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ABSTRACT
In the present paper, we are concerned with a fractional wave equation of Sturm–Liouville type in a general star graph. We first give several existence, uniqueness and regularity results of weak solutions for the one-dimensional case using the spectral theory; we prove the existence and uniqueness of solutions to a quadratic boundary optimal control problem and provide a characterization of the optimal control via the Euler–Lagrange first order optimality conditions. We then investigate the analogous problems for a fractional Sturm–Liouville problem in a general star graph with mixed Dirichlet and Neumann boundary conditions and controls of the velocity. We show the existence and uniqueness of minimizers, and by using the first order optimality conditions with the Lagrange multipliers, we are able to characterize the optimal controls.

KEYWORDS
Caputo fractional derivative; Riemann-Liouville fractional derivative; initial-boundary value problems; Sturm-Liouville equations; optimal control; optimality system; optimality conditions

AMS Subject Classification 35L20, 49J45, 49J20, 26A33.

1. Introduction and position of the problem

Classical Sturm–Liouville theory is the theory of real second-order linear ordinary differential equations of the form:

$$(\beta y')' + qy = \lambda \omega y$$

where y is the unknown physical quantity, β, q, ω are suitable functions, λ is a parameter and the function ω is called the weight function. With appropriate boundary conditions, λ and y appear as eigenvalue and eigenfunction, respectively, of the adjoint operator. There is an exhaustive literature corresponding to this type of equations: we refer for instance to [2] and the references therein.

The last fourth decades, Many researchers have focused their attention on fractional Sturm–Liouville problems which are obtained by replacing the ordinary derivatives with fractional derivatives, and this follows from the fact that many phenomena

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which occur in various fields in science as well as in engineering can be more accurately described by means of fractional order derivatives. For example, in [23], the orthogonality of solutions to fractional Sturm-Liouville eigen-problems involving a left-sided Riemann-Liouville fractional derivative and a right-sided Caputo fractional derivative of the same order was proved. Using variational approach, the existence of a countable set of orthogonal solutions and eigenvalues to a fractional Sturm-Liouville eigen-problem involving left-sided and right-sided Caputo fractional derivative of the same order was established. [38] studied a fractional Sturm-Liouville eigen-problem involving a left-sided Caputo fractional derivative and a right-sided Riemann-Liouville fractional derivative of the same order; They proved that analytical solutions are non polynomial functions which are orthogonal with respect to the weighted function associated to the problem.

Over the past fifty years, hyperbolic partial differential equations have been studied extensively [12,20,21]. In [21] for instance, the authors deal with a wave equation with damping in which one of the boundary conditions is of a dynamic nature; they consider damping effect of Kelvin-Voigt type in both the partial differential equation and the dynamic boundary condition, studying the problem within the framework of B-evolutions theories and fractional powers, they proved the existence of a strong unique solution. In [32], the authors study the motion of a one-dimensional continuum whose deformation is described by a strain measure of nonlocal type; They use the Caputo fractional derivatives and a linear relation between stress and strain measure to obtain an integro-differential equation of motion which is solved in the space of tempered distributions by using the Fourier and Laplace transforms. In complex or viscoelastic media [31], the time fractional derivative used to describe the wave equation for a vibrating string is taken in the Caputo sense; the solution is obtained in terms of the Mittag-Leffler type functions and complete set of eigenfunctions of the Sturm–Liouville problem by using the method of separation of variables and the Laplace transform method. A fractional generalization of the wave equation that describes propagation of damped waves is considered in [33]; In contrast to the fractional diffusion-wave equation, the fractional wave equation contains fractional derivatives of the same order $\alpha,\ 1 \leq \alpha \leq 2$, both in space and in time; the authors show that this feature is a decisive factor for inheriting some crucial characteristics of the wave equation like a constant propagation velocity, its gravity and mass centers. The fundamental solution of the fractional wave equation is determined and shown to be a spatial probability density function evolving in time, all whose moments of order less than α are finite.

A lot has been achieved in the area of optimal control of evolution equations: Agrawal detained the first record of the formulation of the fractional optimal control problem. He presented in [13] a general formulation and proposed a numerical method to solve such problems. In his work, the fractional derivative was defined in the Riemann–Liouville sense, and the formulation was obtained through the fractional variation principle and the Lagrange multiplier technique. J. Lions in [41] has studied several optimal control of system governed by hyperbolic equations. The author in [35] applied the classical control theory to a fractional diffusion equation involving a Riemann–Liouville fractional derivative in a bounded domain by interpreting the Euler–Lagrange first-order optimality condition with an adjoint problem defined through a right fractional Caputo derivative. The author obtained an optimality system for the optimal control. Control problems governed by a linear wave equation are analyzed in [34]; the space of vector measures $\mathcal{M}(\Omega_c, L^2(I))$ is chosen as control space and the support of the controls is time-independent which is desired in many applications, regularity results for the linear wave equation are proven and used to show the well-posedness of the control problem. Deriving first order optimality conditions, they elaborate structural properties of the optimal control and finally the optimal control problem is used to solve an inverse source problem.

In this paper, we are interested in solving the following fractional optimal control problem on a star graph:

$$\min_{v\in\mathcal{U}_{ad}} \sum_{i=1}^{n} \left(\frac{1}{2}\int_a^{b_i} \left| y^i(T) - z_d^{0,i} \right|^2 \mathrm{d}x + \frac{1}{2}\int_a^{b_i} \left| y_t^i(T) - z_d^{1,i} \right|^2 \mathrm{d}x + \frac{N}{2}\int_a^{b_i} |v_i|^2 \,\mathrm{d}x \right), \tag{1.1}$$

where $z_d^0 = (z_d^{0,i})_i \in \mathbb{V}$, $z_d^1 = (z_d^{1,i})_i \in \mathbb{L}^2$, $\mathcal{U}_{ad}$ is a closed and convex subset of $\mathbb{L}^2$, and $y = (y^i)_i$ satisfies the following wave equation involving a fractional Sturm-Liouville operator on a star graph:

$$\left\{ \begin{array}{lcll} y_{tt}^i + \mathcal{D}_{b_i^-}^{\alpha}(\beta^i \mathbb{D}_{a^+}^{\alpha} y^i) + q^i y^i & = & f^i, & \text{in } (a,b_i)\times(0,T),\ i=1,\dots,n, \\ I_{a^+}^{1-\alpha} y^i(a^+,\cdot) & = & I_{a^+}^{1-\alpha} y^j(a^+,\cdot) & \text{in } (0,T),\ i\neq j=1,\dots,n, \\ \displaystyle\sum_{i=1}^{n} \beta^i(a)\mathbb{D}_{a^+}^{\alpha} y^i(a^+,\cdot) & = & 0, & \text{in } (0,T), \\ I_{a^+}^{1-\alpha} y^1(b_1^-,\cdot) & = & 0, & \text{in } (0,T), \\ I_{a^+}^{1-\alpha} y^i(b_i^-,\cdot) & = & h_i, & \text{in } (0,T),\ i=2,\dots,m, \\ \beta^i(b_i)\mathbb{D}_{a^+}^{\alpha} y^i(b_i^-,\cdot) & = & g_i, & \text{in } (0,T),\ i=m+1,\dots,n, \\ y^i(\cdot,0) & = & y^{0,i}, & \text{in } (a,b_i),\ \ i=1,\dots,n, \\ y_t^i(\cdot,0) & = & v_i, & \text{in } (a,b_i),\ \ i=1,\dots,n. \end{array} \right. \tag{1.2}$$

Here, $T>0$, y_{tt}^i denotes the second order derivative with respect to the time of y^i, the operators $\mathbb{D}_{a^+}^{\alpha}$, $\mathcal{D}_{b_i^-}^{\alpha}$, $i=1,\dots,n$, and $I_{a^+}^{\alpha}$ $(0<\alpha<1)$ are respectively Left Riemann-Liouville fractional derivative of order α, the right Caputo fractional derivative of order α and the left Riemann-Liouville integral of order α, we refer to Section 2 for the precise definitions. The real valued functions $\beta^i \in \mathcal{C}^1([a,b_i])$, $q^i \in \mathcal{C}([a,b_i])$, $i=1,\dots,n$, satisfy suitable conditions, f^i belongs to $L^2((0,T)\times L^2(a,b_i))$, $g_i, h_i \in H^2(0,T)$, $i=1,\dots,n$ the controls $v_i, \in L^2(a,b_i)$, $i=1,\dots n$. The setting should, therefore, indicate that we are looking at a star graph, rooted at b_1^-, where we have a fixed Dirichlet-type boundary condition and controls via initial velocity. After deriving some well-posedness results (existence and uniqueness of weak solutions) of the system (1.2) in the general star graph, we show the existence and uniqueness of minimizers to the optimal control problem (1.1)-(1.2), and give the associated optimality conditions by using the method of Lagrange multipliers.

In order to study the general star graph, we shall first of all restraint our equation to the one dimensional optimal control problem given below:

$$\min_{v\in\mathcal{U}_{ad}} \left(\frac{1}{2}\int_a^{b} \left| y(T) - z_d^{0} \right|^2 \mathrm{d}x + \frac{1}{2}\int_a^{b} \left| y_t(T) - z_d^{1} \right|^2 \mathrm{d}x + \frac{N}{2}\int_a^{b} |v|^2 \,\mathrm{d}x \right), \tag{1.3}$$

where $z_d^0 \in D(\mathcal{E})$, $z_d^1 \in L^2(a,b)$, $N>0$, $\mathcal{U}_{ad}$ is a closed and convex subset of $L^2(a,b)$, and $y = y(v)$ satisfies the following fractional Sturm–Liouville wave equation:

$$\begin{cases} y_{tt} + \mathcal{D}^{\alpha}_{b^-}\, (\beta\, \mathbb{D}^{\alpha}_{a^+})(y) + q\, y & = & f & \text{in } (a,b)\times(0,T), \\ (I^{1-\alpha}_{a^+} y)(a^+,\cdot) & = & 0 & \text{in } (0,T), \\ (\beta \mathbb{D}^{\alpha}_{a^+} y)(b^-,\cdot) & = & g & \text{in } (0,T), \\ y(\cdot,0) & = & y^0 & \text{in } (a,b), \\ y_t(\cdot,0) & = & v & \text{in } (a,b). \end{cases} \tag{1.4}$$

Where the real valued functions $\beta \in \mathcal{C}^1([a,b])$, $q \in \mathcal{C}([a,b])$ satisfy suitable conditions, $f \in L^2(Q)$, $g \in H^2(0,T)$, $y^0 \in D(A)$ (see Remark 2.19) and the control $v \in L^2(a,b)$. After proving existence, uniqueness and regularity results of the state equation (1.4) and the associated dual system, we show the existence and uniqueness of minimizers of the optimal control problem (1.3)-(1.4), and characterize the associated first order optimality conditions by using the classical Euler-Lagrange first order optimality conditions.

The rest of the paper is structured as follows. In Section 2, we give some preliminary results that will be used in the proof of our main results. In Section 3, we show first that the homogeneous and non-homogeneous fractional Sturm-Liouville equations on a single edge have unique weak solutions using the spectral theory. These results are contained in Theorems 3.2 and 3.9, respectively. The regularity of solutions has been also investigated. We conclude this section by proving that the quadratic optimal control problem associated to the evolution equation involving a fractional Sturm-Liouville on one edge has a unique optimal control, and we give the associated optimality system that characterizes this control (see Theorem 3.11). The same investigation is done for the wave equation involving a fractional Sturm-Liouville on the graph in Section 4. We study the existence and regularity of the weak solution. The mains results of this section are contained in Theorems 4.6, 4.9, 4.10 and 4.11.

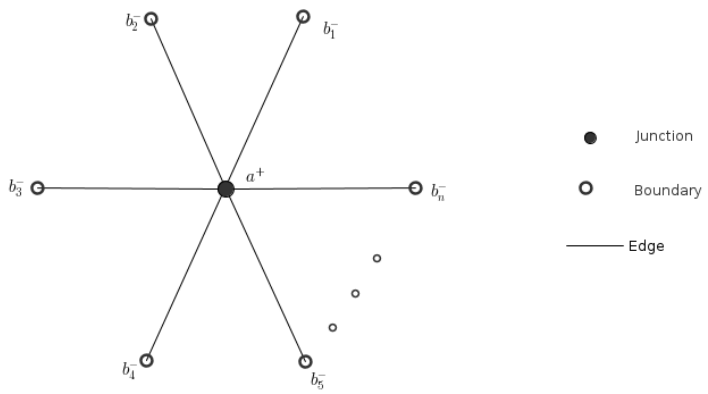

Figure 1. A star graph with n edges

2. Preliminaries

In this section, we introduce some notations, give the function spaces needed to study our problems, recall some known results, and prove some intermediate results that are needed in the proofs of our main results. We start with fractional integrals and derivatives.

Definition 2.1. [6,9] Let z a complex such that $Re(z) > 0$. Then the Gamma function, noted Γ, is given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Definition 2.2. [6,9] Let x and y be two complexes such that $Re(x) > 0$ and $Re(y) > 0$. The Beta function is given by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

Definition 2.3. [4] The left, and right Riemann–Liouville fractional integrals of order $\alpha \in (0,1)$ of f are defined, respectively, by:

$$\begin{aligned}
(I_{a^+}^{\alpha} f)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t), \quad (x > a) \\
(I_{b^-}^{\alpha} f)(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \mathrm{d}t, \quad (x < b),
\end{aligned}$$

provided that the integrals exist.

Lemma 2.4. *[5] Let $0 < \alpha < 1$, $1 < p < 1/\alpha$, $q = p/(1-\alpha p)$. Then, there is a constant $C = C(\alpha, p, q, a, b) > 0$ such that for any $\rho \in L^p(a,b)$, we have*

$$\begin{aligned}
\| I_{a^+}^{\alpha} \rho \|_{L^q(a,b)} &\leq C \| \rho \|_{L^p(a,b)}, \\
\| I_{b^-}^{\alpha} \rho \|_{L^q(a,b)} &\leq C \| \rho \|_{L^p(a,b)}.
\end{aligned}$$

Remark 2.5. Since the continuous embedding $L^2(a,b) \hookrightarrow L^p(a,b)$ holds for every $1 \leq p \leq 2$, it follows from Lemma 2.4 that for every $0 < \alpha < 1$ there is a constant $C > 0$ such that for every $\rho \in L^2(a,b)$,

$$\| I_{a^+}^{\alpha} \rho \|_{L^2(a,b)} \leq C \| \rho \|_{L^2(a,b)}. \tag{2.1}$$

Definition 2.6. [4] The left, and right Riemann–Liouville fractional derivatives of order $\alpha \in (0,1)$ of f are defined, respectively, by:

$$\begin{aligned}
(\mathbb{D}_{a^+}^{\alpha} f)(x) &= D(I_{a^+}^{1-\alpha} f)(x) &&= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x (x-t)^{-\alpha} f(t) \mathrm{d}t, \quad (x > a) \\
(\mathbb{D}_{b^-}^{\alpha} f)(x) &= -D(I_{b^-}^{1-\alpha} f)(x) &&= \frac{-1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b (t-x)^{-\alpha} f(t) \mathrm{d}t, \quad (x < b),
\end{aligned} \tag{2.2}$$

provided that the integrals exist.

