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Claire Joseph, Maryse Moutamal

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**Abstract**

A quasi-boundary method is used to study an ill-posed, time-fractional diffusion equation involving the fractional Riemann-Liouville derivative. In particular, we consider an ill-posed problem for a family of well-posed problems, and prove, by means of eigenfunction expansions, that the solutions of the latter problems converge to the solutions associated with the former problem. The analysis presented includes providing conditions for the rate of the convergence.

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## 1 Introduction

Let $d \in \mathbb{N}^*$ and $\Omega$ be a bounded open subset of $\mathbb{R}^d$, for a boundary $\partial\Omega$ of class $C^2$. For $T > 0$, we set $Q = \Omega \times (0,T), \Sigma = \partial\Omega \times (0,T)$ and consider the fractional diffusion equation:

$$\left\{ \begin{array}{rcll} D^{\alpha}_{RL} y(x,t) - \Delta y(x,t) &=& f(x,t) & \text{in } Q, \\ y(\sigma,t) &=& 0 & \text{on } \Sigma, \\ I^{1-\alpha} y(x,T) &=& y^1(x) & \text{on } \Omega, \end{array} \right. \quad (1.1)$$ FVP

where $0 < \alpha < 1$, $f \in L^2(Q)$, $y^1 \in H^1_0(\Omega)$. The operators $I^{1-\alpha} y$ and $D^{\alpha}_{RL} y$ are, respectively, the Riemann-Liouville fractional integral of order $1-\alpha$, and the left Riemann-Liouville fractional derivative of order $\alpha$ of $y$.

The fractional diffusion equation has been of significant interest for many decades. The equation (specifically the time-fractional diffusion equation) is obtained by replacing the first order time derivative of the classical diffusion equation with a time fractional derivative. In comparison with the conventional first order derivative, Left Riemann-Liouville fractional derivatives are characterised by a convolution integral (see Definition 2.6). This shows that the (fractional) derivative depends on the behavior of function $y$ over the interval $[0,t]$. This is the reason why researchers speak about the memory effect associated with a fractional derivative, and why, in this context fractional derivatives are used in other fields such as Physics, Biology or Economics, where the memory association of a field is mandatory.

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The model compounded in Equation (1.1), can be used to investigate environmental phenomenon. In such cases, one might not have all the necessary information on complete the model. In the case considered here, the initial condition is missing so that Equation (1.1) appears as an ill-posed backward time-fractional diffusion equation. The problem compounded in Equation (1.1) does not satisfies the Hadamard conditions. This is because we cannot prove that (1.1) admits a unique solution which depends continuously of $y^1$. Nowadays, there are many methods such as inverse methods that may be used to approach such ill-posed problems. In this paper, we use a Quasi-boundary method which was originally introduced by the quasi-reversibility method developed in [8].

The quasi-boundary method is based on perturbing the final condition. Some researches on the topic have shown that this method gives better numerical results than the quasi-reversibility method. For example, in [20], Yang et *al.* apply the quasi-boundary method to approximate an inverse problem for identifying the initial data for a time-fractional diffusion equation on a pherically symmetric domain. Jayakumar [5] use a modified quasi-boundary method to solve a non-homogeneous time fractional diffusion problem involving the left fractional Caputo derivative. More recently, Huynh et *al.* [3] applied a modified quasi-boundary method for a fractional diffusion equation involving the Caputo-Fabrizio fractional derivative.

We refer to [6, 9, 16, 18, 1, 10, 17, 19] and references therein for more information in regard to the quasi-reversibility method and quasi-boundary method. In this context, the Riemann-Liouville and Caputo based fractional derivatives are closely related. The best of the authors' knowledge, and, judging from the open literature available, there are no studies on the quasi-boundary method for fractional diffusion equations involving the Riemann-Liouville fractional derivative. In this paper, we approach the ill-posed problem compounded in Equation (1.1) through a family of well-posed problems. More precisely, we consider, for any $\beta > 0$, the following quasi-boundary value problem:

$$\left\{\begin{array}{rcll} D^{\alpha}_{RL} y_\beta(x,t) - \Delta y_\beta(x,t) & = & f(x,t) & (x,t) \in Q, \\ y_\beta(\sigma,t) & = & 0 & (\sigma,t) \in \Sigma, \\ I^{1-\alpha} y_\beta(x,T) + \beta I^{1-\alpha} y_\beta(x,0^+) & = & y^1(x) & x \in \Omega, \end{array}\right.$$

and prove that the family of solutions $y_\beta$ converge to the solution of Equation (1.1) in an appropriate Hilbert space, specifying the rate of convergence.

This paper is structured as follows: In Section 2, we provide some definitions on fractional operators, examples of their properties and some preliminary results. In Section 3, we use the spectral method to prove the existence and uniqueness of the solution of the problem. The convergence results are provided in Section 4.

## 2 Preliminaries

prelim

In this section, we recall some basic definitions and results on fractional integration and derivative.

gamma

**Definition 2.1** *[7, 13] Let $z$ be a complex number such that $Re(z) > 0$. Then the Gamma function is given by*

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt.$$

**Remark 2.1** *It follows from the definition above that*

$$\Gamma(z+1) = z\Gamma(z).$$

def2 **Definition 2.2** *[7, 13] Let $x$ and $y$ be two complex numbers such that $Re(x) > 0$ and $Re(y) > 0$. The Beta function is given by*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

**Remark 2.2** *One can prove [7, 13] that*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.1}$$ beta_formule

**Definition 2.3** *[7, 13] Let $\alpha > 0$ and $\beta > 0$. Then, the two-parameter Mittag-Leffler function is given by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ z \in \mathbb{C}. \tag{2.2}$$ mittag

Thus, we have

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \alpha)}, \ z \in \mathbb{C}. \tag{2.3}$$ Eaa

In what follows, we set

$$E_{\alpha,1}(t) = E_\alpha(t).$$

majora **Theorem 2.1** *[13, 15] Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$. We consider that $\mu$ satisfies*

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}.$$

*In this case, there exists a constant $C = C(\alpha, \beta, \mu) > 0$, such that*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \qquad \mu \leq |arg(z)| \leq \pi.$$

**Definition 2.4** *[2, 7] Let $\alpha, \beta, \rho \in \mathbb{C}$ such that $Re(\alpha) > 0$ and $Re(\beta) > 0$. Then, the generalized Mittag-Leffler function is defined by*

$$\mathcal{E}^{\rho}_{\alpha,\beta}(t) = \sum_{n=0}^{+\infty} \frac{(\rho)_n t^n}{\Gamma(\alpha n + \beta) n!}, \qquad \text{for all } t \in \mathbb{C},$$

*where $(\rho)_n = \rho(\rho+1)\dots(\rho+n-1)$.*

**Remark 2.3** *Note that, when $\rho = 1$ we obtain*

$$\mathcal{E}^{1}_{\alpha,\beta}(t) = E_{\alpha,\beta}(t),$$

*where $E_{\alpha,\beta}$ is the classical Mittag-Leffler function defined in (2.2).*

The following result gives the Laplace transform of the generalized Mittag-Leffler function.

**Lemma 2.1** *[2] Let $\alpha, \beta, \rho$ be complexes such that $Re(\alpha) > 0, Re(\rho) > 0$ and $Re(\beta) > 0$. Then, we have*

$$\mathcal{L}^{-1}\left\{\frac{s^{\rho-1}}{s^\alpha + as^\beta + b}; t\right\} = t^{\alpha-\rho} \sum_{k=0}^{+\infty} (-a)^k t^{(\alpha-\beta)k} \mathcal{E}^{k+1}_{\alpha,\alpha+(\alpha-\beta)k-\rho+1}(-bt^\alpha), \tag{2.4}$$ Lpmittag

*where $|as^\beta/(s^\alpha + b)| < 1$ and $\mathcal{L}^{-1}$ is the inverse Laplace transform.*

IRL **Definition 2.5** *[7, 13] Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function, and $\alpha > 0$. Then, the expression*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t > 0,$$

*is called the Riemann-Liouville integral of order $\alpha$ of the function $f$.*

def26 **Definition 2.6** *[7, 13] Let $f : \mathbb{R}_+ \to \mathbb{R}$. The left Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ of $f$ is defined as*

$$D^\alpha_{RL} f(t) = \frac{1}{\Gamma(1-\alpha)} . \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s)ds, \quad t > 0,$$

*provided that the integral exists.*

**Remark 2.4** *From Definition 2.5, we see that :*

$$D^\alpha_{RL} f(t) = \frac{d}{dt} I^{1-\alpha} f(t). \tag{2.5}$$ DandI

We also with the right Caputo fractional derivative given by:

**Definition 2.7** *[7, 13] Let $f : \mathbb{R}_+ \to \mathbb{R}$, $0 < \alpha < 1$. The right Caputo fractional derivative of order $\alpha$ of $f$ is defined by*

$$\mathcal{D}^\alpha_C f(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f'(s)ds, \quad 0 < t < T, \tag{2.6}$$

*provided that the integral exists.*

We consider a result obtained through integration by parts, which involves the left Riemann-Liouville fractional derivative and the right Caputo fractional derivative.

