) \right)^2 \right) dt \\ &= \sum_{i=1}^{+\infty} \lambda_i \left(I^{1-\alpha} y_{\beta i}(0) - y_i^0 \right)^2 \int_0^T t^{2\alpha-2} E_{\alpha, \alpha}^2(-\lambda_i t^\alpha) dt \\ &\leq C^2 \sum_{i=1}^{+\infty} \lambda_i \left(I^{1-\alpha} y_{\beta i}(0) - y_i^0 \right)^2 \int_0^T t^{2\alpha-2} dt \\ &\leq \frac{C^2 T^{2\alpha-1}}{2\alpha-1} \left\| I^{1-\alpha} y_\beta(0) - y^0 \right\|_{H_0^1(\Omega)}^2. \end{aligned}$$

This implies that y_β converges to y in $L^2((0, T); H_0^1(\Omega))$ because $\lim_{\beta \rightarrow 0} I^{1-\alpha} y_\beta(0) = y^0$.

On the other hand, we have

$$I^{1-\alpha} y_i(T) = y_i^0 E_\alpha(-\lambda_i T^\alpha) + \int_0^T E_\alpha(-\lambda_i (T-u)^\alpha) f_i(u) du,$$

and

$$I^{1-\alpha} y_{\beta i}(T) = I^{1-\alpha} y_{\beta i}(0) E_\alpha(-\lambda_i T^\alpha) + \int_0^T E_\alpha(-\lambda_i (T-u)^\alpha) f_i(u) du.$$

Hence, we obtain

$$\begin{aligned} \left\| I^{1-\alpha} y_\beta(T) - I^{1-\alpha} y(T) \right\|_{H_0^1(\Omega)}^2 &= \sum_{i=1}^{+\infty} \lambda_i \left(I^{1-\alpha} y_{\beta i}(0) - y_i^0 \right)^2 E_\alpha^2(-\lambda_i T^\alpha) \\ &\leq C^2 \left\| I^{1-\alpha} y_\beta(0) - y^0 \right\|_{H_0^1(\Omega)}^2. \end{aligned}$$

This implies that

$$I^{1-\alpha}y_\beta(T) \rightarrow I^{1-\alpha}y(T) \text{ strongly in } H_0^1(\Omega)$$

and since, from Theorem 4.1

$$I^{1-\alpha}y_\beta(T) \rightarrow y^1 \text{ strongly in } H_0^1(\Omega),$$

the uniqueness of the limit allows us to conclude that $I^{1-\alpha}y(T) = y^1$ and y is a solution of the problem compounded in Equation (1.1).

Step 2: We show that if the problem given by Equation (1.1) admits a solution y then $I^{1-\alpha}y_\beta(0)$ converges in $H_0^1(\Omega)$.

Let y be a solution of the problem associated with Equation (1.1), then as in the proof of existence in Theorem 3.1, we know that $y_i = (y(t), w_i)_{L^2(\Omega)}$ is a solution of the ordinary differential equation

$$\begin{cases} D_{RL}^\alpha y_i(t) + \lambda_i y_i(t) = f_i(t), & t \in [0, T], \\ I^{1-\alpha} y_i(T) = y_i^1. \end{cases} \quad (4.1) \quad \boxed{\text{edo2}}$$

Using the Laplace transform of the first equation in Equation (4.1), we obtain

$$y_i(t) = I^{1-\alpha} y_i(0) t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds. \quad (4.2) \quad \boxed{\text{sol_fvp_1}}$$

Observing that

$$I^{1-\alpha}(t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i t^\alpha)) = E_\alpha(-\lambda_i t^\alpha)$$

and

$$I^{1-\alpha} \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds \right) = \int_0^t f_i(u) E_\alpha(-\lambda_i (t-u)^\alpha) du,$$

we have

$$I^{1-\alpha} y_i(t) = I^{1-\alpha} y_i(0) E_\alpha(-\lambda_i t^\alpha) + \int_0^t f_i(u) E_\alpha(-\lambda_i (t-u)^\alpha) du.$$

and because $I^{1-\alpha} y_i(T) = y_i^1$, we can write

$$I^{1-\alpha} y_i(0) E_\alpha(-\lambda_i T^\alpha) + \int_0^T f_i(u) E_\alpha(-\lambda_i (T-u)^\alpha) du = y_i^1,$$

from which, we deduce that

$$I^{1-\alpha} y_i(0) = \frac{y_i^1 - \int_0^T f_i(u) E_\alpha(-\lambda_i (T-u)^\alpha) du}{E_\alpha(-\lambda_i T^\alpha)}.$$