Definition 2.7. [1] The left, and right-sided Caputo fractional derivative of order $\alpha \in (0,1)$ of f are defined respectively, by:

$$\begin{aligned} (\mathcal{D}^{\alpha}_{a^+}f)(x) &= (I^{1-\alpha}_{a^+}Df)(x) &&= \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha}f'(t)\mathrm{d}t, \quad (x>a) \\ (\mathcal{D}^{\alpha}_{b^-}f)(x) &= -(I^{1-\alpha}_{b^-}Df)(x) &&= \frac{-1}{\Gamma(1-\alpha)}\int_x^b (t-x)^{-\alpha}f'(t)\mathrm{d}t \quad (x<b) \end{aligned} \tag{2.3}$$

provided that the integrals exist.

Noticing that the Caputo fractional derivative is too demanding, its existence requires the function f to be absolutely continuous on $[a,b]$ which is equivalent to $f \in W^{1,1}(a,b)$. We refer to the monograph [5] for the precise conditions on f for which the integrals in (2.3) exist.

Definition 2.8. [40][Laplace Transform] Let $f : \mathbb{R}_+ \to \mathbb{R}$, the Laplace transform of the function f is defined by

$$(\mathcal{L}f)(s) = \mathcal{L}[f(t)](s) = \hat{f}(s) := \int_0^{\infty} e^{-ts}f(t)\mathrm{d}t \qquad s \in \mathbb{C}.$$

Thus, the inverse Laplace transform of $\mathcal{L}^{-1}[\hat{f}(s)](t) = f(t)$.

We have the following observations that will be useful for the existence results.

Remark 2.9. In view of the Definition 2.8, we have the following results.

(1) If $f : \mathbb{R}_+ \to \mathbb{R}$ is such that its second derivative f'' exists, then

$$\mathcal{L}[f''(t)](s) = s^2\hat{f}(s) - s\lim_{t\to 0} f(t) - \lim_{t\to 0} f'(t) \quad s \in \mathbb{C}. \tag{2.4}$$

(2) Let $s,t>0$ and $w \in \mathbb{R}$. Then inverse Laplace transform of the real functions $s \mapsto \dfrac{w}{s^2+w^2}$ and $s \mapsto \dfrac{s}{s^2+w^2}$ are

$$\mathcal{L}^{-1}\left(\frac{w}{s^2+w^2}\right) = \sin(wt). \tag{2.5}$$

and

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+w^2}\right) = \cos(wt). \tag{2.6}$$

respectively.

(3) Let $s,t>0$, and $g,f : \mathbb{R}_+ \to \mathbb{R}$. Then the inverse Laplace transform of $\hat{f} \times \hat{g}$ is given by

$$\mathcal{L}^{-1}\left(\hat{f}(s) \times \hat{g}(s)\right) = (f \star g)(t), \tag{2.7}$$

where $\star$ denotes the convolution product.

We recall the Leibniz Integral Rule.

Lemma 2.10. *[8][Leibniz Integral Rule], let a, b be two differentiable real functions such that $-\infty < a(x), b(x) < +\infty$, then*

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = b'(x)f(x,b(x)) - a'(x)f(x,a(x)) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt \quad (2.8)$$

provided that the integral is differentiable.

We have the following results; for the proof, we refer to the references [5,6]. Let $c_0, d_0 \in \mathbb{R}$. Let $f : [a,b] \to \mathbb{R}$ have the representation

$$f(x) = \frac{c_0}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I^{\alpha}_{a^+}\varphi(x) \ \text{ for a.e. } x \in [a,b], \quad (2.9)$$

and let also $g : [a,b] \to \mathbb{R}$ have the representation

$$g(x) = \frac{d_0}{\Gamma(\alpha)}(b-x)^{\alpha-1} + I^{\alpha}_{b^-}\psi(x) \ \ ; \text{ for a.e. } \in [a,b], \quad (2.10)$$

where φ and ψ belong to $L^2(a,b)$. From now on, we denote by $AC^{\alpha,2}_{a^+}$ and $AC^{\alpha,2}_{b^-}$ the spaces of all functions f and g having the representations (2.9) and (2.10), respectively, with $\varphi, \psi \in L^2(a,b)$.

Remark 2.11. We have the following:

$$\mathbb{D}^{\alpha}_{a^+} f \in L^2(a,b) \iff f \in AC^{\alpha,2}_{a^+}, \quad (2.11a)$$
$$\mathbb{D}^{\alpha}_{b^-} f \in L^2(a,b) \iff f \in AC^{\alpha,2}_{b^-}. \quad (2.11b)$$

For more details on these spaces, and the proof of (2.11a)-(2.11b), we refer to [10]. We set

$$H^{\alpha}_{a^+}(a,b) = AC^{\alpha,2}_{a^+} \cap L^2(a,b), \quad (2.12a)$$
$$H^{\alpha}_{b^-}(a,b) = AC^{\alpha,2}_{b^-} \cap L^2(a,b). \quad (2.12b)$$

It follows from the definitions of $AC^{\alpha,2}_{a^+}$ and $AC^{\alpha,2}_{b^-}$ that,

$$\rho \in H^{\alpha}_{a^+}(a,b) \Leftrightarrow \rho \in L^2(a,b) \text{ and } \mathbb{D}^{\alpha}_{a^+}\rho \in L^2(a,b), \quad (2.13a)$$
$$\rho \in H^{\alpha}_{b^-}(a,b) \Leftrightarrow \rho \in L^2(a,b) \text{ and } \mathbb{D}^{\alpha}_{b^-}\rho \in L^2(a,b). \quad (2.13b)$$

For any $0 < \alpha < 1$, we endow $H^{\alpha}_{a^+}(a,b)$ with the inner product:

$$(\varphi, \psi)_{H^{\alpha}_{a^+}(a,b)} = \int_a^b \varphi\psi \mathrm{d}x + \int_a^b \mathbb{D}^{\alpha}_{a^+}\varphi \mathbb{D}^{\alpha}_{a^+}\psi \mathrm{d}x. \quad (2.14)$$

Then, $H^{\alpha}_{a^+}(a,b)$ endowed with the norm

$$\|\varphi\|^2_{H^{\alpha}_{a^+}(a,b)} = \|\varphi\|^2_{L^2(a,b)} + \|\mathbb{D}^{\alpha}_{a^+}\varphi\|^2_{L^2(a,b)}, \quad (2.15)$$

is a Hilbert space (see e.g. [10]). Moreover, the norm on $H^{\alpha}_{a^+}(a,b)$ given by (2.15) is equivalent to the norm given by

$$|||\varphi|||^2_{H^{\alpha}_{a^+}(a,b)} = \left| I^{1-\alpha}_{a^+}\varphi(a^+,\cdot)\right|^2 + \|\mathbb{D}^{\alpha}_{a^+}\varphi\|^2_{L^2(a,b)} \ . \tag{2.16}$$

In other words, there are two constants $0 < C_1 \le C_2$ such that

$$C_1\, |||\varphi|||_{H^{\alpha}_{a^+}(a,b)} \le \|\varphi\|_{H^{\alpha}_{a^+}(a,b)} \le C_2\, |||\varphi|||_{H^{\alpha}_{a^+}(a,b)} \quad \forall \varphi \in H^{\alpha}_{a^+}(a,b). \tag{2.17}$$

Lemma 2.12. *[10, corollary 32] Let $1/2 < \alpha < 1$. Then, the following continuous embedding is compact*

$$H^{\alpha}_{a^+}(a,b) \hookrightarrow L^2(a,b). \tag{2.18}$$

In order to assure the well posedness of the boundary value problems considered in this paper, we introduce the space $\mathcal{V}$ defined as follows:

$$\mathcal{V} = \Big\{ y \in H^{\alpha}_{a^+}(a,b) : \ \mathcal{D}^{\alpha}_{b^-}(\beta \mathbb{D}^{\alpha}_{a^+} y) \in H^{1-\alpha}_{b^-}(a,b) \Big\}, \tag{2.19}$$

with $H^{1-\alpha}_{b^-}(a,b)$ defined as in (2.12b). Then, $\mathcal{V}$ is a closed subspace of $H^{\alpha}_{a^+}(a,b)$. The space $\mathcal{V}$ defined in (2.19) endowed with the norm

$$\begin{aligned} \|y\|_{\mathcal{V}} &:= \Big(\|y\|^2_{H^{\alpha}_{a^+}(a,b)} + \|\mathcal{D}^{\alpha}_{b^-}(\beta\mathbb{D}^{\alpha}_{a^+}y)\|^2_{H^{1-\alpha}_{b^-}(a,b)} \Big)^{1/2} \\ &= \Big(\|y\|^2_{H^{\alpha}_{a^+}(a,b)} + \|I^{1-\alpha}_{b^-}(\beta\mathbb{D}^{\alpha}_{a^+}y)')\|^2_{L^2(a,b)} + \|(\beta\mathbb{D}^{\alpha}_{a^+}\rho)'\|^2_{L^2(a,b)} \Big)^{1/2}, \end{aligned} \tag{2.20}$$

and the associated scalar product

$$\begin{aligned} (\phi,\psi)_{\mathcal{V}} := &\int_a^b \phi\psi\, dx + \int_a^b \mathbb{D}^{\alpha}_{a^+}\phi \mathbb{D}^{\alpha}_{a^+}\psi\, dx + \int_a^b I^{1-\alpha}_{b^-}(\beta\mathbb{D}^{\alpha}_{a^+}\phi)' I^{1-\alpha}_{b^-}(\beta\mathbb{D}^{\alpha}_{a^+}\psi)'\, dx \\ &+ \int_a^b (\beta\mathbb{D}^{\alpha}_{a^+}\phi)'(\beta\mathbb{D}^{\alpha}_{a^+}\psi)'\, dx, \end{aligned} \tag{2.21}$$

is a Hilbert space [39].

The following trace results are useful for some calculations in the upcoming sections.

Lemma 2.13. *Let $1/2 < \alpha < 1$, $\gamma = 1-\alpha$, $\beta \in \mathcal{C}^1([a,b])$ and $T > 0$; for every $x_0 \in [a,b]$, the followings assertions holds:*

(1) Let $\rho \in L^2((0,T); H^{\alpha}_{a^+}(a,b))$. Then the function $I^{1-\alpha}_{a^+}\rho(x_0,\cdot)$ exists and belong to $L^2(0,T)$. Moreover, there is a constant $C = C(a,b) > 0$ such that

$$\|I^{1-\alpha}_{a^+}(\rho)(x_0,\cdot)\|^2_{L^2(0,T)} \le C\|\rho\|^2_{L^2((0,T);H^{\alpha}_{a^+}(a,b))}. \tag{2.22}$$

(2) Let $\rho \in L^2((0,T);\mathcal{V})$. Then, the function $\mathbb{D}^{\alpha}_{a^+}\rho(x_o,\cdot)$ exists and belongs to $L^2(0,T)$. Moreover there is a constant $C = C(a,b,\alpha) > 0$ such that

$$\|\mathbb{D}^{\alpha}_{a^+}\rho(x_o,\cdot)\|_{L^2(0,T)} \le C\|\rho\|_{L^2((0,T);\mathcal{V})}. \tag{2.23}$$

Proof. We proceed exactly as for the proof of Lemma 2.3 in [29]. $\square$

Remark 2.14. Notice that if $\rho \in L^2((0,T);\mathcal{V})$, then from Lemma 2.13 we have that the traces $I_{a^+}^{1-\alpha}\rho(a^+,\cdot)$, $I_{a^+}^{1-\alpha}\rho(b^-,\cdot)$, $\beta\mathbb{D}_{a^+}^{\alpha}\rho(a^+,\cdot)$ and $\beta\mathbb{D}_{a^+}^{\alpha}\rho(b^-,\cdot)$, exist, and belong to $L^2(0,T)$.

Next, we introduce the following integration by parts formulas.

Lemma 2.15. *[13,14] Let $\beta \in \mathcal{C}([a,b])$. Let also $y,\phi : [a,b] \to \mathbb{R}$ be such that $y,\phi \in \mathcal{V}$. Then*

$$\int_a^b \mathcal{D}_{b^-}^{\alpha}(\beta\mathbb{D}_{a^+}^{\alpha}y)(s)\phi(s)ds = -\left[(\beta\mathbb{D}_{a^+}^{\alpha}y)(s)I_{a^+}^{1-\alpha}(\phi)(s)\right]_{s=a}^{s=b} + \left[I_{a^+}^{1-\alpha}(y)(s)\beta(s)\mathbb{D}_{a^+}^{\alpha}(\phi)(s)\right]_{s=a}^{s=b} + \int_a^b y(s)\mathcal{D}_{b^-}^{\alpha}(\beta\mathbb{D}_{a^+}^{\alpha}(\phi))(s)ds. \tag{2.24}$$

Assumption 2.16. We assume that the functions $\beta \in \mathcal{C}^1([a,b])$ and $q \in \mathcal{C}([a,b])$ are such that $\|\beta\|_\infty := \max_{x\in[a,b]} |\beta(x)| \geq \beta_0$ and $\|q\|_\infty := \max_{x\in(a,b)} |q(x)| \geq q_0$, for some $\beta_0 > 0,\ q_0 > 0$.

Lemma 2.17. *Let $0 < \alpha < 1$, and β, q satisfy Assumption 2.16, we define the space $D(\mathcal{E})$ as*

$$D(\mathcal{E}) = \{\phi \in H_{a^+}^{\alpha}(a,b)\ : I_{a^+}^{1-\alpha}\phi(a^+,\cdot) = 0\}.$$

$D(\mathcal{E})$ is a closed subspace of $H_{a^+}^{\alpha}(a,b)$, and the bilinear form $\mathcal{E}(\cdot,\cdot)$ defined on $D(\mathcal{E}) \times D(\mathcal{E})$ by:

$$\mathcal{E}(\phi,\varphi) = \int_a^b \beta(x)\mathbb{D}_{a^+}^{\alpha}\phi(x)\mathbb{D}_{a^+}^{\alpha}\varphi(x)\,dx + \int_a^b q(x)\phi(x)\varphi(x)\,dx,$$

is continuous and coercive on $D(\mathcal{E}) \times D(\mathcal{E})$.

Proof. We proceed as in [29], using on $D(\mathcal{E})$ the norm induced by that on $H_{a^+}^{\alpha}(a,b)$ given by (2.15). $\square$

Remark 2.18. Note that $D(\mathcal{E})$ endowed with the norm

$$\|\phi\|_{D(\mathcal{E})} = (\mathcal{E}(\phi,\phi))^{1/2}, \quad \forall\phi \in D(\mathcal{E}) \tag{2.25}$$

is a Hilbert space.