integrale1 **Lemma 2.2** *[11] Let $0 < \alpha < 1$, $y \in \mathcal{C}^\infty(\bar{Q})$ and $\varphi \in \mathcal{C}^\infty(\bar{Q})$. Then, we have,*

$$\begin{aligned}
&\int_0^T \int_\Omega (D^\alpha_{RL} y(x,t) - \Delta y(x,t))\varphi(x,t)dxdt = \\
&\int_\Omega \varphi(x,T) I^{1-\alpha} y(x,T)dx - \int_\Omega \varphi(x,0) I^{1-\alpha} y(x,0)dx \\
&+ \int_0^T \int_{\partial\Omega} y(\sigma,t) \frac{\partial \varphi}{\partial v}(\sigma,t)d\sigma dt - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial v}(\sigma,t)\varphi(\sigma,t)d\sigma dt \\
&+ \int_\Omega \int_0^T y(x,t)(-\mathcal{D}^\alpha_C \varphi(x,t) - \Delta\varphi(x,t))dxdt,
\end{aligned} \tag{2.7}$$

*where $\mathcal{D}^\alpha_C$ is the right Caputo fractional of order $0 < \alpha < 1$.*

integrale0T **Corollary 2.1** *[11] Let $\mathbb{D}(0,T)$ be the set of $C^\infty$ functions on $(0,T)$ with compact support. Then for all $\varphi \in \mathbb{D}(0,T)$,*

$$\int_0^T D^\alpha_{RL} y(t)\varphi(t)dt = -\int_0^T y(t)\mathcal{D}^\alpha_C \varphi(t)dt,$$

*where $\mathcal{D}^\alpha_C$ is the right fractional Caputo derivative.*

article1 **Theorem 2.2** *[12] Let* $1/2 < \alpha < 1$, $y^0 \in H_0^1(\Omega)$ *and* $f \in L^2(Q)$. *Then, the problem*

$$\begin{cases} D_{RL}^{\alpha} y(x,t) - \Delta y(x,t) &= f(x,t) & \text{in } Q, \\ y(\sigma,t) &= 0 & \text{on } \Sigma, \\ I^{1-\alpha} y(x,0) &= y^0 & \text{in } \Omega. \end{cases}$$

*has a unique solution* $y \in L^2((0,T), H_0^1(\Omega))$. *Moreover,* $I^{1-\alpha} y \in C([0,T], H_0^1(\Omega))$, *and, there exists a constant* $C > 0$, *such that the following estimations hold:*

$$\|y\|_{L^2((0,T);H_0^1(\Omega))} \leq \Delta \left( \|y^0\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q)} \right), \tag{2.8}$$

$$\|I^{1-\alpha} y\|_{C([0,T];H_0^1(\Omega))} \leq \Pi \left( \|y^0\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q)} \right), \tag{2.9}$$

*with*

$$\Delta = \max \left( C \sqrt{\frac{2T^{2\alpha-1}}{2\alpha - 1}}, \frac{C}{\alpha} \sqrt{\frac{2T}{\lambda_1}} \right),$$

*and*

$$\Pi = \sup \left( C\sqrt{2}, C \sqrt{\frac{2T^{1-\alpha}}{(1-\alpha)}} \right).$$

On other hand, it is well-known that $(-\Delta)$ is a symmetric uniform elliptic operator. Thus, it admits real eigenvalues, $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$ with $\lambda_k \to \infty$ when $k \to \infty$. Moreover, there exists an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$ , where $w_k \in H_0^1(\Omega)$ is an eigenfunction corresponding to $\lambda_k$: $-\Delta w_k = \lambda_k w_k$. Further, we have,

$$\int_\Omega \nabla \varphi(x) \cdot \nabla \psi(x) dx = \lambda_k \int_\Omega \varphi(x) \psi(x) dx, \qquad \forall p \in H_0^1(\Omega). \tag{2.10}$$ valprop1

In what follows, for all $\varphi, \psi \in L^2(\Omega)$, we denote

$$(\varphi, \psi)_{L^2(\Omega)} = \int_\Omega \varphi(x) \psi(x) dx,$$

as the inner product in $L^2(\Omega)$ and $\|\varphi\|_{L^2(\Omega)}$ as the associated norm.
We set

$$a(\varphi, \psi) = \int_\Omega \nabla \varphi(x) \cdot \nabla \psi(x) dx, \quad \forall \varphi, \psi \in H_0^1(\Omega). \tag{2.11}$$ formebi1

Then, the bilinear functional $a(.,.)$ defines an inner product on $H_0^1(\Omega)$, and we have

$$\|\varphi\|^2_{H_0^1(\Omega)} = a(\varphi, \varphi), \tag{2.12}$$ equivalence1

which is a norm on $H_0^1(\Omega)$. Since $\left\{ \frac{w_k}{\sqrt{\lambda_k}} \right\}_{k=1}^{\infty}$ is an orthonormal basis of $H_0^1(\Omega)$ for the inner product $a(.,.)$, we can write

$$||\phi||^2_{H_0^1(\Omega)} = \sum_{i=1}^{+\infty} \lambda_i (\phi, w_i)^2_{L^2(\Omega)}, \qquad \forall \phi \in H_0^1(\Omega). \tag{2.13}$$ H101

# 3 Approximate problem

`approx`

In this section, using eigenfunctions expansions of the Laplace operator, we prove the existence and uniqueness of solution to the approximate problem given by

$$\left\{\begin{array}{rcll} D^{\alpha}_{RL} y_\beta(x,t) - \Delta y_\beta(x,t) &=& f(x,t) & (x,t)\in Q,\\ y_\beta(\sigma,t) &=& 0 & (\sigma,t)\in\Sigma,\\ I^{1-\alpha}y_\beta(x,T) + \beta I^{1-\alpha} y_\beta(x,0^+) &=& y^1(x) & x\in\Omega, \end{array}\right. \tag{3.1}$$ `eqA1`

where $1/2<\alpha<1, f\in L^2(Q), y^1\in H^1_0(\Omega)$ and $I^{1-\alpha}y_\beta(x,0^+) = \lim_{t\downarrow 0} I^{1-\alpha}y_\beta(x,t)$.

Let us assume that (3.1) has a solution $y_\beta \in C^\infty(\bar{Q})$. If we multiply the first equation in (3.1) by a function $v\in H^1_0(\Omega)$ and integrate by parts over $\Omega$, we obtain

$$\int_\Omega D^{\alpha}_{RL} y_\beta(x,t) v(x)dx + \int_\Omega \nabla y_\beta(x,t)\cdot\nabla v(x)dx = \int_\Omega f(x,t)v(x)dx. \tag{3.2}$$ `chin`

Observing that $(D^{\alpha}_{RL}y_\beta(t),v) = D^{\alpha}_{RL}(y_\beta(t),v)$ and using (2.11), problem (3.1) becomes for all $t\in(0,T)$:

$$\left\{\begin{array}{rcll} D^{\alpha}_{RL}(y_\beta(t),v)_{L^2(\Omega)} + a(y_\beta(t),v) &=& (f(t),v)_{L^2(\Omega)} & \text{in } \Omega, \quad \forall v\in H^1_0(\Omega),\\ y_\beta(t) &=& 0 & \text{on } \partial\Omega,\\ I^{1-\alpha}y_\beta(x,T) + \beta I^{1-\alpha}y_\beta(x,0^+) &=& y^1 & \text{in } \Omega. \end{array}\right. \tag{3.3}$$ `eqB1`

`eq11` We can then consider the following problem : Given $1/2<\alpha<1, y^1\in H^1_0(\Omega)$ and $f\in L^2(Q)$, find

$$y_\beta \in L^2((0,T),H^1_0(\Omega)), \tag{3.4a}$$

$$I^{1-\alpha}y_\beta \in C([0,T];H^1_0(\Omega)), \tag{3.4b}$$

`diff1` such that

$$D^{\alpha}_{RL}(y_\beta(t),v)_{L^2(\Omega)} + a(y_\beta(t),v) = (f(t),v)_{L^2(\Omega)} \quad \forall t\in(0,T), \quad \forall v\in H^1_0(\Omega), \tag{3.5a}$$ `eq21`

$$I^{1-\alpha}y_\beta(T) + \beta I^{1-\alpha}y_\beta(0^+) = y^1 \quad \text{in } \Omega. \tag{3.5b}$$ `eq22`

In this context, the following existence and uniqueness theorems hold.