Thus, we can write

$$\begin{aligned} y(t) &= \sum_{i=1}^{+\infty} \left\{ \frac{y_i^1 - \int_0^T E_\alpha(-\lambda_i (T-u)^\alpha) f_i(u) du}{E_\alpha(-\lambda_i T^\alpha)} t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i t^\alpha) \right\} w_i \\ &+ \sum_{i=1}^{+\infty} \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds \right\} w_i \end{aligned} \quad (4.3) \quad \boxed{\text{sol_fvp_final}}$$

and

$$I^{1-\alpha}y(t) = \sum_{i=1}^{+\infty} \left\{ \frac{y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du}{E_\alpha(-\lambda_i T^\alpha)} E_\alpha(-\lambda_i t^\alpha) + \int_0^t f_i(u) E_\alpha(-\lambda_i(t-u)^\alpha) du \right\} w_i. \quad (4.4) \quad \boxed{\text{I_fvp}}$$

Let $\beta, \gamma > 0$. Then, from (3.14), we have

$$I^{1-\alpha}y_\beta(0) - I^{1-\alpha}y_\gamma(0) = \frac{(\gamma - \beta) \left(y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du \right)}{\beta\gamma + (\beta + \gamma)E_\alpha(-\lambda_i T^\alpha) + E_\alpha^2(-\lambda_i T^\alpha)}.$$

As $I^{1-\alpha}y(0) \in H_0^1(\Omega)$, we choose $N > 0$ such as

$$\forall \epsilon > 0, \quad \sum_{i=N+1}^{+\infty} \lambda_i |I^{1-\alpha}y_i(0)|^2 < \frac{\epsilon}{2}.$$

This means that

$$\forall \epsilon > 0, \quad \sum_{i=N+1}^{+\infty} \lambda_i \left| \frac{y_i^1 - \int_0^T f_i(u) E_\alpha(-\lambda_i(T-u)^\alpha) du}{E_\alpha(-\lambda_i T^\alpha)} \right|^2 < \frac{\epsilon}{2},$$

and we have

$$\begin{aligned} \|I^{1-\alpha}y_\beta(0) - I^{1-\alpha}y_\gamma(0)\|_{H_0^1(\Omega)}^2 &= \sum_{i=1}^{+\infty} \lambda_i \left| \frac{(\gamma - \beta) \left(y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du \right)}{\beta\gamma + (\beta + \gamma)E_\alpha(-\lambda_i T^\alpha) + E_\alpha^2(-\lambda_i T^\alpha)} \right|^2 \\ &\leq \frac{(\gamma - \beta)^2}{(\beta\gamma)^2} \sum_{i=1}^N \lambda_i \left(y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du \right)^2 \\ &\quad + \frac{(\gamma - \beta)^2}{(\beta + \gamma)^2} \sum_{i=N+1}^{+\infty} \lambda_i \frac{\left(y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du \right)^2}{E_\alpha^2(-\lambda_i T^\alpha)} \\ &\leq \frac{(\gamma - \beta)^2}{\beta\gamma^2} \sum_{i=1}^N \lambda_i \left(y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u) du \right)^2 \\ &\quad + \frac{(\gamma - \beta)^2 \epsilon}{(\beta + \gamma)^2 2}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\|I^{1-\alpha}y_\beta(0) - I^{1-\alpha}y_\gamma(0)\|_{H_0^1(\Omega)}^2 &\leq \left(\frac{\gamma - \beta}{\beta\gamma}\right)^2 \left(2 \sum_{i=1}^N \lambda_i |y_i^1|^2 + \frac{2C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^N \int_0^T |f_i|^2(u) du\right) \\
&+ \left(\frac{\gamma - \beta}{\beta + \gamma}\right)^2 \frac{\epsilon}{2}. \\
&\leq \left(\frac{2}{\beta^2} + \frac{2}{\gamma^2}\right) \left(2 \sum_{i=1}^N \lambda_i |y_i^1|^2 + \frac{2C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^N \int_0^T |f_i|^2(u) du\right) \\
&+ 2\epsilon.
\end{aligned}$$

Since

$$\left(2 \sum_{i=1}^N \lambda_i |y_i^1|^2 + \frac{2C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^N \int_0^T |f_i|^2(u) du\right) < \infty \quad \text{and} \quad \lim_{\gamma, \beta \rightarrow \infty} \left(\frac{2}{\beta^2} + \frac{2}{\gamma^2}\right) = 0,$$

we deduce that

$$\lim_{\gamma, \beta \rightarrow \infty} \|I^{1-\alpha}y_\beta(0) - I^{1-\alpha}y_\gamma(0)\|_{H_0^1(\Omega)} = 0.$$

This implies that the sequence $\{I^{1-\alpha}y_\beta(0)\}$ is of Cauchy and thus it converges in $H_0^1(\Omega)$. ■

5 Conclusion

In this work, we have considered an ill-posed problem associated with a family of well-posed problems and prove, using spectral methods, that the solutions of the latter problems converge to the solution of the former problem in an appropriate Hilbert space. This analysis is useful if we want to control an ill-posed problem which will be the subject of future work. Moreover, the convergence results obtained can be used to find a numerical solution for problem compounded in Equation (1.1).