Remark 2.19. Let A be the self-adjoint operator in $L^2(a,b)$ associated with the form $\mathcal{E}(\cdot,\cdot)$ in the sense that

$$\begin{cases} D(A) = \left\{u \in D(\mathcal{E}) :\ \mathcal{D}_{b^-}^{\alpha}(\beta\mathbb{D}_{a^+}^{\alpha}u) \in L^2(a,b) :\ (I_{a^+}^{1-\alpha}u)(a) = (\beta\mathbb{D}_{a^+}^{\alpha}u)(b) = 0\right\} \\ Au = \mathcal{D}_{b^-}^{\alpha}(\beta\mathbb{D}_{a^+}^{\alpha}u) + qu. \end{cases} \tag{2.26}$$

and

$$(Au,v)_{L^2(\Omega)} = \mathcal{E}(u,v)\ \ \forall u,v \in D(A). \tag{2.27}$$

It follows from the embedding (2.18) that for $1/2 < \alpha < 1$, the operator A has a compact resolvent. Let $(\lambda_n)_n$ be the eigenvalues of A with associated eigenfunctions (φ_n). Considering Assumption 2.16, It follows from the coercivity and the nonnegativity of $\mathcal{E}(\cdot,\cdot)$ that

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots \quad \text{with} \quad \lim_{n\to\infty} \lambda_n = +\infty.$$

Moreover, as we can assume without lost of generality that the eigenfunctions $(\varphi_n)_{n=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$, we have that $\left(\frac{\varphi_n}{\sqrt{\lambda_n}}\right)_{n=1}^{\infty}$ is also an orthonormal basis of $D(A)$ for the scalar product $\mathcal{E}$. Hence, we obtain that

$$\|\phi\|^2_{D(A)} = \|\phi\|^2_{D(\mathcal{E})} = \mathcal{E}(\phi,\phi) = \sum_{n=1}^{\infty} \lambda_n (\phi, \varphi_n)^2_{L^2(\Omega)} \quad \forall\, \phi \in D(A). \tag{2.28}$$

3. Boundary optimal control problems on a single edge

In this section, we consider the following initial-boundary value fractional Sturm–Liouville parabolic equation:

$$\begin{cases} y_{tt} + \mathcal{D}^{\alpha}_{b^-}\left(\beta\, \mathbb{D}^{\alpha}_{a^+} y\right) + q\, y &= f \quad \text{in} \quad Q = (a,b)\times(0,T),\\ (I^{1-\alpha}_{a^+} y)(a^+,\cdot) &= 0 \quad \text{in} \quad (0,T),\\ (\beta \mathbb{D}^{\alpha}_{a^+} y)(b^-,\cdot) &= g \quad \text{in} \quad (0,T),\\ y(\cdot,0) &= y^0 \quad \text{in} \quad (a,b),\\ y_t(\cdot,0) &= v \quad \text{in} \quad (a,b), \end{cases} \tag{3.1}$$

where $f \in L^2((0,T); D(\mathcal{E})^\star)$ and $y^0 \in D(A)$, $g \in L^2(0,T)$ and $v \in L^2(a,b)$. The function β and q satisfy Assumption 2.16.

We are concerned with the following optimal control problem:

$$\min_{v\in\mathcal{U}_{ad}} J(v), \tag{3.2}$$

where the functional J is given by

$$J(v) := \frac{1}{2}\left\|y(T) - z^0_d\right\|^2_{L^2(a,b)} + \frac{1}{2}\left\|y_t(T) - z^1_d\right\|^2_{L^2(a,b)} + \frac{N}{2}\|v\|^2_{L^2(0,T)}. \tag{3.3}$$

Here, $z^0_d \in D(\mathcal{E})$, $z^1_d \in L^2(a,b)$, $N > 0$, $\mathcal{U}_{ad}$ is a closed and convex subspace of $L^2(a,b)$ and $y = y(v)$ satisfies (3.1).

Before going further, we need some existence results.

3.1. Homogeneous fractional Sturm–Liouville parabolic equations in a single edge

Let us consider the following homogeneous fractional Sturm–Liouville parabolic equations:

$$\begin{cases} y_{tt} + \mathcal{D}^{\alpha}_{b^-}(\beta\,\mathbb{D}^{\alpha}_{a^+} y) + q\,y &= f & \text{in} & Q, \\ (I^{1-\alpha}_{a^+} y)(a^+,\cdot) &= 0 & \text{in} & (0,T), \\ (\beta \mathbb{D}^{\alpha}_{a^+} y)(b^-,\cdot) &= 0 & \text{in} & (0,T), \\ y(\cdot,0) &= y^0 & \text{in} & (a,b), \\ y_t(\cdot,0) &= v & \text{in} & (a,b), \end{cases} \tag{3.4}$$

where $1/2 < \alpha < 1$, $f \in L^2((0,T); D(\mathcal{E})^\star)$ and $y^0 \in D(A)$ and $v \in L^2(a,b)$. The function β and q satisfy Assumption 2.16.

From the characterization (2.26) of the operator A, the Cauchy problem (3.4) can be rewritten as the following abstract Cauchy problem:

$$\begin{cases} y_{tt} + Ay &= f & \text{in} & Q, \\ y(\cdot,0) &= y^0 & \text{in} & (a,b), \\ y_t(\cdot,0) &= v & \text{in} & (a,b). \end{cases} \tag{3.5}$$

Next, we give our notion of solutions to the system (3.4), hence to the problem (3.5).

Definition 3.1. Let $f \in L^2((0,T); D(\mathcal{E})^\star)$, $y^0 \in D(A)$, $v \in L^2(a,b)$. A function y is said to be a weak solution of (3.4) in $(0,T)$, $T > 0$, if the following assertions hold.

- Regularity: $y \in \mathcal{C}([0,T], D(A)) \cap \mathcal{C}^1([0,T]; L^2(a,b))$.
- Initial condition: $\begin{array}{rcccl} y(\cdot,0) &=& y^0 & \text{in} & (a,b), \\ y_t(\cdot,0) &=& v & \text{in} & (a,b). \end{array}$
- Variational identity: $\langle \dfrac{\partial^2 y}{\partial t^2}, \varphi\rangle_{D(\mathcal{E})^\star, D(\mathcal{E})} + \mathcal{E}(y(t,\cdot),\varphi) = \langle f(t,\cdot),\varphi\rangle_{D(\mathcal{E})^\star, D(\mathcal{E})}$, for every $\varphi \in D(A)$, and a.e. $t \in (0,T)$.

We have the following existence result.

Theorem 3.2. *Let $1/2 < \alpha < 1$. Assume that Assumption 2.16 holds. Then for every $f \in L^2((0,T); D(\mathcal{E})^\star)$, $y^0 \in D(A)$ and $v \in L^2(a,b)$, the system* (3.4) *(Hence, the Cauchy problem* (3.5)*) has a unique weak solution $y \in \mathcal{C}([0,T], D(A)) \cap \mathcal{C}^1([0,T], L^2(a,b))$ given by*

$$\begin{aligned} y(t,\cdot) = &\sum_{n=1}^{\infty} \Big\{ \frac{1}{\sqrt{\lambda_n}} \langle v, \varphi_n\rangle_{L^2(a,b)} \sin(\sqrt{\lambda_n} t) + \langle y^0, \varphi_n\rangle_{L^2(a,b)} \cos(\sqrt{\lambda_n} t) \\ &+ \frac{1}{\sqrt{\lambda_n}} \int_0^t \sin(\sqrt{\lambda_n}(t-s)) \langle f(s), \varphi_n\rangle_{D(\mathcal{E})^\star, D(\mathcal{E})}\, \mathrm{d}s \Big\} \varphi_n. \end{aligned} \tag{3.6}$$

Moreover, there exists $C_1 > 0$ and $C_2 > 0$ such that,

$$\|y\|_{\mathcal{C}([0,T];D(\mathcal{E}))} \le C_1 \left(\|v\|_{L^2(a,b)} + \left\|y^0\right\|_{D(\mathcal{E})} + \|f\|_{L^2(Q)} \right), \tag{3.7a}$$

$$\left\|y'\right\|_{C([0,T],L^2(a,b))} \le C_2 \left(\|v\|_{L^2(a,b)} + \left\|y^0\right\|_{D(\mathcal{E})} + \|f\|_{L^2(Q)} \right). \tag{3.7b}$$

Proof. Let $\varphi_n \in D(A)$ be the sequence of eigenfunctions defined in Remark 2.19. Then, from the abstract Cauchy problem (3.5) and (2.27), we have that

$$\left\{\begin{array}{ll} \dfrac{\partial^2}{\partial t^2}(y(t),\varphi_n)_{L^2(\Omega)} + \mathcal{E}(y(t),\varphi_n) & = \langle f(t),\varphi_n\rangle_{D(\mathcal{E})^\star,D(\mathcal{E})} \quad \text{for} \quad t\in(0,T),\\ (y(0),\varphi_n)_{L^2(\Omega)} & = (y^0,\varphi_n)_{L^2(\Omega)},\\ \partial_t(y(0),\varphi_n)_{L^2(\Omega)} & = (v,\varphi_n)_{L^2(\Omega)}, \end{array}\right. \tag{3.8}$$

which can be rewritten as

$$\left\{\begin{array}{rl} \dfrac{\partial^2}{\partial t^2}y_n(t) + \lambda_n y_n(t) & = f_n(t) \quad \text{for} \quad t\in(0,T),\\ y_n(0) & = y_n^0,\\ y_n'(0) & = v_n, \end{array}\right. \tag{3.9}$$

where $y_n(t) = (y(t),\varphi_n)_{L^2(\Omega)}$, $f_n(t) = \langle f(t),\varphi_n\rangle_{D(\mathcal{E})^\star,D(\mathcal{E})}$, $y_n^0 = (y^0,\varphi_n)_{L^2(\Omega)}$, and $v_n = (v,\varphi_n)_{L^2(\Omega)}$.

Remark 3.3. Note that f and y^0 being respectively in $L^2(Q)$ and $D(A)$, we have

$$\|y^0\|^2_{D(\mathcal{E})} = \sum_{n=1}^{\infty}\lambda_n|y_n^0|^2 \text{ and } \|f\|^2_{L^2((0,T);D(\mathcal{E})} = \int_0^T |f_n(t)|^2 dt.$$

If we take the Laplace transform $(3.9)_1$ and use (2.4), we have that

$$\hat{y}_n(s) = \frac{v_n}{s^2+\lambda_n} + \frac{s y_n^0}{s^2+\lambda_n} + \frac{\hat{f}_n(s)}{s^2+\lambda_n}.$$

Therefore, if we take the inverse Laplace transform of this latter identity while using (2.5)-(2.7), we obtain that

$$y_n(t) = \frac{v_n}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}t) + y_n^0\cos(\sqrt{\lambda_n}t) + \frac{1}{\sqrt{\lambda_n}}\int_0^t \sin(\sqrt{\lambda_n}(t-s))f_n(s)\,\mathrm{d}s. \tag{3.10}$$

In other to prove that the series

$$y(t) = \sum_{n=1}^{\infty} y_n(t)\varphi_n = \sum_{n=1}^{\infty}\left\{\frac{v_n}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}t) + y_n^0\cos(\sqrt{\lambda_n}t) + \frac{1}{\sqrt{\lambda_n}}\int_0^t \sin(\sqrt{\lambda_n}(t-s))f_n(s)\,\mathrm{d}s\right\}\varphi_n \tag{3.11}$$

converges, we proceed in several steps.
Step 1. We approached problem (3.8).

Let V_m be the subspace of $D(A)$ generated by $\varphi_1,\cdots,\varphi_m$, we set $y_m^0 = \sum_{n=1}^{m} y_n^0\varphi_n$ and $v_m = \sum_{n=1}^{m} v_n\varphi_n$, we consider the following approached problem:

find $y_m(t) = \sum_{n=1}^{m} y_n \varphi_n \in V_m$ solution of

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}(y_m(t), \varphi_k)_{L^2(\Omega)} + \mathcal{E}(y_m(t), \varphi_k) & = \langle f(t), \varphi_k \rangle_{D(\mathcal{E})^\star, D(\mathcal{E})} \quad \text{for} \quad t \in (0,T), \quad 1 \le k \le m, \\ (y_m(0), \varphi_k)_{L^2(\Omega)} & = (y_m^0, \varphi_k)_{L^2(\Omega)}, \qquad 1 \le k \le m, \\ \partial_t (y_m(0), \varphi_k)_{L^2(\Omega)} & = (v_m, \varphi_k)_{L^2(\Omega)}, \qquad 1 \le k \le m. \end{cases} \tag{3.12}$$

Using the fact that $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$ and $\left\{\frac{\varphi_k}{\sqrt{\lambda_k}}\right\}_{k=1}^{\infty}$ is an orthonormal basis of $D(A)$, we obtain from (3.12) that y_n is solution of (3.9) for $1 \le n \le m$. It then follows from (3.10) that y_m is given by

$$y_m(t) = \sum_{n=1}^{m} \left\{ \frac{v_n}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n} t) + y_n^0 \cos(\sqrt{\lambda_n} t) + \frac{1}{\sqrt{\lambda_n}} \int_0^t \sin(\sqrt{\lambda_n}(t-s)) f_n(s)\, \mathrm{d}s \right\} \varphi_n. \tag{3.13}$$

Step 2. We show that (y_m) and (y_m') are Cauchy sequences respectively in $\mathcal{C}([0,T]; D(A))$ and $\mathcal{C}([0,T]; L^2(a,b))$.