`existeyb` **Theorem 3.1** *Let $1/2<\alpha<1$ and $a(.,.)$ be the bilinear form defined by (2.11). Then, the approximate problem (3.4)-(3.5) has a unique solution $y_\beta\in L^2((0,T),H^1_0(\Omega))$ given by*

$$\begin{aligned} y_\beta(t) &= \sum_{i=1}^{+\infty}\left\{\frac{y^1_i - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du}{\beta + E_\alpha(-\lambda_i T^\alpha)} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)\right. \\ &+ \left. \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)f_i(s)ds\right\} w_i. \end{aligned} \tag{3.6}$$ `solution11`

*where $\lambda_i$ is the eigenvalue of the operator $-\Delta$ corresponding to the eigenfunction $w_i$. $E_{\alpha,\alpha}$ as given in (2.3), $y^1_i = (y^1,w_i)$ and $f_i(t) = (f(t),w_i)$ are respectively, the i-th component of $y^1$ and $f(t)$ in the* *orthonormal basis* $\{w_i\}_{i=1}^{\infty}$ *of* $L^2(\Omega)$. *Moreover,* $I^{1-\alpha}y_\beta \in C([0,T], H_0^1(\Omega))$ *and there exists a constant* $C > 0$ *such that,*

$$\|y_\beta\|_{L^2((0,T),H_0^1(\Omega))} \leq \Pi \left(\|y^1\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q)}\right), \tag{3.7}$$

estimation_yb

*and*

$$\left\|I^{1-\alpha}y_\beta\right\|_{C([0,T],H_0^1(\Omega))} \leq \Theta \left(\|y^1\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q)}\right), \tag{3.8}$$

estimation_Iyb

*where*

$$\Pi = \max\left(\frac{2C}{\beta}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}, \sqrt{\frac{2C^2T^\alpha}{\beta^2(1-\alpha)(2\alpha-1)} + \frac{4C^2T^\alpha}{\alpha-\frac{1}{2}}}\right)$$

*and*

$$\Theta = \sup\left(\frac{2C}{\beta}, \sqrt{\frac{2C^4T^{1-\alpha}}{\beta^2(1-\alpha)} + \frac{2CT^{1-\alpha}}{1-\alpha}}\right).$$

**Proof.** If we replace $v$ by $w_i$ in (3.5a) and use the fact that

$$a(y_\beta(t), w_i) = \lambda_i(y_\beta(t), w_i)_{L^2(\Omega)} = \lambda_i y_{\beta i},$$

we deduce from (3.5) that $y_{\beta i}$ is a solution of the ordinary differential equation

$$\left\{\begin{array}{rcl} D^\alpha_{RL}y_{\beta i}(t) + \lambda_i y_{\beta i}(t) &=& f_i(t), \quad t \in (0,T), \\ I^{1-\alpha}y_{\beta i}(T) + \beta I^{1-\alpha}y_{\beta i}(0^+) &=& y_i^1, \end{array}\right. \tag{3.9}$$

edo11

where $y_i^1 = (y^1, w_i)$.
Now, using the Laplace transform, we obtain from the first equation of (3.9) that,

$$\hat{D}^\alpha_{RL}y_{\beta i}(s) + \lambda_i \hat{y_{\beta i}}(s) = \hat{f}_i(s), \tag{3.10}$$

eqlaplace1

where

$$\begin{array}{rcl} \hat{D}^\alpha_{RL}y_{\beta i}(s) &=& \mathcal{L}(D^\alpha_{RL}y_{\beta i}(t))(s), \\ \hat{y_{\beta i}}(s) &=& \mathcal{L}(y_{\beta i}(t))(s), \\ \hat{f}_i(s) &=& \mathcal{L}(f_i(t))(s) \end{array}$$

and $\mathcal{L}$ denotes the Laplace transform operator.
From (2.5), we have

$$\hat{D}^\alpha_{RL}y_{\beta i}(s) = -I^{1-\alpha}y_{\beta i}(0^+) + s^\alpha \hat{y_{\beta i}}(s),$$

which, combining with (3.10), gives

$$-I^{1-\alpha}y_{\beta i}(0^+) + s^\alpha \hat{y_{\beta i}}(s) + \lambda_i \hat{y_{\beta i}}(s) = \hat{f}_i(s).$$

Hence,

$$\hat{y_{\beta i}}(s) = I^{1-\alpha}y_{\beta i}(0^+) \times \frac{1}{s^\alpha + \lambda_i} + \hat{f}_i(s) \times \frac{1}{s^\alpha + \lambda_i},$$

and it follows from (2.4) that

$$y_{\beta i}(t) = I^{1-\alpha}y_{\beta i}(0^+)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)f_i(s)ds.$$

Therefore,

$$\begin{aligned} I^{1-\alpha} y_{\beta i}(t) &= I^{1-\alpha}(I^{1-\alpha} y_{\beta i}(0) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha)) \\ &+ I^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds\right), \\ &= A + B \end{aligned}$$

where

$$A = I^{1-\alpha}(I^{1-\alpha} y_{\beta i}(0) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha)),$$

$$B = I^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds\right).$$

Let us now compute $A$ and $B$. We have,

$$\begin{aligned} A &= I^{1-\alpha}(I^{1-\alpha} y_{\beta i}(0) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha)) \\ &= I^{1-\alpha} y_{\beta i}(0) \times I^{1-\alpha}(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha)) \\ &= I^{1-\alpha} y_{\beta i}(0) \times \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i s^\alpha) ds\right) \\ &= \frac{I^{1-\alpha} y_{\beta i}(0)}{\Gamma(1-\alpha)} \sum_{k=0}^{+\infty} \frac{(-\lambda_i)^k}{\Gamma(\alpha k+\alpha)} \int_0^t (t-s)^{-\alpha} s^{\alpha-1} s^{\alpha k} ds \\ &= \frac{I^{1-\alpha} y_{\beta i}(0)}{\Gamma(1-\alpha)} \sum_{k=0}^{+\infty} \frac{(-\lambda_i)^k t^{\alpha k-1}}{\Gamma(\alpha k+\alpha)} \int_0^1 (1-u)^{-\alpha} u^{\alpha-1+\alpha k} t du \\ &= \frac{I^{1-\alpha} y_{\beta i}(0)}{\Gamma(1-\alpha)} \sum_{k=0}^{+\infty} \frac{(-\lambda_i)^k t^{\alpha k}}{\Gamma(\alpha k+\alpha)} B(1-\alpha, \alpha k+\alpha), \end{aligned}$$

which in view of (2.1) gives

$$A = I^{1-\alpha} y_{\beta i}(0) \sum_{k=0}^{+\infty} \frac{(-\lambda_i t^\alpha)^k}{\Gamma(\alpha k+1)} = I^{1-\alpha} y_{\beta i}(0) E_\alpha(-\lambda_i t^\alpha). \tag{3.11}$$

CalA

On the other hand,

$$\begin{aligned}
B &= I^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^{\alpha})f_i(s)ds\right)\\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\left(\int_0^s (s-u)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(s-u)^{\alpha})f_i(u)du\right)ds\\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t f_i(u)\left(\int_u^t (t-s)^{-\alpha}(s-u)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(s-u)^{\alpha})ds\right)du\\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t f_i(u)\sum_{k=0}^{+\infty}\frac{(-\lambda_i)^k}{\Gamma(\alpha k+\alpha)}\left(\int_u^t (t-s)^{-\alpha}(s-u)^{\alpha-1+\alpha k}ds\right)du\\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t f_i(u)\sum_{k=0}^{+\infty}\frac{(-\lambda_i)^k(t-u)^{\alpha k}}{\Gamma(\alpha k+\alpha)}\left(\int_0^1 (1-z)^{-\alpha}z^{\alpha-1+\alpha k}dz\right)du\\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t f_i(u)\left(\sum_{k=0}^{+\infty}\frac{(-\lambda_i)^k(t-u)^{\alpha k}}{\Gamma(\alpha k+\alpha)}B(1-\alpha,\alpha k+\alpha)\right)du\\
&= \int_0^t f_i(u)\left(\sum_{k=0}^{+\infty}\frac{(-\lambda_i(t-u)^{\alpha})^k}{\Gamma(\alpha k+1)}\right)du = \int_0^t f_i(u)E_{\alpha}(-\lambda_i(t-u)^{\alpha})du.
\end{aligned}$$

Thus,

$$B = \int_0^t f_i(u)E_{\alpha}(-\lambda_i(t-u)^{\alpha})du \tag{3.12}$$

CalB

Finally, adding (3.11) to (3.12), we obtain

$$I^{1-\alpha}y_{\beta i}(t) = I^{1-\alpha}y_{\beta i}(0)E_{\alpha}(-\lambda_i t^{\alpha}) + \int_0^t f_i(u)E_{\alpha}(-\lambda_i(t-u)^{\alpha})du. \tag{3.13}$$

I1-a

Hence,

$$I^{1-\alpha}y_{\beta i}(T) = I^{1-\alpha}y_{\beta i}(0)E_{\alpha}(-\lambda_i T^{\alpha}) + \int_0^T f_i(u)E_{\alpha}(-\lambda_i(T-u)^{\alpha})du.$$

From $(3.9)_2$, we have that,

$$I^{1-\alpha}y_{\beta i}(0)E_{\alpha}(-\lambda_i T^{\alpha}) + \int_0^T f_i(u)E_{\alpha}(-\lambda_i(T-u)^{\alpha})du + \beta I^{1-\alpha}y_{\beta i}(0) = y_i^1,$$

from which, we deduce that

$$I^{1-\alpha}y_{\beta i}(0) = \frac{y_i^1 - \displaystyle\int_0^T E_{\alpha}(-\lambda_i(T-u)^{\alpha})f_i(u)du}{\beta + E_{\alpha}(-\lambda_i T^{\alpha})}. \tag{3.14}$$

Iyb0

Finally, we obtain

$$y_{\beta i}(t) = \left[\frac{y_i^1 - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du}{\beta + E_\alpha(-\lambda_i T^\alpha)}\right] t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)$$
$$+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)f_i(s)ds.$$

The rest of proof can be done in three steps.