References

- denche** [1] M. DENCHE, K. BESSILA, *A modified quasi-boundary value method for ill-posed problems*, J. Math. Anal. Appl. 301 (2005), pp. 419-426.
- Mittag** [2] H.J. HAUBOLD, A. M. MATHAI, AND R. K. SAXENA, *Mittag-Leffler Functions and Their Applications*, Hindawi Publishing Corporation, Journal of Applied Mathematics, Volume 2011, Article ID 298628, 51 pages.
- baleanu** [3] LE NHAT HUYNH, NGUYEN HOANG LUC , DUMITRU BALEANU AND LE DINH LONG, *Recovering the space source term for the fractional-diffusion equation with Caputo-Fabrizio derivative*, Journal of inequalities and applications, 2021, Vol.2021 (1), p.1-20.

- [4] NGUYEN HUY TUAN, LE DINH LONG, VAN THINH NGUYEN AND THANH TRAN, *On a final value problem for the time-fractional diffusion equation with inhomogeneous source*, Inverse Problems In Sciences And Engineering, 2017, Vol.25, No 9, 1367-1395.
- [5] KOKILA JAYAKUMAR, *Modified quasi-boundary value method for the multidimensional nonhomogeneous backward time fractional diffusion equation*, Mathematical methods in the applied sciences, 2021, Vol.44 (10), p.8363-8378.
- [6] ERDAL KARAPINAR, DEVENDRA KUMAR, RATHINASAMY SAKTHIVEL, NGUYEN HOANG LUC AND N.H. CAN, *Identifying the space source term problem for time-space-fractional diffusion equation*, Advances in difference equations, Vol. 2020 (1), p. 1-23.
- [7] A.A. KILBAS, H.M. SRIVASTAVA AND J.J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.
- [8] R. LATTÈS AND J.L. LIONS, *The Method of Quasireversibility : Applications to Partial Differential Equations*, Elsevier, New York, 1969.
- [9] J.J. LIU, M. YAMAMOTO, *A backward problem for the time-fractional diffusion equation*, Applicable Analysis, Vol. 89, No. 11, November 2010, 1769-1788.
- [10] R. METZLER AND J. KLAFTER, *Boundary value problems for fractional diffusion equations*, Physica A 278 (2000), pp. 107-125.
- [11] G. MOPHOU, *Optimal control of fractional diffusion equation*, Computers and Mathematics with Applications 61 (2011) 68-78.
- [12] G. MOPHOU, S. TAO AND C. JOSEPH, *Initial value/boundary value problem for composite fractional relaxation equation*, Applied Mathematics and Computation. 257 (2015) 134-144.
- [13] I.PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [14] T.R. PRABHAKAR, *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohama Mathematical Journal, vol. 19, pp. 7-15, 1971.
- [15] K. SAKAMOTO AND M. YAMAMOTO, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. Appl. 382 (2011) 426-447.
- [16] WANXIA SHI, XIANGTUAN XIONG, XUEMIN XUE, *A fractional-order quasi-reversibility method to a backward problem for the time fractional diffusion equation*, Journal of Computational and Applied Mathematics, 394 (2021) 113552.
- [17] R.E. SHOWALTER, *The final value problem for evolution equations*, J. Math. Anal. Appl. 47 (1974), pp. 563-572.
- [18] JUN-GANG WANG, TING WEI, *Quasi-reversibility method to identify a space-dependent source for the time-fractional diffusion equation*, Applied Mathematical Modelling, 39 (2015) 6139-6149.
- [19] FAN YANG, CHU-LI FU, *The quasi-reversibility regularization method for identifying the unknown source for time fractional diffusion equation*, Applied Mathematical Modelling, 39 (2015) 1500-1512.

- yang2 [20] FAN YANG, NI WANG, XIAO-XIAO LI AND CAN-YUN HUANG, *A quasi-boundary regularization method for identifying the initial value of time-fractional diffusion equation on spherically symmetric domain*, Journal of inverse and ill-posed problems, 2019, Vol.27 (5), p.609-621.
- fan [21] FAN YANG, YAN ZHANG, XIAO-XIAO LI AND CAN-YUN HUANG, *The quasi-boundary value regularization method for identifying the initial value with discrete random noise*, Boundary Value Problems (2018) 2018:108.