Let m and p be two entire numbers such that $1 \le m \le p$. Then we have,

$$y_p(t) - y_m(t) = \sum_{i=m+1}^{p} y_i(t) \varphi_i, \; \forall\, t \in [0,T].$$

$$\begin{aligned} \text{Therefore, } \mathcal{E}(y_p(t) - y_m(t), y_p(t) - y_m(t)) &= \sum_{i=m+1}^{p} \lambda_i \left| y_i(t) \right|^2 \\ &\le 2 \sum_{i=m+1}^{p} \lambda_i \left(\frac{v_i}{\sqrt{\lambda_i}} \sin(\sqrt{\lambda_i} t) + y_i^0 \cos(\sqrt{\lambda_i} t) \right)^2 \\ &\quad + 2 \sum_{i=m+1}^{p} \left(\int_0^t \sin(\sqrt{\lambda_i}(t-s)) f_i(s)\, \mathrm{d}s \right)^2, \end{aligned}$$

which applying Cauchy-Schwarz inequality gives

$$\begin{aligned} \mathcal{E}(y_p(t) - y_m(t), y_p(t) - y_m(t)) &\le 2 \sum_{i=m+1}^{p} \lambda_i \left(\frac{|v_i|^2}{\lambda_i} + \left| y_i^0 \right|^2 \right) \\ &\quad + 2 \sum_{i=m+1}^{p} \left(\int_0^t \sin^2(\sqrt{\lambda_i}(t-s))\, ds \right) \left(\int_0^t |f_i(s)|^2\, \mathrm{d}s \right) \\ &\le 2 \sum_{i=m+1}^{p} \left(|v_i|^2 + \lambda_i \left| y_i^0 \right|^2 \right) + 2t \sum_{i=m+1}^{p} \left(\int_0^t |f_i(s)|^2\, \mathrm{d}s \right). \end{aligned}$$

$$\begin{aligned}\text{Hence, } \|y_p - y_m\|^2_{\mathcal{C}([0,T];D(\mathcal{E}))} &= \sup_{t\in[0,T]} |\mathcal{E}(y_p(t) - y_m(t), y_p(t) - y_m(t))| \\ &\leq 2\sum_{i=m+1}^{p} \left(|v_i|^2 + \lambda_i \left|y_i^0\right|^2\right) + 2T\sum_{i=m+1}^{p} \left(\int_0^T |f_i(s)|^2 \,\mathrm{d}s\right).\end{aligned} \tag{3.14}$$

On the other hand, using Lemma 2.10 we have

$$y'_p(t) - y'_m(t) = \sum_{i=m+1}^{p} \left\{ v_i \cos(\sqrt{\lambda_i}t) - \sqrt{\lambda_i} y_i^0 \sin(\sqrt{\lambda_i}t) + \int_0^t \cos(\sqrt{\lambda_i}(t-s)) f_i(s)\,\mathrm{d}s \right\} \varphi_i.$$

$$\begin{aligned}\text{Thus, } \left\|y'_p(t) - y'_m(t)\right\|^2_{L^2(a,b)} \leq\ & 2\sum_{i=m+1}^{p} \left(v_i \cos(\sqrt{\lambda_i}t) - \sqrt{\lambda_i} y_i^0 \sin(\sqrt{\lambda_i}t)\right)^2 \\ &+2\sum_{i=m+1}^{p} \left(\int_0^t \cos(\sqrt{\lambda_i}(t-s)) f_i(s)\,\mathrm{d}s\right)^2.\end{aligned}$$

Applying again Cauchy-Schwarz and Young inequalities, we obtain that

$$\left\|y'_p - y'_m\right\|^2_{\mathcal{C}([0,T],L^2(a,b))} \leq 2\sum_{i=m+1}^{p} \left(|v_i|^2 + \lambda_i \left|y_i^0\right|^2\right) + 2T\sum_{i=m+1}^{p} \int_0^T |f_i(s)|^2 \,\mathrm{d}s. \tag{3.15}$$

It follows from (3.14), (3.15) and Remark 3.3 that, (y_m) and (y'_m) are Cauchy sequences respectively in $\mathcal{C}([0,T];D(A))$ and $\mathcal{C}([0,T];L^2(a,b))$ thus there exists $y \in \mathcal{C}([0,T];D(A)) \cap \mathcal{C}^1([0,T];L^2(a,b))$ such that

$$y_m \to y \text{ in } \mathcal{C}([0,T];D(A)) \text{ and } y'_m \to y' \text{ in } \mathcal{C}([0,T];L^2(a,b)). \tag{3.16}$$

Remark 3.4. Since (y_m) and (y'_m) are Cauchy sequences respectively in $\mathcal{C}([0,T];D(A))$ and $\mathcal{C}([0,T],L^2(a,b))$, we have that they are bounded in $\mathcal{C}([0,T];D(A))$ and $\mathcal{C}([0,T],L^2(a,b))$ respectively. Therefore, $(0,T)$ being bounded, we have that (y_m) and (y'_m) are bounded in $L^2((0,T);D(A))$ and $L^2(Q)$ and we can write:

$$y_m \to y \text{ weakly in } L^2((0,T);D(A)), \tag{3.17a}$$

$$y'_m \to y' \text{ weakly in } L^2(Q). \tag{3.17b}$$

Step 3. We prove that y is solution of (3.5)(or equivalently (3.4).

Let $\mathcal{C}_c^\infty(0,T)$ be the space of C^∞ function in $(0,T)$ with compact support and let $\mu \geq 1$. If we multiply the first equation in (3.12) by $\rho \in \mathcal{C}_c^\infty(0,T)$ and integrate by part over $(0,T)$, we have for all $m > \nu$

$$\begin{aligned}\int_0^T \langle f(t), \varphi\rangle_{D(\mathcal{E})^\star, D(\mathcal{E})} \rho(t)dt &= \int_0^T \frac{\partial^2}{\partial t^2}(y_m(t), \varphi)_{L^2(\Omega)} \rho(t)dt + \int_0^T \mathcal{E}(y_m(t), \varphi)\rho(t)dt \\ &= \int_0^T (y_m(t), \varphi)_{L^2(\Omega)} \frac{\partial^2}{\partial t^2}\rho(t)dt \int_0^T \mathcal{E}(y_m(t), \varphi)\rho(t)\,dt \quad \forall \varphi \in V_\mu.\end{aligned}$$

Passing this latter identity through the limit while using (3.17a), we get $\forall \varphi \in$

$V_\mu, \forall \rho \in \mathcal{C}_c^\infty(0,T)$

$$\int_0^T \langle f(t), \varphi\rangle_{D(\mathcal{E})^\star, D(\mathcal{E})}\rho(t)dt = \int_0^T (y(t),\varphi)_{L^2(\Omega)}\frac{\partial^2}{\partial t^2}\rho(t)dt + \int_0^T \mathcal{E}(y(t),\varphi)\rho(t)\,dt,$$

which after an integration by parts, gives $\forall \rho \in \mathcal{C}_c^\infty(0,T)$, $\forall \varphi \in V_\mu$,

$$\int_0^T \langle f(t), \varphi\rangle_{D(\mathcal{E})^\star, D(\mathcal{E})}\rho(t)dt = \int_0^T \frac{\partial^2}{\partial t^2}(y(t),\varphi)_{L^2(\Omega)}\rho(t)dt + \int_0^T \mathcal{E}(y(t),\varphi_)\rho(t)dt,.$$

And since $\cup_{\mu\geq 1}V_\mu$ is dense in $D(A)$, we can write $\forall \varphi \in D(A), \forall \rho \in \mathcal{C}_c^\infty(0,T)$,

$$\int_0^T \langle f(t), \varphi\rangle_{D(\mathcal{E})^\star, D(\mathcal{E})}\rho(t)dt = \int_0^T \frac{\partial^2}{\partial t^2}(y(t),\varphi)_{L^2(\Omega)}\rho(t)dt + \int_0^T \mathcal{E}(y(t),\varphi)\rho(t)dt,$$

therefore, $\forall \phi \in D(A)$,

$$\frac{\partial^2}{\partial t^2}(y(t),\phi)_{L^2(\Omega)} + \mathcal{E}(y(t),\phi) = \langle f(t),\phi\rangle_{D(\mathcal{E})^\star, D(\mathcal{E})}\ ,\forall t \in (0,T). \tag{3.18}$$

And from (3.16), we have that

$$y_m(0) \to y(0) \text{ in } D(A) \text{ and } y_m'(0) \to y'(0) \text{ in } L^2(a,b).$$

But $y_m(0) = \sum_{i=1}^m y_i^0\varphi_i \to \sum_{i=1}^\infty y_i^0\varphi_i = y^0$, and $y_m'(0) = \sum_{i=1}^m v_i\varphi_i \to \sum_{i=1}^\infty v_i\varphi_i = v$ we therefore deduce from the uniqueness of the limit that

$$y(0) = y^0 \text{ in } (a,b) \text{ and } y'(0) = v \text{ in } (a,b). \tag{3.19}$$

It follows from (3.18) and (3.19) that y is solution of (3.4) in the sense of Definition 3.1.

Step 4. We show estimations (3.7a) and (3.7b).

Using (3.13) and proceeding exactly as in (3.14) and (3.15), we obtain that

$$\begin{aligned}
\|y_m\|^2_{\mathcal{C}([0,T];D(\mathcal{E}))} &= \sup_{t\in[0,T]} |\mathcal{E}(y_m(t),y_m(t))| \\
&\leq 2\sum_{i=1}^m \left(|v_i|^2 + \lambda_i\left|y_i^0\right|^2\right) + 2T\sum_{i=1}^m\left(\int_0^T |f_i(s)|^2\,\mathrm{d}s\right) \\
&\leq 2\|v\|^2_{L^2(\Omega)} + 2\left\|y^0\right\|^2_{D(\mathcal{E})} + 2T\|f\|^2_{L^2(Q)}.
\end{aligned}$$

$$\begin{aligned}
\text{and}\quad \|y_m'\|^2_{\mathcal{C}([0,T],L^2(a,b))} &\leq 2\sum_{i=1}^m \left(|v_i|^2 + \lambda_i\left|y_i^0\right|^2\right) + 2T\sum_{i=1}^m\int_0^T |f_i(s)|^2\,\mathrm{d}s \\
&\leq 2\|v\|^2_{L^2(\Omega)} + 2\left\|y^0\right\|^2_{D(\mathcal{E})} + 2T\|f\|^2_{L^2(Q)}.
\end{aligned}$$

Because of Remark 3.3. Therefore, using (3.16) while passing these latter estimations to the limit when $m \to +\infty$, we have (3.7a) and (3.7b).

$\square$

We have the following result which proves that the energy does not vary with respect to the time.

Proposition 3.5. *Let $f \equiv 0$, $y^0 \in D(A), v \in L^2(a,b)$. The energy of a weak solution y of* (3.4) *is the continuous function defined by*

$$E_y(t) = \frac{1}{2}\int_a^b (y_t(t,x))^2 + \beta(x)(\mathbb{D}^\alpha_{a+}y(x,t))^2 + q(y^2(x,t))\ \mathrm{d}x,$$

which satisfies the following so called energy estimate,

$$E_y(t) = E_y(0). \tag{3.20}$$

Proof. If we multiply (3.4) by y_t and integrate by parts over $(0,s)\times(a,b)$, we obtain

$$\begin{aligned}
0 &= \int_0^s \int_a^b y_{tt}y_t\, dxdt + \int_0^t \int_a^b \mathcal{D}^\alpha_{b-}(\beta\mathbb{D}^\alpha_{a+}y)y_t\, dxdt + \int_0^t \int_a^b qyy_t\, dxdt \\
&= \frac{1}{2}\int_0^s \int_a^b \partial_t\, |y_t(t)|^2\ dx\mathrm{d}t + \frac{1}{2}\int_0^s \int_a^b \partial_t \left|\sqrt{\beta}\mathbb{D}^\alpha_{a+}y(t)\right|^2 dxdt + \frac{1}{2}\int_0^s \int_a^b \partial_t(\sqrt{q}y(t))^2\, dxdt \\
&= \frac{1}{2}\int_a^b y_t^2(s) - y_t^2(0)\, dx + \frac{1}{2}\int_a^b \left|\sqrt{\beta}\mathbb{D}^\alpha_{a+}y(s)\right|^2 - \left|\sqrt{\beta}\mathbb{D}^\alpha_{a+}y(0)\right|^2 dx \\
&+ \frac{1}{2}\int_a^b |\sqrt{q}y(s)|^2 - \left|\sqrt{q}y^0\right|^2 dx \\
&= E_y(s) - E_y(0).
\end{aligned}$$

$\square$

Remark 3.6. Note that, we can prove the uniqueness of the weak solution to (3.4) by using the non variation of the energy.

Remark 3.7. Note that if y^0 belongs to $L^2(a,b)$, then the homogeneous problem (3.4) admits a unique weak solution $y \in L^2((0,T);D(\mathcal{E})) \cap \mathcal{C}^1([0,T],L^2(a,b))$.

3.2. Non Homogeneous wave equation

Now we consider the non-homogeneous problem given in (3.1).

Lemma 3.8. *Let β, q satisfy Assumption 2.16, let $g \in H^2(0,T)$, with $g(0) = 0$, we consider the function w defined by*

$$w(x,t) = \frac{g(t)}{\beta(b)\Gamma(\alpha+1)}(x-a)^\alpha,$$

then the functions w, w_{tt} and $\mathcal{D}^\alpha_{b-}\beta\mathbb{D}^\alpha_{a+}w$ belong to $L^2(Q)$ and the function w_t belongs to $\mathcal{C}([0,T];L^2(a,b))$, furthermore, $w \in \mathcal{C}([0,T],D(\mathcal{E}))$.

Proof. Since $g \in L^2(0,T)$ and $(x-a)^\alpha \in L^2(a,b)$, while using $\mathbb{D}^\alpha_{a+}w(x,t) = \frac{g(t)}{\beta(b)}$, it follows that

$$\begin{array}{rl}\|w\|_{L^2(Q)} & \leq C(\alpha,\beta_0)\,\|g\|_{L^2(0,T)}\,,\ \ \|w\|_{\mathcal{C}([0,T];D(\mathcal{E}))} \leq C(\alpha,\beta_0)\,\|g\|_{\mathcal{C}([0,T])} \\ \|w_t\|_{\mathcal{C}([0,T];L^2(a,b))} & \leq C(\alpha,\beta_0)\,\|g_t\|_{\mathcal{C}([0,T])}\,,\ \ \|w_{tt}\|_{L^2(Q)} \leq C(\alpha,\beta_0)\,\|g_{tt}\|_{L^2(0,T)}\end{array} \tag{3.21}$$

. and $\left\|\mathcal{D}^{\alpha}_{b^-}\beta\mathbb{D}^{\alpha}_{a^+}w\right\|_{L^2(Q)} = \left\|I^{1-\alpha}_{b^-}\partial_x\left(\beta\mathbb{D}^{\alpha}_{a^+}w\right)\right\|_{L^2(Q)} \leq C(\alpha,\beta_0,\|\beta'\|_\infty)\,\|g\|_{L^2(0,T)}$. $\square$

If we set $z(x,t) = y(x,t) - w(x,t)$, then z satisfies the homogeneous system

$$\left\{\begin{array}{lcllr} z_{tt} + \mathcal{D}^{\alpha}_{b^-}\beta\mathbb{D}^{\alpha}_{a^+}z + qz & = & f - w_{tt} - \mathcal{D}^{\alpha}_{b^-}\beta\mathbb{D}^{\alpha}_{a^+}w - qw & \text{in} & Q, \\ I^{1-\alpha}_{a^+}z(a^+,\cdot) & = & 0 & \text{in} & (0,T), \\ \beta(b)\mathbb{D}^{\alpha}_{a^+}z(b^-,\cdot) & = & 0 & \text{in} & (0,T), \\ z(\cdot,0) & = & y^0 & \text{in} & (a,b), \\ z_t(.,0) & = & v - w_t(\cdot,0) & \text{in} & (a,b). \end{array}\right. \tag{3.22}$$

Theorem 3.9. *Let $1/2 < \alpha < 1$, $f \in L^2(Q)$, $g \in H^2(0,T)$ with $g(0) = 0$, $y^0 \in D(A)$ and $v \in L^2(a,b)$, then the problem (3.1) admits a unique weak solution $y \in \mathcal{C}([0,T],D(\mathcal{E})) \cap \mathcal{C}^1([0,T],L^2(a,b))$. Moreover, the following estimates hold true*

$$\|y\|_{\mathcal{C}([0,T];D(\mathcal{E}))} \leq C(\alpha,\beta_0,\left\|\beta'\right\|_\infty,\overline{q})\left(\|v\|_{L^2(a,b)} + \left\|y^0\right\|_{D(\mathcal{E})} + \|g\|_{H^2(0,T)} + \|f\|_{L^2(Q)}\right) \tag{3.23a}$$

$$\left\|y'\right\|_{C([0,T],L^2(a,b))} \leq C(\alpha,\beta_0,\left\|\beta'\right\|_\infty,\overline{q})\left(\|v\|_{L^2(a,b)} + \left\|y^0\right\|_{D(\mathcal{E})} + \|g\|_{H^2(0,T)} + \|f\|_{L^2(Q)}\right), \tag{3.23b}$$

Proof. From Theorem 3.2, we have that the homogeneous problem (3.22) admits a unique weak solution $z \in \mathcal{C}([0,T],D(A)) \cap \mathcal{C}^1([0,T],L^2(a,b))$ given by

$$z(x,t) = \sum_{n=1}^{\infty}\left\{\frac{\tilde{v}_n}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}t) + y^0_n\cos(\sqrt{\lambda_n}t) + \frac{1}{\sqrt{\lambda_n}}\int_0^t \sin(\sqrt{\lambda_n}(t-s))\tilde{f}_n(s)\,\mathrm{d}s\right\}\varphi_n, \tag{3.24}$$

where $\tilde{f} = f - w_{tt} - \mathcal{D}^{\alpha}_{b^-}\beta\mathbb{D}^{\alpha}_{a^+}w - qw$ and $\tilde{v} = v - w_t(\cdot,0)$.