**Step 1**: We give the formulation of a solution to an approximate problem associated to Equations (3.4)-(3.5).
Let $V_m$ be a subspace of $H_0^1(\Omega)$ generated by the $w_1, w_2, \ldots, w_m$.
Consider the following approximate problem associated to Equations $(3.4) - (3.5)$:
Find $y_{\beta m} : t \in (0,T] \to y_{\beta m}(t) \in V_m$, the solution of

$$D_{RL}^\alpha (y_{\beta m}(t), v)_{L^2(\Omega)} + a(y_{\beta m}(t), v) = (f(t), v)_{L^2(\Omega)}, \forall v \in V_m, \tag{3.15}$$ eq111

$$I^{1-\alpha}y_{\beta m}(T) + \beta I^{1-\alpha}y_{\beta m}(0) = y_m^1 = \sum_{i=1}^m y_{\beta i}^{\ 1} w_i. \tag{3.16}$$ eq121

As $y_{\beta m}(t) \in V_m$, we have

$$y_{\beta m}(t) = \sum_{i=1}^m (y(t), w_i)_{L^2(\Omega)} w_i = \sum_{i=1}^m y_{\beta i}(t) w_i.$$

Proceeding as per the computation of $y_\beta$, we show that $y_{\beta m}$ is a solution of the problem given by Equations $(3.15) - (3.16)$ and obtain,

$$y_{\beta m}(t) = \sum_{i=1}^m \left\{ \frac{y_i^1 - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du}{\beta + E_\alpha(-\lambda_i T^\alpha)} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)\right\} w_i$$
$$+ \sum_{i=1}^m \left\{ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)f_i(s)ds \right\} w_i. \tag{3.17}$$ solm1

**Step 2:** We show that the sequences $(y_{\beta m})$ and $(I^{1-\alpha}y_{\beta m})$ are respectively, Cauchy sequences in $L^2((0,T); H_0^1(\Omega))$ and $C([0,T]; H_0^1(\Omega))$.
Let $m$ and $p$ be two integers such that $p > m \geq 1$. Then, from (3.17)

$$y_{\beta p}(t) - y_{\beta m}(t) = \sum_{i=m+1}^p y_{\beta i}(t) w_i.$$

Set $b_i = \dfrac{y_i^1 - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha) f_i(u)du}{\beta + E_\alpha(-\lambda_i T^\alpha)}$. Then, we have that,

$$\begin{aligned}
a(y_{\beta p}(t) - y_{\beta m}(t), y_{\beta p}(t) - y_{\beta m}(t)) &= \sum_{i=m+1}^{p} \lambda_i [y_{\beta i}(t)]^2 \\
&\leq 2 \sum_{i=m+1}^{p} \lambda_i t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha) |b_i|^2 \\
&+ 2 \sum_{i=m+1}^{p} \lambda_i \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds \right\}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|y_{\beta p}(t) - y_{\beta m}(t)\|^2_{L^2((0,T);H_0^1(\Omega))} &= \int_0^T a(y_{\beta p}(t) - y_{\beta m}(t), y_{\beta p}(t) - y_{\beta m}(t)) dt \\
&\leq A_p + B_p,
\end{aligned}$$

with

$$\begin{aligned}
A_p &= 2 \sum_{i=m+1}^{p} \lambda_i |b_i|^2 \int_0^T t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha) dt, \\
B_p &= 2 \sum_{i=m+1}^{p} \int_0^T \lambda_i \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds \right\}^2 dt.
\end{aligned}$$

Note that from Theorem 2.1, we know that there exists a generic constant $C > 0$ such that

$$\begin{aligned}
A_p &= 2 \sum_{i=m+1}^{p} \lambda_i |b_i|^2 \int_0^T t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha) dt \\
&\leq 2C^2 \sum_{i=m+1}^{p} \lambda_i |b_i|^2 \int_0^T t^{2\alpha-2} dt \\
&\leq 2C^2 \sum_{i=m+1}^{p} \lambda_i |b_i|^2 \left[ \frac{t^{2\alpha-1}}{2\alpha-1} \right]_0^T \quad \leq \quad \frac{2C^2 T^{2\alpha-1}}{2\alpha-1} \sum_{i=m+1}^{p} \lambda_i |b_i|^2.
\end{aligned}$$

**Remark 3.1** *From the latter estimation, we see that we have to take* $1/2 < \alpha < 1$ *to give a sense to our computation.*

Using again Theorem 2.1 and noting that $1-\alpha \neq 0$, we obtain

$$\begin{aligned}
\sum_{i=m+1}^{p} \lambda_i |b_i|^2 &= \sum_{i=m+1}^{p} \lambda_i \left| \frac{y_i^1 - \displaystyle\int_0^T E_\alpha(-\lambda_i (T-u)^\alpha) f_i(u) du}{\beta + E_\alpha(-\lambda_i T^\alpha)} \right|^2 \\
&\leq \frac{1}{\beta^2} \sum_{i=m+1}^{p} \lambda_i \left| y_i^1 - \int_0^T E_\alpha(-\lambda_i (T-u)^\alpha) f_i(u) du \right|^2 \\
&\leq \frac{2}{\beta^2} \sum_{i=m+1}^{p} \lambda_i |y_i^1|^2 + \frac{2}{\beta^2} \sum_{i=m+1}^{p} \lambda_i \left| \int_0^T E_\alpha(-\lambda_i (T-u)^\alpha) f_i(u) du \right|^2 \\
&\leq \frac{2}{\beta^2} \sum_{i=m+1}^{p} \lambda_i |y_i^1|^2 + \frac{2C^2}{\beta^2} \left[ \frac{-(T-u)^{1-\alpha}}{1-\alpha} \right]_0^T \sum_{i=m+1}^{p} \int_0^T |f_i(u)|^2 du.
\end{aligned}$$

Consequently,

$$\sum_{i=m+1}^{p} \lambda_i |b_i|^2 \leq \frac{2}{\beta^2} \sum_{i=m+1}^{p} \lambda_i |y_i^1|^2 + \frac{2C^2 T^{1-\alpha}}{\beta^2 (1-\alpha)} \sum_{i=m+1}^{p} \int_0^T |f_i(u)|^2 du. \tag{3.18}$$

sum_bi

and we have that

$$A_p \leq \frac{4C^2 T^{2\alpha-1}}{\beta^2 (2\alpha-1)} \sum_{i=m+1}^{p} \lambda_i |y_i^1|^2 + \frac{4C^4 T^\alpha}{\beta^2 (1-\alpha)(2\alpha-1)} \left[ \sum_{i=m+1}^{p} \int_0^T |f_i(u)|^2 du \right]. \tag{3.19}$$

CalAP

On the other hand, using the Cauchy-Schwartz inequality,

$$\begin{aligned}
B_p &= 2 \sum_{i=m+1}^{p} \int_0^T \lambda_i \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds \right\}^2 dt \\
&= 2 \sum_{i=m+1}^{p} \int_0^T \lambda_i \left\{ \int_0^t \left[ (t-s)^{\frac{\alpha}{2}-\frac{1}{4}} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) \right] \left[ (t-s)^{\frac{\alpha}{2}-\frac{3}{4}} f_i(s) \right] ds \right\}^2 dt \\
&\leq 2 \sum_{i=m+1}^{p} \int_0^T \lambda_i \left[ \int_0^t (t-s)^{\alpha-\frac{1}{2}} E_{\alpha,\alpha}^2(-\lambda_i (t-s)^\alpha) ds \right] \left[ \int_0^t (t-s)^{\alpha-\frac{3}{2}} |f_i(s)|^2 ds \right] dt,
\end{aligned}$$

which in view of Theorem 2.1 gives

$$\begin{aligned} B_p &\leq 2C^2 \sum_{i=m+1}^{p} \int_0^T \left[\int_0^t (t-s)^{-1/2} ds\right] \left[\int_0^t (t-s)^{\alpha-\frac{3}{2}} |f_i(s)|^2 ds\right] dt \\ &\leq 4C^2 T^{1/2} \sum_{i=m+1}^{p} \int_0^T \int_0^t (t-s)^{\alpha-\frac{3}{2}} |f_i(s)|^2 ds dt \\ &= 4C^2 T^{1/2} \sum_{i=m+1}^{p} \int_0^T |f_i(s)|^2 \int_s^T (t-s)^{\alpha-\frac{3}{2}} dt ds \\ &\leq \frac{4C^2 T^\alpha}{\alpha - \frac{1}{2}} \sum_{i=m+1}^{p} \int_0^T |f_i(s)|^2 ds. \end{aligned}$$

Thus,

$$B_p \leq \frac{4C^2 T^\alpha}{\alpha - \frac{1}{2}} \sum_{i=m+1}^{p} \int_0^T |f_i(s)|^2 ds \qquad (3.20) \quad \boxed{\texttt{CalBP}}$$

Adding (3.19) to (3.20), we obtain

$$\begin{aligned} \|y_{\beta p}(t) - y_{\beta m}(t)\|^2_{L^2((0,T);H_0^1(\Omega))} &\leq A_p + B_p \\ &\leq \frac{4C^2 T^{2\alpha-1}}{\beta^2(2\alpha-1)} \sum_{i=m+1}^{p} \lambda_i |y_i^1|^2 \\ &+ \left[\frac{4C^2 T^\alpha}{\beta^2(1-\alpha)(2\alpha-1)} + \frac{4C^2 T^\alpha}{\alpha - \frac{1}{2}}\right] \left[\sum_{i=m+1}^{p} \int_0^T |f_i(s)|^2 ds\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \|y_{\beta p}(t) - y_{\beta m}(t)\|_{L^2((0,T);H_0^1(\Omega))} &\leq \frac{2C}{\beta} \sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}} \left(\sum_{i=m+1}^{p} \lambda_i |y_i^1|^2\right)^{1/2} \\ &+ \sqrt{\frac{4C^2 T^\alpha}{\beta^2(1-\alpha)(2\alpha-1)} + \frac{4C^2 T^\alpha}{\alpha - \frac{1}{2}}} \left(\sum_{i=m+1}^{p} \int_0^T |f_i(s)|^2 ds\right)^{1/2}. \end{aligned} \qquad (3.21) \quad \boxed{\texttt{normyb}}$$

In view of Equation (3.13), we have

$$I^{1-\alpha}(y_{\beta p}(t) - y_{\beta m}(t)) = \sum_{i=m+1}^{p} |b_i| E_{\alpha,1}(-\lambda_i t^\alpha) w_i + \sum_{i=m+1}^{p} \left\{\int_0^t f_i(u) E_{\alpha,1}(-\lambda_i (t-u)^\alpha) du\right\} w_i,$$

from which we deduce that,

$$\begin{aligned}
\|I^{1-\alpha}(y_{\beta p}(t)-y_{\beta m}(t))\|^2_{H^1_0(\Omega)} &\le a(I^{1-\alpha}(y_{\beta p}(t)-y_{\beta m}(t)), I^{1-\alpha}(y_{\beta p}(t)-y_{\beta m}(t)))\\
&\le 2\sum_{i=m+1}^{p}\lambda_i|b_i|^2E^2_{\alpha,1}(-\lambda_i t^\alpha)\\
&+ 2\sum_{i=m+1}^{p}\lambda_i\left\{\int_0^t f_i(u)E_{\alpha,1}(-\lambda_i(t-u)^\alpha)du\right\}^2.
\end{aligned}$$

If we set

$$\begin{aligned}
C_p &= 2\sum_{i=m+1}^{p}\lambda_i|b_i|^2E^2_{\alpha,1}(-\lambda_i t^\alpha),\\
Z_p &= 2\sum_{i=m+1}^{p}\lambda_i\left\{\int_0^t f_i(u)E_{\alpha,1}(-\lambda_i(t-u)^\alpha)du\right\}^2,
\end{aligned}$$

we have from Theorem 2.1, (3.18) and the Cauchy-Schwartz inequality that,

$$\begin{aligned}
C_p &\le 2C^2\sum_{i=m+1}^{p}\lambda_i|b_i|^2\\
&\le 2C^2\left(\frac{2}{\beta^2}\sum_{i=m+1}^{p}\lambda_i|y_i^1|^2+\frac{2C^2T^{1-\alpha}}{\beta^2(1-\alpha)}\sum_{i=m+1}^{p}\int_0^T|f_i(u)|^2du\right)\\
&\le \frac{4C^2}{\beta^2}\sum_{i=m+1}^{p}\lambda_i|y_i^1|^2+\frac{4C^4T^{1-\alpha}}{\beta^2(1-\alpha)}\left(\sum_{i=m+1}^{p}\int_0^T|f_i(u)|^2du\right)
\end{aligned}$$

and

$$\begin{aligned}
Z_p &\le 2\sum_{i=m+1}^{p}\lambda_i\left(\int_0^t E^2_{\alpha,1}(-\lambda_i(t-u)^\alpha)du\right)\left(\int_0^t|f_i(u)|^2du\right)\\
&\le 2C\sum_{i=m+1}^{p}\left(\int_0^t(t-u)^{-\alpha}du\right)\left(\int_0^t|f_i(u)|^2du\right)\\
&\le \frac{2Ct^{1-\alpha}}{1-\alpha}\sum_{i=m+1}^{p}\left(\int_0^t|f_i(u)|^2du\right).
\end{aligned}$$

Using the estimations of $C_p$ and $Z_p$, we obtain

$$\begin{aligned}\|I^{1-\alpha}(y_{\beta p}(t)-y_{\beta m}(t))\|^2_{H^1_0(\Omega)} &\leq \frac{4C^2}{\beta^2}\sum_{i=m+1}^{p}\lambda_i|y^1_i|^2 \\ &+ \left(\frac{4C^4T^{1-\alpha}}{\beta^2(1-\alpha)}+\frac{2Ct^{1-\alpha}}{1-\alpha}\right)\left(\sum_{i=m+1}^{p}\int_0^t|f_i(u)|^2du\right).\end{aligned}$$

Thus,

$$\begin{aligned}\sup_{t\in[0,T]}\|I^{1-\alpha}(y_{\beta p}(t)-y_{\beta m}(t))\|_{H^1_0(\Omega)} &\leq \frac{2C}{\beta}\left(\sum_{i=m+1}^{p}\lambda_i|y^1_i|^2\right)^{1/2} \\ &+ \sqrt{\frac{4C^4T^{1-\alpha}}{\beta^2(1-\alpha)}+\frac{2CT^{1-\alpha}}{1-\alpha}}\left(\sum_{i=m+1}^{p}\int_0^t|f_i(u)|^2du\right)^{1/2}.\end{aligned} \tag{3.22}$$

supI

As $y^1\in H^1_0(\Omega)$ and $f\in L^2(Q)$,

$$\lim_{m,p\to+\infty}\left(\sum_{i=m+1}^{p}\lambda_i|y^1_i|^2\right)^{1/2} = \lim_{m,p\to+\infty}\left(\sum_{i=m+1}^{p}\int_0^t|f_i(u)|^2du\right)^{1/2} = 0.$$

Then, from Equation (3.21) and Equation (3.22), we obtain

$$\lim_{m,p\to+\infty}\int_0^T\|y_{\beta p}(t)-y_{\beta m}(t)\|^2_{H^1_0(\Omega)}dt=0$$

and

$$\sup_{t\in[0,T]}\|I^{1-\alpha}(y_{\beta p}(t)-y_{\beta m}(t))\|_{H^1_0(\Omega)}=0.$$

Consequently, $(y_{\beta m})$ and $(I^{1-\alpha}y_{\beta m})$ are Cauchy sequences in $L^2((0,T);H^1_0(\Omega))$ and $C([0,T],H^1_0(\Omega))$ respectively. This implies that

$$y_{\beta m}\to y_\beta \quad \text{in } L^2((0,T);H^1_0(\Omega)), \tag{3.23}$$

limy1

and

$$I^{1-\alpha}y_{\beta m}\to\xi \quad \text{in } C([0,T];H^1_0(\Omega)).$$

Since $y_\beta\in L^2((0,T),H^1_0(\Omega))$ and $I^{1-\alpha}y_\beta$ are continuous, we have $\xi=I^{1-\alpha}y_\beta$ and

$$I^{1-\alpha}y_{\beta m}\to I^{1-\alpha}y_\beta \quad \text{in } C([0,T];H^1_0(\Omega)). \tag{3.24}$$

limI1

**Step 3:** We show that $y_\beta$ satisfies (3.4) – (3.5).

Let $\varphi\in\mathbb{D}(0,T)$ and $\mu\geq 1$ an integer. Then, from (3.15), we have for all $m\geq\mu$,

$$\begin{aligned}\int_0^T(f(t),v)_{L^2(\Omega)}\varphi(t)dt &= \int_0^T D^\alpha_{RL}(y_{\beta m}(t),v)_{L^2(\Omega)}\varphi(t)dt \\ &+ \int_0^T a(y_{\beta m}(t),v)\varphi(t)dt, \quad \forall v\in V_\mu,\end{aligned}$$

which according to Corollary 2.1 implies that,

$$\begin{aligned}\int_0^T (f(t),v)_{L^2(\Omega)}\varphi(t)dt &= -\int_0^T (y_{\beta m}(t),v)_{L^2(\Omega)}\mathcal{D}_C^\alpha\varphi(t)dt \\ &+ \int_0^T a(y_{\beta m}(t),v)\varphi(t)dt, \quad \forall v\in V_\mu.\end{aligned}$$

Therefore, passing to the limit and using (3.23), we obtain

$$\begin{aligned}\int_0^T (f(t),v)_{L^2(\Omega)}\varphi(t)dt &= -\int_0^T (y_{\beta}(t),v)_{L^2(\Omega)}\mathcal{D}_C^\alpha\varphi(t)dt \\ &+ \int_0^T a(y_{\beta}(t),v)\varphi(t)dt, \quad \forall v\in V_\mu.\end{aligned}$$

Since $\cup_{\mu\geq 1}V_\mu$ is dense in $H_0^1(\Omega)$ because $(w_i)$ is a base of $H_0^1(\Omega)$, we have for all $v\in H_0^1(\Omega)$ that

$$\begin{aligned}\int_0^T (f(t),v)_{L^2(\Omega)}\varphi(t)dt &= -\int_0^T (y_{\beta}(t),v)_{L^2(\Omega)}\mathcal{D}_C^\alpha\varphi(t)dt \\ &+ \int_0^T a(y_{\beta}(t),v)\varphi(t)dt, \quad \forall v\in H_0^1(\Omega).\end{aligned}$$

Using, once again, Corollary 2.1, we can write

$$\begin{aligned}\int_0^T (f(t),v)_{L^2(\Omega)}\varphi(t)dt &= \int_0^T D_{RL}^\alpha(y_{\beta}(t),v)_{L^2(\Omega)}\varphi(t)dt \\ &+ \int_0^T a(y_{\beta}(t),v)\varphi(t)dt, \quad \forall v\in H_0^1(\Omega).\end{aligned}$$

This implies that for all $v\in H_0^1(\Omega)$,

$$(f(t),v)_{L^2(\Omega)}\varphi(t) = D_{RL}^\alpha(y_\beta(t),v)_{L^2(\Omega)}\varphi(t) + a(y_\beta(t),v)\varphi(t), \quad \forall t\in(0,T).$$