Since $w \in \mathcal{C}^1([0,T],D(\mathcal{E}))$, using the fact that the function y can be uniquely determined by the equality $y(x,t) = z(x,t)+w(x,t)$, we have that $y \in \mathcal{C}([0,T],D(\mathcal{E}))\cap \mathcal{C}^1([0,T],L^2(a,b))$.

Thus, according to estimates (3.7a) and (3.7b), and using (3.21), it follows that

$$\begin{array}{rcl}\|y\|_{C([0,T];D(\mathcal{E}))} & \leq & \|z\|_{\mathcal{C}([0,T];D(\mathcal{E}))} + \|w\|_{C([0,T];D(\mathcal{E}))} \\ & \leq & C_1\left(\|\tilde{v}\|_{L^2(a,b)} + \left\|y^0\right\|_{D(\mathcal{E})} + \left\|\tilde{f}\right\|_{L^2(Q)}\right) + C(\alpha,\beta_0)\,\|g\|_{L^2(0,T)} \\ & \leq & C(\alpha,\beta_0,\left\|\beta'\right\|_\infty,\overline{q})\left(\|v\|_{L^2(a,b)} + \left\|y^0\right\|_{D(\mathcal{E})} + \|f\|_{L^2(Q)} + \|g\|_{H^2(0,T)}\right)\end{array}$$

and

$$\begin{aligned}
\|y\|_{\mathcal{C}^1([0,T];L^2(a,b))} &\leq \|z\|_{\mathcal{C}^1([0,T];L^2(a,b))} + \|w\|_{\mathcal{C}^1([0,T];L^2(a,b))} \\
&\leq C_2 \left(\|\tilde{v}\|_{L^2(a,b)} + \left\|y^0\right\|_{D(\mathcal{E})} + \left\|\tilde{f}\right\|_{L^2(Q)} \right) + C(\alpha, \beta_0) \|g_t\|_{\mathcal{C}([0,T])} \\
&\leq C(\alpha, \beta_0, \left\|\beta'\right\|_\infty, \overline{q}) \left(\|v\|_{L^2(a,b)} + \left\|y^0\right\|_{D(\mathcal{E})} + \|f\|_{L^2(Q)} + \|g\|_{H^2(0,T)} \right).
\end{aligned}$$

The proof is concluded. □

3.3. Study of the optimal control problem on one edge

We are concerned with optimal control problem (3.1)-(3.3). Note that, y solution to (3.1) being in $\mathcal{C}([0,T], D(\mathcal{E})) \cap \mathcal{C}^1([0,T], L^2(a,b))$, the cost function J in (3.3) is well defined.

Theorem 3.10. *Assume that Assumption 2.16 holds. Let $1/2 < \alpha < 1$. Then, there exists a unique solution $u \in \mathcal{U}_{ad}$ of the optimal control problem* (3.1)-(3.3)*.*

Proof. From (3.3), we have We have that:

$$\begin{aligned}
J(v) &= \frac{1}{2} \left\|y(v;T) - z_d^0\right\|^2_{L^2(a,b)} + \frac{1}{2} \left\|y_t(v;T) - z_d^1\right\|^2_{L^2(a,b)} + \frac{N}{2} \|v\|^2_{L^2(a,b)} \\
&= \frac{1}{2}\pi(v,v) - L(v) + \frac{1}{2} \left\|y(0;T) - z_d^0\right\|^2_{L^2(a,b)} + \frac{1}{2} \left\|y_t(0;T) - z_d^1\right\|^2_{L^2(a,b)}
\end{aligned} \tag{3.25}$$

where the symmetric bilinear functional $\pi(\cdot,\cdot)$ and the linear functional $L(\cdot)$ defined on $\mathcal{U}_{ad}$ is given by

$$\pi(v,u) = \langle \psi(v;T), \psi(u;T)\rangle_{L^2(a,b)} + \langle \psi_t(v;T), \psi_t(u;T)\rangle_{L^2(a,b)} + N\langle u, v\rangle_{L^2(a,b)}$$

$$\text{and } \; L(u) = \langle \psi(u;T), z_d^0 - y(0;T)\rangle_{L^2(a,b)} + \langle \psi_t(u;T), z_d^1 - y_t(0;T)\rangle_{L^2(a,b)}$$

where $\psi(v) = y(v) - y(0)$ satisfies the system

$$\left\{\begin{array}{lcll}
\psi_{tt} + \mathcal{D}^\alpha_{b^-} (\beta\, \mathbb{D}^\alpha_{a^+}\psi) + q\, \psi &=& 0 & \text{in} \quad (a,b)\times(0,T), \\
(I^{1-\alpha}_{a^+}\psi)(a^+,\cdot) &=& 0 & \text{in} \quad (0,T), \\
(\beta\mathbb{D}^\alpha_{a^+}\psi)(b^-,\cdot) &=& 0 & \text{in} \quad (0,T), \\
\psi(\cdot,0) &=& 0 & \text{in} \quad (a,b). \\
\psi_t(\cdot,0) &=& v & \text{in} \quad (a,b).
\end{array}\right. \tag{3.26}$$

It follow from Theorem 3.2 that $\psi \in \mathcal{C}([0,T], D(A)) \cap \mathcal{C}^1([0,T], L^2(a,b))$ and there exist $\Delta_1 > 0$ and $\Delta_2 > 0$ such that

$$\|\psi\|_{\mathcal{C}([0,T];D(\mathcal{E}))} \leq \Delta_1 \|v\|_{L^2(a,b)} \tag{3.27a}$$

$$\|\psi_t\|_{C([0,T],L^2(a,b))} \leq \Delta_2 \|v\|_{L^2(a,b)}. \tag{3.27b}$$

Using Cauchy-Schwartz inequality, we obtain for all $u, v \in \mathcal{U}_{ad}$ that,

$$\begin{aligned}|\pi(u,v)| &\leq \|\psi(v;T)\|_{L^2(a,b)} \|\psi(u;T)\|_{L^2(a,b)} + \|\psi_t(v;T)\|_{L^2(a,b)} \|\psi_t(u;T)\|_{L^2(a,b)} \\ &+N \|u\|_{L^2(a,b)} \|v\|_{L^2(a,b)} \\ &\leq \left(\Delta_1^2 + \Delta_2^2 + N\right) \|u\|_{L^2(a,b)} \|v\|_{L^2(a,b)} .\end{aligned}$$

This means that $\pi(\cdot,\cdot)$ is continuous on $\mathcal{U}_{ad} \times \mathcal{U}_{ad}$. On the other hand, we have for all $v \in \mathcal{U}_{ad}$,

$$\pi(v,v) = \|\psi(v;T)\|^2_{L^2(a,b)} + \|\psi_t(v;T)\|^2_{L^2(a,b)} + N\|v\|^2_{L^2(a,b)} \geq N\|v\|^2_{L^2(a,b)} .$$

And using Cauchy-Schwartz inequality,

$$\begin{aligned}|L(v)| &\leq \|\psi(v;T)\|_{L^2(a,b)} \left\|z_d^0 - y(0;T)\right\|_{L^2(a,b)} + \|\psi_t(v;T)\|_{L^2(a,b)} \left\|z_d^1 - y_t(0;T)\right\|_{L^2(a,b)} \\ &\leq C(\Delta_1 + \Delta_2) \|v\|_{L^2(a,b)} .\end{aligned}$$

This prove that $\pi(\cdot,\cdot)$ is coercive on $\mathcal{U}_{ad}$ and the functional $L(.)$ is continuous on $\mathcal{U}_{ad}$.

In short we have proved that the symmetric bilinear functional $\pi(\cdot,\cdot)$ is continuous and coercive on $\mathcal{U}_{ad} \times \mathcal{U}_{ad}$ and the linear functional $L(.)$ is continuous on $\mathcal{U}_{ad}$. Therefore, the Lax-Milgram allows to say that there exists a unique $u \in \mathcal{U}_{ad}$ such that

$$\frac{1}{2}\pi(u,u) - L(u) = \inf_{v\in\mathcal{U}_{ad}} \frac{1}{2}\pi(v,v) - L(v),$$

which in view of the expression of J given by (3.25) implies that

$$J(u) = \underset{v\in\mathcal{U}_{ad}}{inf}\ J(v)$$

because $\frac{1}{2}\left\|y(0;T) - z_d^0\right\|^2_{L^2(a,b)} + \frac{1}{2}\left\|y_t(0;T) - z_d^1\right\|^2_{L^2(a,b)}$ does not depend on v. □

Theorem 3.11. *Let $1/2 < \alpha < 1$. Let β, q satisfy Assumption 2.16. Let $u \in \mathcal{U}_{ad}$ be the optimal control for the problem* (3.1)-(3.3). *Then, there exists a unique $p \in \mathcal{C}^1([0,T]; L^2(a,b))$ such that the triple $(y = y(u), u, p)$ satisfies*

$$\left\{\begin{array}{lllll} y_{tt} + \mathcal{D}^\alpha_{b^-}\left(\beta\, \mathbb{D}^\alpha_{a^+} y\right) + q\, y &=& f & in & Q, \\ I^{1-\alpha}_{a^+} y(a^+,\cdot) &=& 0 & in & (0,T), \\ \beta \mathbb{D}^\alpha_{a^+} y(b^-,\cdot) &=& g & in & (0,T), \\ y(\cdot,0) &=& y^0 & in & (a,b), \\ y_t(\cdot,0) &=& u & in & (a,b), \end{array}\right. \tag{3.28}$$

$$\left\{\begin{array}{lllll} p_{tt} + \mathcal{D}^\alpha_{b^-}\left(\beta\, \mathbb{D}^\alpha_{a^+} p\right) + q\, p &=& 0 & in & Q, \\ I^{1-\alpha}_{a^+} p(a^+,\cdot) &=& 0 & in & (0,T), \\ \beta \mathbb{D}^\alpha_{a^+} p(b^-,\cdot) &=& 0 & in & (0,T), \\ p(\cdot,T) &=& y_t(\cdot,T) - z_d^1 & in & (a,b), \\ p_t(\cdot,T) &=& -(y(\cdot,T) - z_d^0) & in & (a,b) \end{array}\right. \tag{3.29}$$

and

$$\int_a^b \Big(Nu + p(\cdot,0)\Big)(v-u)\;dx \geq 0 \quad \forall v \in \mathcal{U}_{ad}. \tag{3.30}$$

Proof. Since $u \in \mathcal{U}_{ad}$ is the optimal control for the problem (3.1)-(3.3), we have that $y = y(u)$ is solution of (3.28). To complete the proof of the theorem, we write the Euler-Lagrange first order optimality condition that characterizes the optimal control u. That is,

$$\lim_{\theta \to 0} \frac{J(u+\theta(v-u)) - J(u)}{\theta} \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \tag{3.31}$$

which after some straightforward calculations gives

$$\begin{aligned} \langle z(v-u;T), y(u;T) - z_d^0\rangle_{L^2(a,b)} + \langle z_t(v-u;T), y_t(u;T) - z_d^1\rangle_{L^2(a,b)} + \\ N\int_a^b u\,(v-u)dx \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \end{aligned} \tag{3.32}$$

where $z = z(v-u) \in \mathcal{C}([0,T], D(A)) \cap \mathcal{C}^1([0,T], L^2(a,b))$ is the solution of

$$\left\{\begin{array}{lcll} z_{tt} + \mathcal{D}^{\alpha}_{b^-}\,(\beta\,\mathbb{D}^{\alpha}_{a^+} z) + q\,z & = & 0 & \text{in} \quad Q, \\ I^{1-\alpha}_{a^+} z(a^+,\cdot) & = & 0 & \text{in} \quad (0,T), \\ \beta\mathbb{D}^{\alpha}_{a^+} z(b^-,\cdot) & = & 0 & \text{in} \quad (0,T), \\ z(\cdot,0) & = & 0 & \text{in} \quad (a,b), \\ z_t(\cdot,0) & = & v-u & \text{in} \quad (a,b). \end{array}\right. \tag{3.33}$$

To interpret (3.32), we use the adjoint state given by (3.29). Since $y(T) - z_d^0 \in L^2(a,b)$ and $z_d^1 - y_t(T) \in L^2(a,b)$, we have from Remark 3.7 that there exists a unique adjoint state $p \in L^2((0,T); D(\mathcal{E})) \cap \mathcal{C}^1([0,T], L^2(a,b))$ solution to (3.29). So, by multiplying the first equation in (3.33) by p solution of (3.29), and integrate by parts over Q, we get

$$\langle z_t(v-u;T), y_t(u;T) - z_d^1\rangle_{L^2(a,b)} + \langle z(v-u;T), y(u;T) - z_d^0\rangle_{L^2(a,b)} - \int_a^b (v-u)p(\cdot,0)dx = 0. \tag{3.34}$$

Combining (3.32)-(3.34), we get (3.30). The proof is complete. $\square$

4. Boundary optimal control problems on general star graphs

For this section, we generalize the wave equation on a general star graph. For every real number $T > 0$, we consider the following fractional Sturm-Liouville problem:

$$\left\{\begin{array}{llll} y^i_{tt} + \mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+} y^i) + q^i y^i & = & f^i & \text{in } \ Q_i,\ i = 1, \dots, n, \\ I^{1-\alpha}_{a^+} y^i(a^+, \cdot) & = & I^{1-\alpha}_{a^+} y^j(a^+, \cdot) & \text{in } \ (0, T),\ i \neq j = 1, \dots, n, \\ \displaystyle\sum_{i=1}^{n} \beta^i(a) \mathbb{D}^{\alpha}_{a^+} y^i(a^+, \cdot) & = & 0 & \text{in } \ (0, T), \\ I^{1-\alpha}_{a^+} y^1(b_1^-, \cdot) & = & 0 & \text{in } \ (0, T), \\ I^{1-\alpha}_{a^+} y^i(b_i^-, \cdot) & = & g_i & \text{in } \ (0, T),\ i = 2, \dots, m, \\ \beta^i(b_i) \mathbb{D}^{\alpha}_{a^+} y^i(b_i^-, \cdot) & = & h_i & \text{in } \ (0, T),\ i = m+1, \dots, n, \\ y^i(0) & = & y^{0,i} & \text{in } \ (a, b_i), \quad i = 1, \dots, n, \\ y^i_t(0) & = & v_i & \text{in } \ (a, b_i), \quad i = 1, \dots, n, \end{array}\right. \tag{4.1}$$

where $a \in \mathbb{R}$ and $b_i \in \mathbb{R}$, $i = 1, \cdots, n$, $Q_i = (a, b_i) \times (0, T)$, $1/2 < \alpha < 1$, $f_i \in L^2(Q_i)$, $g_i, h_i \in H^2(0, T)$, $i = 1, \cdots n$ and the control functions $v_i \in L^2(a, b_i)$, $i = 1, \cdots n.$. From now on, we set

$$\mathbb{L}^2 := \prod_{i=1}^{n} L^2(a, b_i) \text{ and } \mathbb{H}^{\alpha}_{a} := \prod_{i=1}^{n} H^{\alpha}_{a^+}(a, b_i).$$

Then, we respectively endow $\mathbb{L}^2$ and $\mathbb{H}^{\alpha}_{a}$ with the norms

$$\|\rho\|^2_{\mathbb{L}^2} = \sum_{i=1}^{n} \|\rho^i\|^2_{L^2(a, b_i)}, \quad \rho = (\rho^i)_i \in \mathbb{L}^2 \tag{4.2}$$

and

$$\|\rho\|^2_{\mathbb{H}^{\alpha}_{a}} = \sum_{i=1}^{n} \|\rho^i\|^2_{H^{\alpha}_{a^+}(a, b_i)} = \sum_{i=1}^{n} \left(\left\|\rho^i\right\|^2_{L^2(a, b_i)} + \left\|\mathbb{D}^{\alpha}_{a^+} \rho^i\right\|^2_{L^2(a, b_i)} \right), \quad \rho = (\rho^i)_i \in \mathbb{H}^{\alpha}_{a}. \tag{4.3}$$

We define the space $\mathcal{V}_i$ by

$$\mathcal{V}_i := \left\{ \rho^i \in H^{\alpha}_{a^+}(a, b_i) : \mathcal{D}^{\alpha}_{b^-}(\beta^i \mathbb{D}^{\alpha}_{a^+} \rho^i) \in H^{1-\alpha}_{b^-}(a, b_i) \right\} \tag{4.4}$$

and we set

$$\mathbb{V} = \left\{ (\rho^i)_i \in \prod_{i=1}^{n} \mathcal{V}_i : \sum_{i=1}^{n} \beta^i(a) \mathbb{D}^{\alpha}_{a^+} \rho^i(a^+, \cdot) = 0, I^{1-\alpha}_{a^+} \rho^i(a^+, \cdot) = I^{1-\alpha}_{a^+} \rho^j(a^+, \cdot), i \neq j,\ i, j = 1, \dots, n. \right\}. \tag{4.5}$$

Remark 4.1. Note that $\mathbb{V} \subset \mathbb{H}^{\alpha}_{a}$. Therefore, we endow $\mathbb{V}$ with norm on $\mathbb{H}^{\alpha}_{a}$ given by (4.3), i.e.:

$$\|\rho\|_{\mathbb{V}} = \|\rho\|_{\mathbb{H}^{\alpha}_{a}} \quad \forall \rho = (\rho^i) \in \mathbb{V}.$$

As for the case of a single edge studied in the previous section, to investigate the minimization problem (4.1)-(4.27), we need some preliminary results. We start with some existence and regularity results.

We make the following assumption.

Assumption 4.2. We assume that the functions $\beta^i \in \mathcal{C}^1([a,b_i])$, $i=1,\cdots,n$ and $q^i \in \mathcal{C}([a,b_i])$, $i=1,\cdots,n$ are such that

$$\underline{q^0} := \min_{1\le i\le n} q^i, \quad \underline{\beta^0} := \min_{1\le i\le n} \beta^i, \tag{4.6a}$$

$$\overline{q} := \max_{1\le i\le n} \|q^i\|_\infty, \quad \overline{\beta} := \max_{1\le i\le n} \|\beta^i\|_\infty, \tag{4.6b}$$

where $\|\beta^i\|_\infty := \max\limits_{x\in[a,b_i]} |\beta^i(x)|$ and $\|q^i\|_\infty := \sup\limits_{x\in(a,b_i)} |q^i(x)|$.

Then, we define the self adjoint operator $\mathcal{A}$ as follows: For $\rho = (\rho^i)_i$, we let

$$\mathcal{A}\rho = ((\mathcal{A}\rho)^i)_i, \ \ i=1,\dots,n, \tag{4.7}$$

where each component is given by

$$(\mathcal{A}\rho)^i = \mathcal{D}^\alpha_{b^-}(\beta^i \mathbb{D}^\alpha_{a^+}\rho^i) + q^i\rho^i, \qquad i=1,\cdots,n, \tag{4.8}$$

with

$$\begin{aligned} D(\mathcal{A}) \ := \ & \Big\{ (\rho^i)_i \in \prod_{i=1}^n \mathcal{V}_i : \sum_{i=1}^n \beta^i(a)\mathbb{D}^\alpha_{a^+}\rho^i(a^+,\cdot) = 0, \\ & I^{1-\alpha}_{a^+}\rho^i(a^+,\cdot) = I^{1-\alpha}_{a^+}\rho^j(a^+,\cdot),\ i\neq j,\ i,j=1,\dots,n, \\ & I^{1-\alpha}_{a^+}\rho^i(b_i^-,\cdot) = 0,\ i=1,\dots,m,\ \beta^i(b_i)\mathbb{D}^\alpha_{a^+}\rho^i(b_i^-,\cdot)=0,\ i=m+1,\dots,n \Big\}. \end{aligned} \tag{4.9}$$

We endowed $D(\mathcal{A})$ with the norm:

$$\|\phi\|^2_{D(\mathcal{A})} := \sum_{i=1}^n \|\phi^i\|^2_{H^\alpha_{a^+}(a,b_i)} = \|\phi\|^2_{\mathbb{H}^\alpha_a}. \tag{4.10}$$

It is clear that $D(\mathcal{A})$ is a closed subspace of $\mathbb{H}^\alpha_a$. For any $\rho,\phi \in D(\mathcal{A})$, we define the bilinear functional $\mathbb{F}(\cdot,\cdot) : D(\mathcal{A})\times D(\mathcal{A}) \to \mathbb{R}$ by:

$$\begin{aligned} \mathbb{F}(\rho,\phi) \ = \ & \sum_{i=1}^n \int_{Q_i} \beta^i(x)\mathbb{D}^\alpha_{a^+}\phi^i(x,t)\mathbb{D}^\alpha_{a^+}\rho^i(x,t)dxdt \\ + \ & \sum_{i=1}^n \int_{Q_i} q^i(x)\phi^i(x,t)\rho^i(x,t)dx\,dt. \end{aligned} \tag{4.11}$$

Lemma 4.3. *under the Assumption 4.2, the symmetric bilinear form $\mathbb{F}(\cdot,\cdot)$ given by (4.11) is continuous and coercive on $D(\mathcal{A})\times D(\mathcal{A})$.*

Proof. For every $\rho,\phi\in D(\mathcal{A})$, we have

$$
\begin{aligned}
|\mathbb{F}(\rho,\phi)| &\leq \sum_{i=1}^{n}\int_0^T \|\beta^i\|_\infty \|\mathbb{D}^\alpha_{a^+}\phi^i(t)\|_{L^2(a,b_i)}\|\mathbb{D}^\alpha_{a^+}\rho^i(t)\|_{L^2(a,b_i)}dt \\
&+ \sum_{i=1}^{n}\int_0^T \|q^i\|_\infty \|\phi^i(t)\|_{L^2(a,b_i)}\|\rho^i(t)\|_{L^2(a,b_i)}\,dt \\
&\leq \left(\max_{1\leq i\leq n}\|\beta^i\|_\infty + \max_{1\leq i\leq n}\|q^i\|_\infty\right)\|\phi\|_{L^2((0,T);\mathbb{H}^\alpha_a)}\|\rho\|_{L^2((0,T);\mathbb{H}^\alpha_a)} \\
&\leq \left(\overline{q}+\overline{\beta}\right)\|\phi\|_{L^2((0,T);\mathbb{H}^\alpha_a)}\|\rho\|_{L^2((0,T);\mathbb{H}^\alpha_a)}.
\end{aligned}
$$

Thus $\mathbb{F}(\cdot,\cdot)$ is continuous on $D(\mathcal{A})\times D(\mathcal{A})$.
Now For every $\phi\in D(\mathcal{A})$,

$$
\begin{aligned}
\mathbb{F}(\phi,\phi) &\geq \sum_{i=1}^{n}\int_0^T \min_{1\geq i\geq n}\beta^i\|\mathbb{D}^\alpha_{a^+}\phi^i(t)\|^2_{L^2(a,b_i)}\,dxdt + \sum_{i=1}^{n}\int_0^T \min_{1\geq i\geq n} q^i\|\phi^i(t)\|^2_{L^2(a,b_i)}\,dt \\
&\geq \left(\min_{1\geq i\geq n}\beta^i + \min_{1\geq i\geq n} q^i\right)\|\phi\|^2_{L^2((0,T);\mathbb{H}^\alpha_a)} \\
&\geq \left(\underline{q}^0+\underline{\beta}^0\right)\|\phi\|^2_{L^2((0,T);\mathbb{H}^\alpha_a)}.
\end{aligned}
$$

This means that $\mathbb{F}(\cdot,\cdot)$ is coercive on $D(\mathcal{A})$. □

Remark 4.4. In view of Lemma 4.3 and the embedding (2.18), the operator $\mathcal{A}$ has a compact resolvent. Let $(\mu_k)_k$ be the eigenvalues of $\mathcal{A}$ with associated eigenfunctions $(\zeta_k)=(\zeta^i_k)_{1\leq i\leq n}$. It follows from the coercivity and the nonnegativity of $\mathbb{F}(\cdot,\cdot)$ that

$$
0<\mu_1\leq\mu_2\leq\cdots\leq\mu_k\leq\cdots \quad \text{with} \quad \lim_{k\to\infty}\mu_k=+\infty.
$$

4.1. Homogeneous boundary fractional Sturm–Liouville parabolic equations in a star graph

We consider the following fractional Sturm-Liouville boundary value problem on a general star graph:

$$
\left\{
\begin{array}{lcll}
y^i_{tt}+\mathcal{D}^\alpha_{b_i^-}(\beta^i\mathbb{D}^\alpha_{a^+}y^i)+q^iy^i &=& f^i & \text{in } \; Q_i,\; i=1,\dots,n,\\
I^{1-\alpha}_{a^+}y^i(a^+,\cdot) &=& I^{1-\alpha}_{a^+}y^j(a^+,\cdot) & \text{in } \; (0,T),\; i\neq j=1,\dots,n,\\
\displaystyle\sum_{i=1}^{n}\beta^i(a)\mathbb{D}^\alpha_{a^+}y^i(a^+,\cdot) &=& 0 & \text{in } \; (0,T),\\
I^{1-\alpha}_{a^+}y^i(b_i^-,\cdot) &=& 0 & \text{in } \; (0,T),\; i=1,\dots,m\\
\beta^i(b_i)\mathbb{D}^\alpha_{a^+}y^i(b_i^-,\cdot) &=& 0 & \text{in } \; (0,T),\; i=m+1,\dots,n,\\
y^i(0) &=& y^{0,i} & \text{in } \; (a,b_i),\;\; i=1,\dots,n,\\
y^i_t(0) &=& v_i & \text{in } \; (a,b_i),\;\; i=1,\dots,n,
\end{array}
\right.
\tag{4.12}
$$

where $f=(f^i)\in L^2((0,T);\mathbb{V}^\star)$, $y^0=(y^{0,i}\in D(\mathcal{A})$, $v=(v_i)\in\mathbb{L}^2$.

Using the definition of the operator $\mathcal{A}$ above, we have that the Cauchy problem (4.12) can be rewritten as the following abstract Cauchy problem

$$
\left\{
\begin{array}{lcl}
y_{tt}+\mathcal{A}y &=& f(t) \quad \text{for} \quad t\in(0,T),\\
y(\cdot,0) &=& y^0,\\
y_t(\cdot,0) &=& v,
\end{array}
\right.
\tag{4.13}
$$

where $y=(y^i)_{i=1,\cdots,n}$, $f=(f^i)_{i=1,\cdots,n}$, $y^0=(y^{0,i})_{i=1,\cdots,n}$, $v=(v_i)_{i=1,\cdots,n}$.

Definition 4.5. A function $y = (y^i)$ is said to be a weak solution of (4.12) in $(0,T)$, $T > 0$, if the following assertions hold.

- Regularity: $y \in C^1([0,T];\mathbb{L}^2) \cap C([0,T];D(\mathcal{A}))$
- Initial condition: $\begin{array}{ll} y^i(\cdot,0) = y^{i,0} & \text{in } (a,b_i),\ i=1,\cdots,n \\ y^i_t(\cdot,0) = v_i & \text{in } (a,b_i),\ i=1,\cdots,n \end{array}$
- Variational identity: $\langle y_{tt}, \zeta\rangle_{\mathbb{V}^\star,\mathbb{V}} + \mathbb{F}(y(t,\cdot),\zeta) = \langle f(t,\cdot),\varphi\rangle_{\mathbb{V}^\star,\mathbb{V}}$ for every $\varphi \in D(\mathcal{A})$, and a.e. $t \in (0,T)$.