From (3.24), we have

$$I^{1-\alpha}y_{\beta m}(0)\to I^{1-\alpha}y_\beta(0) \quad \text{in } H_0^1(\Omega),$$

and

$$I^{1-\alpha}y_{\beta m}(T)\to I^{1-\alpha}y_\beta(T) \quad \text{in } H_0^1(\Omega).$$

But

$$I^{1-\alpha}y_{\beta m}(T)+\beta I^{1-\alpha}y_{\beta m}(0) = \sum_{i=1}^m y_i^1 w_i \to \sum_{i=1}^{+\infty} y_i^1 w_i = y^1.$$

Thus,

$$I^{1-\alpha}y_\beta(T)+\beta I^{1-\alpha}y_\beta(0) = y^1.$$

To complete the proof of Theorem 3.1, we need to prove Equation (3.7) and Equation (3.8). Since $y_\beta$ is the solution of (3.4)-(3.5), we have

$$\begin{aligned}y_\beta(t) &= \sum_{i=1}^{+\infty}\left(\frac{y_i^1 - \int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du}{\beta+E_\alpha(-\lambda_i T^\alpha)}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)\right. \\ &+ \left.\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)f_i(s)ds\right)w_i.\end{aligned}$$

Proceeding as above for estimations on $y_{\beta m}$, we can prove that there exists a constant $C > 0$ such that

$$\begin{aligned}\|y_\beta(t)\|_{L^2((0,T);H_0^1(\Omega))} &\leq \frac{2C}{\beta}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\left(\sum_{i=1}^{+\infty}\lambda_i|y_i^1|^2\right)^{1/2} \\ &+ \sqrt{\frac{4C^2T^\alpha}{\beta^2(1-\alpha)(2\alpha-1)}+\frac{4C^2T^\alpha}{\alpha-\frac{1}{2}}}\left(\sum_{i=1}^{+\infty}\int_0^T|f_i(s)|^2ds\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in[0,T]}\left\|I^{1-\alpha}y_\beta(t)\right\|_{H_0^1(\Omega)} &\leq \frac{2C}{\beta}\left(\sum_{i=1}^{+\infty}\lambda_i|y_i^1|^2\right)^{1/2} \\ &+ \sqrt{\frac{4C^4T^{1-\alpha}}{\beta^2(1-\alpha)}+\frac{2CT^{1-\alpha}}{1-\alpha}}\left(\sum_{i=1}^{+\infty}\int_0^T|f_i(s)|^2ds\right)^{1/2},\end{aligned}$$

from which we deduce, respectively, Equation (3.7) and Equation (3.8). ■

# 4 Convergence results

convergence

In this section we provide some convergence results.

conv_I_y1

**Theorem 4.1** *For all $y^1 \in H_0^1(\Omega)$, we have*

$$\lim_{\beta\to 0}\|I^{1-\alpha}y_\beta(T)-y^1\| = 0.$$

*That is $I^{1-\alpha}y_\beta(T)$ converges to $y^1$ in $H_0^1(\Omega)$.*

**Proof.** Since $y^1 \in H_0^1(\Omega)$, we know that

$$\forall\epsilon>0,\ \exists N_\epsilon\in\mathbb{N} \text{ such that } \sum_{i=N_\epsilon+1}^{+\infty}\lambda_i|y_i^1|^2 < \frac{\epsilon}{2}.$$

Also, since $f \in L^2(Q)$, we know that

$$\forall\epsilon>0,\ \exists N_\epsilon\in\mathbb{N} \text{ such that } \sum_{i=N_\epsilon+1}^{+\infty}\int_0^T|f_i(s)|^2ds < \frac{\epsilon}{2}.$$

Let $\epsilon > 0$ and choose $N > 0$ such that

$$\sum_{i=N+1}^{+\infty}\lambda_i|y_i^1|^2 < \frac{\epsilon}{2} \quad\text{and}\quad \sum_{i=N+1}^{+\infty}\int_0^T|f_i(s)|^2ds < \frac{\epsilon}{2}.$$

Then, we have

$$\begin{aligned}\|I^{1-\alpha}y_\beta(T)-y^1\|^2_{H_0^1(\Omega)} &= a\left(-\beta I^{1-\alpha}y_\beta(0),-\beta I^{1-\alpha}y_\beta(0)\right)\\ &= \beta^2\sum_{i=1}^{+\infty}\lambda_i|b_i|^2\\ &\leq A+B,\end{aligned}$$

where

$$A=2\beta^2\sum_{i=1}^{+\infty}\frac{\lambda_i|y_i^1|^2}{(\beta+E_\alpha(-\lambda_iT^\alpha))^2}.$$

$$B=2\beta^2\sum_{i=1}^{+\infty}\frac{\lambda_i}{(\beta+E_\alpha(-\lambda_iT^\alpha))^2}\left(\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du\right)^2.$$

We firstly have,

$$\begin{aligned}A &= 2\beta^2\sum_{i=1}^{N}\frac{\lambda_i|y_i^1|^2}{(\beta+E_\alpha(-\lambda_iT^\alpha))^2}+2\beta^2\sum_{i=N+1}^{+\infty}\frac{\lambda_i|y_i^1|^2}{(\beta+E_\alpha(-\lambda_iT^\alpha))^2}\\ &\leq \beta^2\sum_{i=1}^{N}\frac{2\lambda_i|y_i^1|^2}{E_\alpha^2(-\lambda_iT^\alpha)}+2\sum_{i=N+1}^{+\infty}\lambda_i|y_i^1|^2\\ &\leq \beta^2\sum_{i=1}^{N}\frac{2\lambda_i|y_i^1|^2}{E_\alpha^2(-\lambda_iT^\alpha)}+\epsilon.\end{aligned}$$

Secondly, using the Cauchy-Schwartz inequality, we can write

$$\begin{aligned}B &= 2\beta^2\sum_{i=1}^{+\infty}\frac{\lambda_i}{(\beta+E_\alpha(-\lambda_iT^\alpha))^2}\left(\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du\right)^2\\ &\leq 2\beta^2\sum_{i=1}^{+\infty}\frac{C^2T^{1-\alpha}}{(1-\alpha)(\beta+E_\alpha(-\lambda_iT^\alpha))^2}\left(\int_0^T|f_i(u)|^2du\right)\\ &\leq \beta^2\sum_{i=1}^{N}\frac{2C^2T^{1-\alpha}}{(1-\alpha)E_\alpha^2(-\lambda_iT^\alpha)}\left(\int_0^T|f_i(u)|^2du\right)+\frac{C^2T^{1-\alpha}}{(1-\alpha)}\epsilon.\end{aligned}$$

Finally, using the estimations of $A$ and $B$, we have

$$\begin{aligned}\|I^{1-\alpha}y_\beta(T)-y^1\|^2_{H_0^1(\Omega)} &\leq \beta^2\left[\sum_{i=1}^{N}\frac{2\lambda_i|y_i^1|^2}{E_\alpha^2(-\lambda_iT^\alpha)}+\sum_{i=1}^{N}\frac{2C^2T^{1-\alpha}}{(1-\alpha)E_\alpha^2(-\lambda_iT^\alpha)}\left(\int_0^T|f_i(u)|^2du\right)\right]\\ &+ \left(1+\frac{C^2T^{1-\alpha}}{(1-\alpha)}\right)\epsilon.\end{aligned}$$

Since

$$\sum_{i=1}^{N} \frac{2\lambda_i |y_i^1|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + \sum_{i=1}^{N} \frac{2C^2 T^{1-\alpha}}{(1-\alpha) E_\alpha^2(-\lambda_i T^\alpha)} \left( \int_0^T |f_i(u)|^2 du \right) < \infty,$$

we choose $\beta$ such that

$$\beta^2 < \epsilon \left[ \sum_{i=1}^{N} \frac{2\lambda_i |y_i^1|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + \sum_{i=1}^{N} \frac{2C^2 T^{1-\alpha}}{(1-\alpha) E_\alpha^2(-\lambda_i T^\alpha)} \left( \int_0^T |f_i(u)|^2 du \right) \right]^{-1}.$$

■

**Theorem 4.2** *Supposing there exists* $\epsilon \in (0,2)$ *such that*

$$D = 2\sum_{i=1}^{+\infty} \frac{\lambda_i |y_i^1|^2}{E_\alpha^\epsilon(-\lambda_i T^\alpha)} + \frac{2C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^{+\infty} \frac{\int_0^T |f_i(s)|^2 ds}{E_\alpha^\epsilon(-\lambda_i T^\alpha)}$$

*converges, then* $\|I^{1-\alpha} y_\beta - y^1\|_{H_0^1(\Omega)}$ *converges to zero with order* $\epsilon^{-2}\beta^\epsilon$.