Theorem 4.6. *Assume that Assumption 4.2 holds. Let $1/2 < \alpha < 1$, $f \in L^2((0,T);\mathbb{V}^\star)$ and $y^0 \in D(\mathcal{A}), v \in \mathbb{L}^2$, the system (4.12) has a unique weak solution $y = (y^i)$ given by*

$$\begin{aligned} y(t,\cdot) =& \sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{\mu_n}}\langle v,\zeta_n\rangle_{\mathbb{L}^2}\sin(\sqrt{\mu_n}t) + \langle y^0,\zeta_n\rangle_{\mathbb{L}^2}\cos(\sqrt{\mu_n}t)\right. \\ &\left. + \frac{1}{\sqrt{\mu_n}}\int_0^t \sin(\sqrt{\mu_n}(t-s))\langle f(s),\zeta_n\rangle_{\mathbb{V}^\star,\mathbb{V}}\,\mathrm{d}s\right\}\zeta_n. \end{aligned} \tag{4.14}$$

Moreover,

$$\frac{1}{2}\|y_t\|^2_{\mathcal{C}([0,T];\mathbb{L}^2)} \le 2T\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \max\left(\overline{\beta},\overline{q}\right)\|y^0\|^2_{\mathbb{H}^\alpha_a} \tag{4.15a}$$

$$\|y\|^2_{\mathcal{C}([0,T];\mathbb{H}^\alpha_a))} \le \frac{2}{\min\left(\underline{\beta^0},\underline{q^0}\right)}\left(2T\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \max\left(\overline{\beta},\overline{q}\right)\|y^0\|^2_{\mathbb{H}^\alpha_a}\right) \tag{4.15b}$$

Proof. For the existence, we proceed exactly as for the one edge case. Let us prove (4.15).
If we multiply equation (4.12) by y_t and integrate by parts over (a,b_i), we obtain for all $t \in [0,T]$ that

$$\begin{aligned} \sum_{i=1}^n\int_a^{b_i} f^i(t)y^i_t(t)\,\mathrm{d}x &= \sum_{i=1}^n\int_a^{b_i} y^i_{tt}(t)y^i_t(t)\,\mathrm{d}x + \sum_{i=1}^n\int_a^{b_i}\mathcal{D}^\alpha_{b_i^-}(\beta^i\mathbb{D}^\alpha_{a^+}y^i)y^i_t(t)\,\mathrm{d}x \\ &+ \sum_{i=1}^n\int_a^{b_i} q^iy^i(t)y^i_t(t)\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{i=1}^n\frac{d}{dt}\int_a^{b_i}\left|y^i_t(t)\right|^2\mathrm{d}x + \frac{1}{2}\sum_{i=1}^n\frac{d}{dt}\int_a^{b_i}\beta^i\left|\mathbb{D}^\alpha_{a^+}y^i(t)\right|^2\mathrm{d}x \\ &+ \frac{1}{2}\sum_{i=1}^n\frac{d}{dt}\int_a^{b_i} q^i\left|y^i(t)\right|^2\mathrm{d}x. \end{aligned}$$

Using Young inequality, we have that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|y_t(t)\|^2_{\mathbb{L}^2} + \frac{1}{2}\sum_{i=1}^n\frac{d}{dt}\int_a^{b_i}\beta^i\left|\mathbb{D}^\alpha_{a^+}y^i(t)\right|^2\mathrm{d}x + \frac{1}{2}\sum_{i=1}^n\frac{d}{dt}\int_a^{b_i} q^i\left|y^i(t)\right|^2\mathrm{d}x \\ &\le \frac{1}{2\delta}\|f(t)\|^2_{\mathbb{L}^2} + \frac{\delta}{2}\sup_{t\in[0,T]}\|y_t(t)\|^2_{\mathbb{L}^2}, \end{aligned}$$

for some $\delta > 0$. Integrating each term of this latter inequality over (0,s,) for $s \in [0, T]$ yields,

$$
\begin{aligned}
&\|y_t(s)\|^2_{\mathbb{L}^2} + \min\left(\underline{\beta^0}, \underline{q^0}\right)\left(\|\mathbb{D}^\alpha_{a^+} y(s)\|^2_{\mathbb{L}^2} + \|y(s)\|^2_{\mathbb{L}^2}\right) \\
&\leq \frac{1}{\delta}\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \delta T\|y_t\|^2_{\mathcal{C}([0,T];\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \max\left(\overline{\beta}, \overline{q}\right)\left(\|\mathbb{D}^\alpha_{a^+} y^0\|^2_{\mathbb{L}^2} + \|y^0\|^2_{\mathbb{L}^2}\right).
\end{aligned}
$$

because Assumption 4.2 holds true. We deduce that,

$$
\begin{aligned}
\|y_t(s)\|^2_{\mathbb{L}^2} &\leq \frac{1}{\delta}\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \delta T\|y_t\|^2_{\mathcal{C}([0,T];\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} \\
&+ \max\left(\overline{\beta}, \overline{q}\right)\|y^0\|^2_{\mathbb{H}^\alpha_a} \\
\min\left(\underline{\beta^0}, \underline{q^0}\right)\|y(s)\|^2_{\mathbb{H}^\alpha_a} &\leq \frac{1}{\delta}\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \delta T\|y_t\|^2_{\mathcal{C}([0,T];\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} \\
&+ \max\left(\overline{\beta}, \overline{q}\right)\|y^0\|^2_{\mathbb{H}^\alpha_a}.
\end{aligned}
$$

Choosing $\delta = \dfrac{1}{2T}$, we obtain that

$$
\begin{aligned}
\frac{1}{2}\|y_t\|^2_{\mathcal{C}([0,T];\mathbb{L}^2)} &\leq 2T\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \max\left(\overline{\beta}, \overline{q}\right)\|y^0\|^2_{\mathbb{H}^\alpha_a} \\
\|y\|^2_{\mathcal{C}([0,T];\mathbb{H}^\alpha_a)} &\leq \frac{2}{\min\left(\underline{\beta^0}, \underline{q^0}\right)}\left(2T\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \max\left(\overline{\beta}, \overline{q}\right)\|y^0\|^2_{\mathbb{H}^\alpha_a}\right).
\end{aligned}
$$

□

4.2. Non-Homogeneous boundary fractional Sturm–Liouville parabolic equations in a star graph

So let us consider the non-homogeneous problem given by (4.1); In what follows, we shall transform that problem into an homogeneous type problem.

Let $(g_i)_{i=2\cdots m} \in \left(H^2(0,T)\right)^{m-1}$ and $(h_i)_{i=m+1\cdots n} \in \left(H^2(0,T)\right)^{n-m}$ be such that $h_i(0) = g_i(0) = 0$, $i = 1, \cdots, n$, we set

$$
\|\overline{g}\|_{\mathcal{C}([0,T])} = \max_{2\leq i\leq m}\|g_i\|_{\mathcal{C}([0,T])}, \quad \|\overline{h}\|_{\mathcal{C}([0,T])} = \max_{m+1\leq i\leq n}\|h_i\|_{\mathcal{C}([0,T])} \tag{4.16a}
$$

$$
\|\overline{g_t}\|_{\mathcal{C}([0,T])} = \max_{2\leq i\leq m}\|g'_i\|_{\mathcal{C}([0,T])}, \quad \|\overline{h}_t\|_{\mathcal{C}([0,T])} = \max_{m+1\leq i\leq n}\|h'_i\|_{\mathcal{C}([0,T])} \tag{4.16b}
$$

We consider the function $w = (w^i)_{i=1,\cdots,n}$, where w^i are giving by :

$$
w^i(x,t) = \begin{cases} 0 & i = 1 \\ \dfrac{2g_i(t)}{\Gamma(\alpha+2)(b_i-a)^2}(x-a)^{\alpha+1} & i = 2, \cdots, m \\ \dfrac{h_i(t)}{\Gamma(\alpha+2)\beta^i(b_i)(b_i-a)}(x-a)^{\alpha+1} & i = m+1, \cdots, n. \end{cases} \tag{4.17}
$$

Lemma 4.7. *Assume that Assumption 4.2 holds. Let $(g_i)_{i=2\cdots m} \in \left(H^2(0,T)\right)^{m-1}$ and $(h_i)_{i=m+1\cdots n} \in \left(H^2(0,T)\right)^{n-m}$ be such that (4.16) holds. Let also $w = (w^i)$ be defined as in (4.17). Then we have that w^i, $\mathcal{D}^\alpha_{b_i^-}(\beta^i\mathbb{D}^\alpha_{a^+}w^i) \in L^2(Q_i)$, $w^i_t \in \mathcal{C}([0,T]; L^2(a,b_i))$ and $w^i_{tt} \in L^2(Q_i)$.*

Moreover,

$$\sup_{t\in[0,T]}\|w(t)\|^2_{\mathbb{H}^\alpha_a}\le C(\alpha,\underline{b},\overline{b},\underline{\beta^0},n)\left(\|\overline{g}\|^2_{\mathcal{C}([0,T])}+\|\overline{h}\|^2_{\mathcal{C}([0,T])}\right),\tag{4.18a}$$

$$\|w_t(0)\|^2_{\mathbb{L}^2}\le\sup_{t\in[0,T]}\|w_t(t)\|^2_{\mathbb{L}^2}\le C(\alpha,\overline{b},\underline{\beta^0},n)\left(\|\overline{g}_t\|^2_{\mathcal{C}([0,T])}+\|\overline{h}_t\|^2_{\mathcal{C}([0,T])}\right),\tag{4.18b}$$

where $\overline{b}=\max_{1\le i\le n}(b_i)$ *and* $\underline{b}=\min_{1\le i\le n}(b_i)$.

Proof. As in Lemma 3.8 we have that $w^i\in L^2(Q_i)$ and $\mathcal{D}^\alpha_{b_i^-}(\beta^i\mathbb{D}^\alpha_{a^+}w^i)\in L^2(Q_i)$. Moreover, g_i, h_i, $i=2,\cdots,m$ being in $H^2(0,T)$, we deduce that w^i_t and w^i_{tt} are in $H^1((0,T),L^2(a,b_i))$ and $L^2(Q_i)$ respectively. Using the continuous embedding of $H^1(0,T)$ into $\mathcal{C}([0,T])$ that $w^i_t\in\mathcal{C}([0,T];L^2(a,b_i))$.

Now, let $\psi(x)=(x-a)^{\alpha+1}$ for all $x\in(a,b)$. Then, we prove that

$$\|\psi\|^2_{H^\alpha_{a^+}(a,b_i)}=\frac{1}{2\alpha+3}(b_i-a)^{2\alpha+3}+\frac{\Gamma^2(\alpha+2)}{3}(b_i-a)^3.$$

We have for all $t\in[0,T]$ that,

$$\left\|\mathcal{D}^\alpha_{b_i^-}\left(\beta^i\mathbb{D}^\alpha_{a^+}w^i(t)\right)\right\|^2_{L^2(a,b_i)}\le C(\alpha)^2\left(\left\|\beta_i'\mathbb{D}^\alpha_{a^+}w^i(t)\right\|^2_{L^2(a,b_i)}+\left\|\beta^i\partial_x\left(\mathbb{D}^\alpha_{a^+}w^i(t)\right)\right\|^2_{L^2(a,b_i)}\right),$$

the latter implies that

$$\sum_{i=1}^n\left\|\mathcal{D}^\alpha_{b_i^-}\left(\beta^i\mathbb{D}^\alpha_{a^+}w^i\right)\right\|^2_{L^2(Q_i)}\le C(\alpha,\overline{\beta},\overline{\beta'},\overline{b},\underline{b})\left(\|g\|^2_{L^2(0,T)}+\|h\|^2_{L^2(0,T)}\right).\tag{4.19}$$

and we also have that for all $t\in(0,T)$

$$\begin{aligned}\|w(t)\|^2_{\mathbb{H}^\alpha_a}&=\sum_{i=2}^m\frac{4g_i^2(t)}{\Gamma^2(\alpha+2)(b_i-a)^4}\|\psi\|^2_{H^\alpha_{a^+}(a,b_i)}+\sum_{i=m+1}^n\frac{h_i^2(t)}{\Gamma^2(\alpha+2)\beta^i(b_i)^2(b_i-a)^2}\|\psi\|^2_{H^\alpha_{a^+}(a,b_i)}.\\ \|w_t(t)\|^2_{\mathbb{L}^2}&=\sum_{i=2}^m\frac{4(g_i'(t))^2}{\Gamma^2(\alpha+2)(b_i-a)^4}\|\psi\|^2_{L^2(a,b_i)}+\sum_{i=m+1}^n\frac{(h_i'(t))^2}{\Gamma^2(\alpha+2)\beta^i(b_i)^2(b_i-a)^2}\|\psi\|^2_{L^2(a,b_i)}.\end{aligned}\tag{4.20}$$

Then using (4.16), we obtain from (4.20) the estimates (4.18a) and (4.18b). □

Observing that

$$\begin{array}{lcll}I^{1-\alpha}_{a^+}w^i(a^+,t)&=&I^{1-\alpha}_{a^+}w^j(a^+,t),&i,j=1,\cdots,n,\\ \displaystyle\sum_{i=1}^n\beta^i(a)\mathbb{D}^\alpha_{a^+}w^i(x,t)&=&0,&\\ I^{1-\alpha}_{a^+}w^i(b_i^-,t)&=&g_i(t),&i=2,\cdots m,\\ \beta^i(b_i)\mathbb{D}^\alpha_{a^+}w^i(b_i^-,t)&=&h^i(t),&i=m+1,\ldots,n,\\ w^i(x,0)&=&0,&i=1,\cdots,n,\end{array}$$

and setting $z=(z^i)_{i=1,\cdots,n}$ where $z^i(x,t)=y^i(x,t)-w^i(x,t)$, it follows that z satisfies the homogeneous system

$$\left\{\begin{array}{llll}
z^i_{tt} + \mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+} z^i) + q^i z^i & = & \tilde{f}^i & \text{in} \quad Q_i,\ i = 1, \dots, n, \\
I^{1-\alpha}_{a^+} z^i(a^+, \cdot) & = & I^{1-\alpha}_{a^+} z^j(a^+, \cdot) & \text{in} \quad (0,T),\ i \neq j = 1, \dots, n, \\
\sum\limits_{i=1}^{n} \beta^i(a) \mathbb{D}^{\alpha}_{a^+} z^i(a^+, \cdot) & = & 0 & \text{in} \quad (0,T), \\
I^{1-\alpha}_{a^+} z^1(b_1^-, \cdot) & = & 0 & \text{in} \quad (0,T), \\
I^{1-\alpha}_{a^+} z^i(b_i^-, \cdot) & = & 0 & \text{in} \quad (0,T),\ i = 2, \dots, m \\
\beta^i(b_i) \mathbb{D}^{\alpha}_{a^+} z^i(b_i^-, \cdot) & = & 0 & \text{in} \quad (0,T),\ i = m+1, \dots, n, \\
z^i(\cdot, 0) & = & y^{0,i} & \text{in} \quad (a, b_i), \quad i = 1, \dots, n \\
z^i_t(\cdot, 0) & = & \tilde{v}_i & \text{in} \quad (a, b_i), \quad i = 1, \dots, n,
\end{array}\right. \tag{4.21}$$

where $\tilde{f}^i \in L^2(Q_i)$ and $\tilde{v}_i \in L^2(a, b_i)$ are given by

$$\tilde{f}^i = f^i - w^i_{tt} - \mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+} w^i) - q^i w^i, i = 1, \cdots, n, \tag{4.22a}$$

$$\tilde{v}_i = v_i - w^i_t(\cdot, 0), \qquad i = 1, \cdots, n. \tag{4.22b}$$

Lemma 4.8. *Assume that Assumption 4.2 holds. Let* $f = (f^i), v = (v^i) \in L^2((0,T); \mathbb{L}^2)$. *Let also* $(g_i)_{i=2\cdots m} \in \left(H^2(0,T)\right)^{m-1}$ *and* $(h_i)_{i=m+1\cdots n} \in \left(H^2(0,T)\right)^{n-m}$ *be such that* (4.16) *holds. Then the functions* $\tilde{f} = (\tilde{f}^i)$ *and* $\tilde{v} = (\tilde{v}_i)$ *defined by* (4.22) *satisfies*

$$\left\|\tilde{f}\right\|_{L^2((0,T);\mathbb{L}^2)} \leq C\left(\alpha, \bar{b}, \underline{b}, \overline{\beta}, \overline{q}\right)\left(\|f\|_{L^2((0,T);\mathbb{L}^2)} + \|g\|_{H^2(0,T)} + \|h\|_{H^2(0,T)}\right) \tag{4.23a}$$

$$\|\tilde{v}\|_{\mathbb{L}^2} \leq C(\alpha, \bar{b}, \underline{\beta^0}, n)\left(\|v\|_{\mathbb{L}^2} + \|\overline{g}_t\|_{C([0,T])} + \left\|\overline{h}_t\right\|_{C([0,T])}\right), \tag{4.23b}$$

wherew $= (w^i)$ *be defined as in* (4.17), $\bar{b} = \max_{1\leq i\leq n}(b_i)$ *and* $\underline{b} = \min_{1\leq i\leq n}(b_i)$.