**Proof.** Let $\epsilon \in (0,2)$ such that $D$ converges and $k \in (0,2)$. We fix a natural integer $i$, and define

$$g_i(\beta) = \frac{\beta^k}{[\beta + E_\alpha(-\lambda_i T^\alpha)]^2}.$$

Differentiating $g_i$ with respect to $\beta$, we obtain

$$\begin{aligned} g_i'(\beta) &= \frac{(k-2)\beta^k + k\beta^{k-1} E_\alpha(-\lambda_i T^\alpha)}{[\beta + E_\alpha(-\lambda_i T^\alpha)]^3} \\ &= \beta^{k-1} \times \frac{(k-2)\beta + k E_\alpha(-\lambda_i T^\alpha)}{[\beta + E_\alpha(-\lambda_i T^\alpha)]^3}. \end{aligned}$$

Observing that $g_i'(\beta) = 0$ if $\beta = 0$ or $(k-2)\beta + kE_\alpha(-\lambda_i T^\alpha) = 0$. We have

$$(k-2)\beta + kE_\alpha(-\lambda_i T^\alpha) = 0 \quad \Leftrightarrow \quad \beta = \frac{k}{2-k} E_\alpha(-\lambda_i T^\alpha).$$

As $g_i(\beta) > 0$, $g_i(0) = 0$ and $\lim_{\beta \to +\infty} g_i(\beta) = 0$. Indeed,

$$\lim_{\beta \to +\infty} g_i(\beta) = \lim_{\beta \to +\infty} \frac{\beta^k}{\beta^2} = \lim_{\beta \to +\infty} \frac{1}{\beta^{2-k}} = 0.$$

We know that $g_i$ achieves its maximum at $\beta_0 = \dfrac{k}{2-k} E_\alpha(-\lambda_i T^\alpha)$. Hence, we have,

$$
\begin{aligned}
g_i(\beta) \le g_i(\beta_0) \quad \Leftrightarrow \quad g_i(\beta) &\le \frac{(\beta_0)^k}{[\beta_0 + E_\alpha(-\lambda_i T^\alpha)]^2} \\
\Leftrightarrow \quad g_i(\beta) &\le \frac{\left(\dfrac{k}{2-k}\right)^k E_\alpha^k(-\lambda_i T^\alpha)}{[\beta_0 + E_\alpha(-\lambda_i T^\alpha)]^2} \\
\Leftrightarrow \quad g_i(\beta) &\le \left(\frac{k}{2-k}\right)^k E_\alpha^{k-2}(-\lambda_i T^\alpha).
\end{aligned}
$$

Since we can write

$$
\begin{aligned}
\|I^{1-\alpha} y_\beta - y^1\|^2_{H_0^1(\Omega)} &\le 2\beta^2 \sum_{i=1}^{+\infty} \frac{\lambda_i |y_i^1|^2}{(\beta + E_\alpha(-\lambda_i T^\alpha))^2} \\
&+ 2\beta^2 \sum_{i=1}^{+\infty} \frac{\lambda_i}{(\beta + E_\alpha(-\lambda_i T^\alpha))^2} \left( \int_0^T E_\alpha(-\lambda_i (T-u)^\alpha) f_i(u) du \right)^2 \\
&= 2\beta^{2-k} \sum_{i=1}^{+\infty} \lambda_i |y_i^1|^2 g_i(\beta) + 2\beta^{2-k} \sum_{i=1}^{+\infty} \lambda_i \left( \int_0^T E_\alpha(-\lambda_i (T-u)^\alpha) f_i(u) du \right)^2 g_i(\beta),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\|I^{1-\alpha} y_\beta - y^1\|^2_{H_0^1(\Omega)} &\le 2\beta^{2-k} \sum_{i=1}^{+\infty} \lambda_i |y_i^1|^2 g_i(\beta) + 2\beta^{2-k} \frac{C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^{+\infty} \left( \int_0^T |f_i(u)|^2 du \right) g_i(\beta) \\
&\le \beta^{2-k} \left(\frac{k}{2-k}\right)^k \left[ 2 \sum_{i=1}^{+\infty} \lambda_i |y_i^1|^2 E_\alpha^{k-2}(-\lambda_i T^\alpha) \right. \\
&+ \left. \frac{2C^2 T^{1-\alpha}}{1-\alpha} \sum_{i=1}^{+\infty} \left( \int_0^T |f_i(u)|^2 du \right) E_\alpha^{k-2}(-\lambda_i T^\alpha) \right].
\end{aligned}
$$

If we choose, $k = 2-\epsilon$ (then $\epsilon = 2-k$), we then obtain

$$\begin{aligned}
\|I^{1-\alpha}y_\beta - y^1\|^2_{H^1_0(\Omega)} &\leq \beta^\epsilon \left(\frac{2-\epsilon}{\epsilon}\right)^{2-\epsilon} \left[2\sum_{i=1}^{+\infty} \lambda_i |y^1_i|^2 E^{-\epsilon}_\alpha(-\lambda_i T^\alpha)\right.\\
&+ \left.\frac{2C^2T^{1-\alpha}}{1-\alpha}\sum_{i=1}^{+\infty}\left(\int_0^T |f_i(u)|^2du\right) E^{-\epsilon}_\alpha(-\lambda_i T^\alpha)\right]\\
&\leq \beta^\epsilon \left(\frac{2}{\epsilon}\right)^{2} \left[2\sum_{i=1}^{+\infty} \lambda_i |y^1_i|^2 E^{-\epsilon}_\alpha(-\lambda_i T^\alpha)\right.\\
&+ \left.\frac{2C^2T^{1-\alpha}}{1-\alpha}\sum_{i=1}^{+\infty}\left(\int_0^T |f_i(u)|^2du\right) E^{-\epsilon}_\alpha(-\lambda_i T^\alpha)\right].
\end{aligned}$$

Since $D$ converges, there exists a constant $K > 0$ such that

$$2\sum_{i=1}^{+\infty} \lambda_i |y^1_i|^2 E^{-\epsilon}_\alpha(-\lambda_i T^\alpha) + \frac{2C^2T^{1-\alpha}}{1-\alpha}\sum_{i=1}^{+\infty}\left(\int_0^T |f_i(u)|^2du\right) E^{-\epsilon}_\alpha(-\lambda_i T^\alpha) < K.$$

which implies that

$$\begin{aligned}
\|I^{1-\alpha}y_\beta(T) - y^1\|^2_{H^1_0(\Omega)} &\leq \beta^\epsilon \left(\frac{2}{\epsilon}\right)^2 K\\
&= \epsilon^{-2}\beta^\epsilon(4K)\\
&= \epsilon^{-2}\beta^\epsilon K'.
\end{aligned}$$

It then suffices to take $K' = 4K$ to achieve the proof. ■

**Theorem 4.3** *For all $y^1 \in H^1_0(\Omega)$, the problem compounded in Equation* (1.1) *has a solution $y$ if and only if the sequence $I^{1-\alpha}y_\beta(0^+)$ converges in $H^1_0(\Omega)$. Furthermore, we have that $y_\beta$ converges to $y$ as $\beta$ tends to zero in $L^2((0,T); H^1_0(\Omega))$.*

**Proof.** We proceed in two steps.
**Step 1:** We show that if $I^{1-\alpha}y_\beta(0)$ converges in $H^1_0(\Omega)$, then the problem (1.1) admits a solution. Assume that $\lim_{\beta\to 0} I^{1-\alpha}y_\beta(0) = y^0$ exists. Since $y^0 \in H^1_0(\Omega)$, we can write

$$y^0 = \sum_{i=1}^{+\infty} y^0_i w_i \quad \text{where} \quad y^0_i = (y^0, w_i).$$

Let $y$ the solution of the following equation

$$\begin{cases}
D^\alpha_{RL} y(x,t) - \Delta y(x,t) = f(x,t) & \text{in } Q,\\
y(\sigma,t) = 0 & \text{on } \Sigma,\\
I^{1-\alpha}y(x,0) = y^0 & \text{in } \Omega.
\end{cases}$$

where $1/2 < \alpha < 1$. Then, from Theorem 2.2, we know that $y \in L^2((0,T); H_0^1(\Omega))$ is given by

$$y(t) = \sum_{i=1}^{+\infty} \left\{ t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) y_i^0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds \right\} w_i.$$

Moreover, $I^{1-\alpha} y \in C([0,T], H_0^1(\Omega))$. Thus, $I^{1-\alpha} y(T) \in H_0^1(\Omega)$ exists.

Now, let $t \in [0,T]$, we have

$$\begin{aligned} y_\beta(t) - y(t) &= \sum_{i=1}^{+\infty} \left[ I^{1-\alpha} y_{\beta i}(0) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds \right] w_i \\ &- \sum_{i=1}^{+\infty} \left[ y_i^0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) f_i(s) ds \right] w_i \\ &= \sum_{i=1}^{+\infty} \left( I^{1-\alpha} y_{\beta i}(0) - y_i^0 \right) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) w_i. \end{aligned}$$

consequently,

$$\begin{aligned} \|y_\beta - y\|^2_{L^2((0,T),H_0^1(\Omega))} &= \int_0^T a(y_\beta(t) - y(t), y_\beta(t) - y(t)) dt \\ &= \int_0^T \left( \sum_{i=1}^{+\infty} \lambda_i \left( I^{1-\alpha} y_{\beta i}(0) - y_i^0 \right)^2 \left( t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) \right)^2 \right) dt \\ &= \sum_{i=1}^{+\infty} \lambda_i \left( I^{1-\alpha} y_{\beta i}(0) - y_i^0 \right)^2 \int_0^T t^{2\alpha-2} E^2_{\alpha,\alpha}(-\lambda_i t^\alpha) dt \\ &\leq C^2 \sum_{i=1}^{+\infty} \lambda_i \left( I^{1-\alpha} y_{\beta i}(0) - y_i^0 \right)^2 \int_0^T t^{2\alpha-2} dt \\ &\leq \frac{C^2 T^{2\alpha-1}}{2\alpha - 1} \left\| I^{1-\alpha} y_\beta(0) - y^0 \right\|^2_{H_0^1(\Omega)}. \end{aligned}$$

This implies that $y_\beta$ converges to $y$ in $L^2((0,T); H_0^1(\Omega))$ because $\lim\limits_{\beta \to 0} I^{1-\alpha} y_\beta(0) = y^0$.
On the other hand, we have

$$I^{1-\alpha} y_i(T) = y_i^0 E_\alpha(-\lambda_i T^\alpha) + \int_0^T E_\alpha(-\lambda_i (T-u)^\alpha) f_i(u) du,$$

and

$$I^{1-\alpha} y_{\beta i}(T) = I^{1-\alpha} y_{\beta i}(0) E_\alpha(-\lambda_i T^\alpha) + \int_0^T E_\alpha(-\lambda_i (T-u)^\alpha) f_i(u) du.$$

Hence, we obtain

$$\begin{aligned} \left\| I^{1-\alpha} y_\beta(T) - I^{1-\alpha} y(T) \right\|^2_{H_0^1(\Omega)} &= \sum_{i=1}^{+\infty} \lambda_i \left( I^{1-\alpha} y_{\beta i}(0) - y_i^0 \right)^2 E^2_\alpha(-\lambda_i T^\alpha) \\ &\leq C^2 \left\| I^{1-\alpha} y_\beta(0) - y^0 \right\|^2_{H_0^1(\Omega)}. \end{aligned}$$

This implies that

$$I^{1-\alpha}y_\beta(T)\to I^{1-\alpha}y(T) \text{ strongly in } H_0^1(\Omega)$$

and since, from Theorem 4.1

$$I^{1-\alpha}y_\beta(T)\to y^1 \text{ strongly in } H_0^1(\Omega),$$

the uniqueness of the limit allows us to conclude that $I^{1-\alpha}y(T)=y^1$ and $y$ is a solution of the problem compounded in Equation (1.1).

**Step 2:** We show that if the problem given by Equation (1.1) admits a solution $y$ then $I^{1-\alpha}y_\beta(0)$ converges in $H_0^1(\Omega)$.

Let $y$ be a solution of the problem associated with Equation (1.1), then as in the proof of existence in Theorem 3.1, we know that $y_i=(y(t),w_i)_{L^2(\Omega)}$ is a solution of the ordinary differential equation

$$\begin{cases} D^\alpha_{RL}y_i(t)+\lambda_i y_i(t) &= f_i(t), \quad t\in[0,T],\\ I^{1-\alpha}y_i(T) &= y_i^1. \end{cases} \tag{4.1}$$ edo2

Using the Laplace transform of the first equation in Equation (4.1), we obtain

$$y_i(t)=I^{1-\alpha}y_i(0)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)f_i(s)ds. \tag{4.2}$$ sol_fvp_1

Observing that

$$I^{1-\alpha}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha))=E_\alpha(-\lambda_i t^\alpha)$$

and

$$I^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)f_i(s)ds\right)=\int_0^t f_i(u)E_\alpha(-\lambda_i(t-u)^\alpha)du,$$

we have

$$I^{1-\alpha}y_i(t)=I^{1-\alpha}y_i(0)E_\alpha(-\lambda_i t^\alpha)+\int_0^t f_i(u)E_\alpha(-\lambda_i(t-u)^\alpha)du.$$

and because $I^{1-\alpha}y_i(T)=y_i^1$, we can write

$$I^{1-\alpha}y_i(0)E_\alpha(-\lambda_i T^\alpha)+\int_0^T f_i(u)E_\alpha(-\lambda_i(T-u)^\alpha)du=y_i^1,$$

from which, we deduce that

$$I^{1-\alpha}y_i(0)=\frac{y_i^1-\displaystyle\int_0^T f_i(u)E_\alpha(-\lambda_i(T-u)^\alpha)du}{E_\alpha(-\lambda_i T^\alpha)}.$$

Thus, we can write

$$\begin{aligned} y(t) &= \sum_{i=1}^{+\infty}\left\{\frac{y_i^1-\displaystyle\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du}{E_\alpha(-\lambda_i T^\alpha)}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)\right\}w_i \\ &+ \sum_{i=1}^{+\infty}\left\{\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)f_i(s)ds\right\}w_i \end{aligned} \tag{4.3}$$ sol_fvp_final

and

$$I^{1-\alpha}y(t)=\sum_{i=1}^{+\infty}\left\{\frac{y_i^1-\displaystyle\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du}{E_\alpha(-\lambda_i T^\alpha)}E_\alpha(-\lambda_i t^\alpha)+\int_0^t f_i(u)E_\alpha(-\lambda_i(t-u)^\alpha)du\right\}w_i. \tag{4.4}$$

I_fvp

Let $\beta,\gamma>0$. Then, from (3.14), we have

$$I^{1-\alpha}y_\beta(0)-I^{1-\alpha}y_\gamma(0) = \frac{(\gamma-\beta)\left(y_i^1-\displaystyle\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du\right)}{\beta\gamma+(\beta+\gamma)E_\alpha(-\lambda_i T^\alpha)+E_\alpha^2(-\lambda_i T^\alpha)}.$$

As $I^{1-\alpha}y(0)\in H_0^1(\Omega)$, we choose $N>0$ such as

$$\forall\epsilon>0,\quad \sum_{i=N+1}^{+\infty}\lambda_i|I^{1-\alpha}y_i(0)|^2<\frac{\epsilon}{2}.$$

This means that

$$\forall\epsilon>0,\quad \sum_{i=N+1}^{+\infty}\lambda_i\left|\frac{y_i^1-\displaystyle\int_0^T f_i(u)E_\alpha(-\lambda_i(T-u)^\alpha)du}{E_\alpha(-\lambda_i T^\alpha)}\right|^2<\frac{\epsilon}{2},$$

and we have

$$\begin{aligned}
\left\|I^{1-\alpha}y_\beta(0)-I^{1-\alpha}y_\gamma(0)\right\|_{H_0^1(\Omega)}^2 &= \sum_{i=1}^{+\infty}\lambda_i\left|\frac{(\gamma-\beta)\left(y_i^1-\displaystyle\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du\right)}{\beta\gamma+(\beta+\gamma)E_\alpha(-\lambda_i T^\alpha)+E_\alpha^2(-\lambda_i T^\alpha)}\right|^2. \\
&\le \frac{(\gamma-\beta)^2}{(\beta\gamma)^2}\sum_{i=1}^{N}\lambda_i\left(y_i^1-\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du\right)^2. \\
&+ \frac{(\gamma-\beta)^2}{(\beta+\gamma)^2}\sum_{i=N+1}^{+\infty}\lambda_i\frac{\left(y_i^1-\displaystyle\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du\right)^2}{E_\alpha^2(-\lambda_i T^\alpha)} \\
&\le \frac{(\gamma-\beta)^2}{\beta\gamma^2}\sum_{i=1}^{N}\lambda_i\left(y_i^1-\int_0^T E_\alpha(-\lambda_i(T-u)^\alpha)f_i(u)du\right)^2 \\
&+ \frac{(\gamma-\beta)^2}{(\beta+\gamma)^2}\frac{\epsilon}{2}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\left\|I^{1-\alpha}y_\beta(0)-I^{1-\alpha}y_\gamma(0)\right\|^2_{H^1_0(\Omega)} &\leq \left(\frac{\gamma-\beta}{\beta\gamma}\right)^2\left(2\sum_{i=1}^{N}\lambda_i|y_i^1|^2+\frac{2C^2T^{1-\alpha}}{1-\alpha}\sum_{i=1}^{N}\int_0^T|f_i|^2(u)du\right)\\
&+ \left(\frac{\gamma-\beta}{\beta+\gamma}\right)^2\frac{\epsilon}{2}.\\
&\leq \left(\frac{2}{\beta^2}+\frac{2}{\gamma^2}\right)\left(2\sum_{i=1}^{N}\lambda_i|y_i^1|^2+\frac{2C^2T^{1-\alpha}}{1-\alpha}\sum_{i=1}^{N}\int_0^T|f_i|^2(u)du\right)\\
&+ 2\epsilon.
\end{aligned}$$

Since

$$\left(2\sum_{i=1}^{N}\lambda_i|y_i^1|^2+\frac{2C^2T^{1-\alpha}}{1-\alpha}\sum_{i=1}^{N}\int_0^T|f_i|^2(u)du\right)<\infty \quad \text{and} \quad \lim_{\gamma,\beta\to\infty}\left(\frac{2}{\beta^2}+\frac{2}{\gamma^2}\right)=0,$$

we deduce that

$$\lim_{\gamma,\beta\to\infty}\left\|I^{1-\alpha}y_\beta(0)-I^{1-\alpha}y_\gamma(0)\right\|_{H^1_0(\Omega)}=0.$$

This implies that the sequence $\left\{I^{1-\alpha}y_\beta(0)\right\}$ is of Cauchy and thus it converges in $H^1_0(\Omega)$. ■

# 5 Conclusion

In this work, we have considered an ill-posed problem associated with a family of well-posed problems and prove, using spectral methods, that the solutions of the latter problems converge to the solution of the former problem in an appropriate Hilbert space. This analysis is useful if we want to control an ill-posed problem which will be the subject of future work. Moreover, the convergence results obtained can be used to find a numerical solution for problem compounded in Equation (1.1).

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