Proof. We have that

$$\|w_{tt}(t)\|_{\mathbb{L}^2} = \sum_{i=2}^{m} \frac{4(g_i''(t))^2}{\Gamma^2(\alpha+2)(b_i-a)^4}\|\psi\|^2_{L^2(a,b_i)} + \sum_{i=m+1}^{n} \frac{(h_i''(t))^2}{\Gamma^2(\alpha+2)\beta^i(b_i)^2(b_i-a)^2}\|\psi\|^2_{L^2(a,b_i)}.$$

Consequently, $\|w_{tt}\|^2_{L^2((0,T);\mathbb{L}^2)} \leq C\left(\alpha, \underline{\beta^0}, \bar{b}\right)\left(\|g''\|^2_{L^2(0,T)} + \|h''\|^2_{L^2(0,T)}\right)$. (4.24)

Combining (4.19) and (4.24), we obtain (4.23a), moreover using (4.18b), we obtain (4.23b). □

Theorem 4.9. *Assume that Assumption 4.2 holds. Let* $1/2 < \alpha < 1$, $f \in L^2(0,T;\mathbb{L}^2)$, $(g_i)_{i=2\cdots m} \in \left(H^2(0,T)\right)^{m-1}$ *and* $(h_i)_{i=m+1\cdots n} \in \left(H^2(0,T)\right)^{n-m}$ *be such that* $g_i(0) = h_i(0) = 0$, $i = 1, \cdots, n$, $y^0 = (y^{0,i}) \in D(\mathcal{A})$ *and* $v = (v_i) \in \mathbb{L}^2$, *then the problem* (4.1) *admit a unique weak solution* $y = (y^i) \in \mathcal{C}([0,T], \mathbb{V}) \cap \mathcal{C}^1([0,T], \mathbb{L}^2)$ *given by*

$$y(x,t) = \sum_{n=1}^{\infty}\left\{\frac{\tilde{v}_n}{\sqrt{\mu_n}}\sin(\sqrt{\mu_n}t) + y^0_n\cos(\sqrt{\mu_n}t) + \frac{1}{\sqrt{\mu_n}}\int_0^t \sin(\sqrt{\mu_n}(t-s))\tilde{f}_n(s)\,\mathrm{d}s\right\}\zeta_n + w(x,t),$$

where $\tilde{f}^i \in L^2(Q_i)$ and $\tilde{v}_i \in L^2(a,b_i)$ are defined as in (4.22). Moreover, there exist $\Delta_1, \Delta_2 = C\left(\alpha, \bar{b}, \underline{b}, \overline{\beta}, \overline{q}, \underline{\beta^0}, n\right) > 0$ such that

$$\begin{aligned} \|y\|_{\mathcal{C}([0,T];\mathbb{H}^\alpha_a)} &\leq \Delta_1 \Big(\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \|y^0\|^2_{\mathbb{H}^\alpha_a} + \|\overline{g}\|^2_{\mathcal{C}([0,T])} \\ &+ \|\overline{h}\|^2_{\mathcal{C}([0,T])} + \|\overline{g}_t\|^2_{\mathcal{C}([0,T])} + \|\overline{h}_t\|^2_{\mathcal{C}([0,T])} \Big) \end{aligned} \tag{4.25a}$$

$$\begin{aligned} \|y'\|_{C([0,T],\mathbb{L}^2)} &\leq \Delta_2 \Big(\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \|y^0\|^2_{\mathbb{H}^\alpha_a} + \|\overline{g}\|^2_{\mathcal{C}([0,T])} \\ &+ \|\overline{h}\|^2_{\mathcal{C}([0,T])} + \|\overline{g}_t\|^2_{\mathcal{C}([0,T])} + \|\overline{h}_t\|^2_{\mathcal{C}([0,T])} \Big) \end{aligned} \tag{4.25b}$$

.

Proof. From Theorem 4.6, we have that the homogeneous problem (4.21) admits a unique weak solution $z \in \mathcal{C}([0,T]; D(\mathcal{A})) \cap \mathcal{C}^1([0,T];\mathbb{L}^2)$.

Since $w \in \mathcal{C}^1([0,T];\mathbb{V})$, using the fact that the function y can be uniquely determined by the equality $y(x,t) = z(x,t) + w(x,t)$, we have that $y \in \mathcal{C}([0,T];\mathbb{V}) \cap \mathcal{C}^1([0,T];\mathbb{L}^2)$.

Since $z \in \mathcal{C}([0,T]; D(\mathcal{A})) \cap \mathcal{C}^1([0,T];\mathbb{L}^2)$ is solution of (4.21) and $w \in \mathcal{C}^1([0,T];\mathbb{V})$, using (4.15), (4.18) and Remark 4.1, we have that,

$$\begin{aligned} \|y\|_{\mathcal{C}([0,T];\mathbb{H}^\alpha_a))} &\leq \|z\|_{\mathcal{C}([0,T];\mathbb{H}^\alpha_a)} + \|w\|_{\mathcal{C}([0,T];\mathbb{H}^\alpha_a))} \\ &\leq \frac{2}{\min\left(\underline{\beta^0}, \underline{q^0}\right)} \left(2T\|\tilde{f}\|^2_{L^2((0,T);\mathbb{L}^2)} + \|\tilde{v}\|^2_{\mathbb{L}^2} + \max\left(\overline{\beta}, \overline{q}\right) \|y^0\|^2_{\mathbb{H}^\alpha_a} \right) \\ &+ C(\alpha, \underline{b}, \bar{b}, \underline{\beta^0}, n) \left(\|\overline{g}\|^2_{\mathcal{C}([0,T])} + \|\overline{h}\|^2_{\mathcal{C}([0,T])} \right) \\ &\leq C\left(\alpha, \bar{b}, \underline{b}, \overline{\beta}, \overline{q}, \underline{\beta^0}, n\right) \Big(\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \|y^0\|^2_{\mathbb{H}^\alpha_a} + \|\overline{g}\|^2_{\mathcal{C}([0,T])} \\ &\quad + \|\overline{h}\|^2_{\mathcal{C}([0,T])} + \|\overline{g}_t\|^2_{\mathcal{C}([0,T])} + \|\overline{h}_t\|^2_{\mathcal{C}([0,T])} \Big) \end{aligned}$$

and

$$\begin{aligned} \|y_t\|_{\mathcal{C}([0,T];\mathbb{L}^2)} &\leq \|z_t\|_{\mathcal{C}([0,T];\mathbb{L}^2)} + \|w_t\|_{\mathcal{C}([0,T];\mathbb{L}^2)} \\ &\leq 2T\|\tilde{f}\|^2_{L^2((0,T);\mathbb{L}^2)} + \|\tilde{v}\|^2_{\mathbb{L}^2} + max\left(\overline{\beta}, \overline{q}\right) \|y^0\|^2_{\mathbb{H}^\alpha_a} \\ &+ \mathcal{C}(\alpha, \bar{b}, \underline{\beta^0}, n) \left(\|\overline{g}_t\|^2_{\mathcal{C}([0,T])} + \|\overline{h}_t\|^2_{\mathcal{C}([0,T])} \right) \\ &\leq C\left(\alpha, \bar{b}, \underline{b}, \overline{\beta}, \overline{q}, \underline{\beta^0}, n\right) \Big(\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \|y^0\|^2_{\mathbb{H}^\alpha_a} + \|\overline{g}\|^2_{\mathcal{C}([0,T])} \\ &\quad + \|\overline{h}\|^2_{\mathcal{C}([0,T])} + \|\overline{g}_t\|^2_{\mathcal{C}([0,T])} + \|\overline{h}_t\|^2_{\mathcal{C}([0,T])} \Big). \end{aligned}$$

The proof is concluded. □

4.3. Existence of minimizers and optimality conditions in a general star graph

We are interested in solving the following optimal control problem:

$$\min_{v \in \mathcal{U}_{ad}} \mathcal{J}(v), \tag{4.26}$$

where

$$\mathcal{J}(v) := \frac{1}{2}\ \left\|y(v;T) - z_d^0\right\|_{\mathbb{L}^2}^2 + \frac{1}{2}\ \left\|y_t(v;T) - z_d^1\right\|_{\mathbb{L}^2}^2 + \frac{N}{2}\left\|v\right\|_{\mathbb{L}^2}^2. \tag{4.27}$$

and $y = (y^i)_i$ satisfies (4.1), $z_d^o = (z_d^{o,i})_i \in \mathbb{V}$, $z_d^1 = (z_d^{1,i})_i \in \mathbb{L}^2$ and $\mathcal{U}_{ad}$ is a closed and convex subset of $\mathbb{L}^2$.

We have the following existence result of optimal controls.

Theorem 4.10. *Let $1/2 < \alpha < 1$. Let $q^i \in \mathcal{C}(a, b_i)$ and $\beta^i \in \mathcal{C}^1([a, b_i])$ satisfy Assumption 4.2. Then, there exists a unique solution $\hat{u} \in \mathcal{U}_{ad}$ of the optimal control problem* (4.26)-(4.27)*.*

Proof. We have that:

$$\mathcal{J}(v) = \frac{1}{2}\pi(v,v) - L(v) + \frac{1}{2}\left\|y(0;T) - z_d^0\right\|_{\mathbb{L}^2}^2 + \frac{1}{2}\left\|y_t(0;T) - z_d^1\right\|_{\mathbb{L}^2}^2$$

$$\begin{aligned} \text{where}\ \ \pi(v,u) = \ &\langle y(v;T) - y(0;T), y(u;T) - y(0;T)\rangle_{\mathbb{L}^2} \\ &+\langle y_t(v;T) - y_t(0;T), y_t(u;T) - y_t(0;T)\rangle_{\mathbb{L}^2} + \frac{N}{2}\langle u, v\rangle_{\mathbb{L}^2} \end{aligned}$$

$$\text{and } L(u) = \langle y(u;T) - y(0;T), z_d^0 - y(0;T)\rangle_{\mathbb{L}^2} + \langle y_t(u;T) - y_t(0;T), z_d^1 - y_t(0;T)\rangle_{\mathbb{L}^2}$$

Proceeding as in the one edge case, we prove that π is a bilinear, symmetric, continue and coercive form on $\mathcal{U}_{ad}$, and that L is a linear continue form on $\mathcal{U}_{ad}$, thus using the Lax-Milgram theorem, we conclude that

$$\exists! u \in \mathcal{U}_{ad};\ \text{such that } \mathcal{J}(u) = \underset{v\in\mathcal{U}_{ad}}{inf}\ \mathcal{J}(v).$$

□

Next, we characterize the optimality conditions.

Theorem 4.11. *Let $1/2 < \alpha < 1$. Let $q^i \in \mathcal{C}([a, b_i])$ and $\beta^i \in \mathcal{C}^1([a, b_i])$ satisfy Assumption 4.2. Let $u = (u_i)_i \in \mathcal{U}_{ad}$ be the optimal control for the minimization problem* (4.26)-(4.27)*. Then, there exists a unique $p \in L^2((0,T);\mathbb{V}) \cap \mathcal{C}^1([0,T];\mathbb{L}^2)$*

such that the triple $(\hat{y}, u, p)$ *satisfies*

$$\begin{cases} \hat{y}^i_{tt} + \mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+}\hat{y}^i) + q^i\hat{y}^i & = f^i, & in \quad Q_i,\ i = 1,\dots,n, \\ I^{1-\alpha}_{a^+}\hat{y}^i(a^+,\cdot) & = I^{1-\alpha}_{a^+}\hat{y}^j(a^+,\cdot) & in \quad (0,T),\ i \neq j = 1,\dots,n, \\ \sum_{i=1}^n \beta^i(a)\mathbb{D}^{\alpha}_{a^+}\hat{y}^i(a^+,\cdot) & = 0, & in \quad (0,T), \\ I^{1-\alpha}_{a^+}\hat{y}^1(b_1^-,\cdot) & = 0, & in \quad (0,T), \\ I^{1-\alpha}_{a^+}\hat{y}^i(b_i^-,\cdot) & = g^i, & in \quad (0,T),\ i = 2,\dots,m \\ \beta^i(b_i)\mathbb{D}^{\alpha}_{a^+}\hat{y}^i(b_i^-,\cdot) & = h_i, & in \quad (0,T),\ i = m+1,\dots,n, \\ \hat{y}^i(\cdot,0) & = y^{0,i}, & in \quad (a,b_i), \quad i = 1,\dots,n, \\ \hat{y}^i_t(\cdot,0) & = u^i, & in \quad (a,b_i), \quad i = 1,\dots,n, \end{cases} \tag{4.28}$$

and

$$\begin{cases} p^i_{tt} + \mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+}p^i) + q^i p^i & = 0, & in \quad Q_i,\ i = 1,\dots,n, \\ I^{1-\alpha}_{a^+}p^i(a^+,\cdot) & = I^{1-\alpha}_{a^+}p^j(a^+,\cdot) & in \quad (0,T),\ i \neq j = 1,\dots,n, \\ \sum_{i=1}^n \beta^i(a)\mathbb{D}^{\alpha}_{a^+}p^i(a^+,\cdot) & = 0, & in \quad (0,T), \\ I^{1-\alpha}_{a^+}p^i(b_i^-,\cdot) & = 0, & in \quad (0,T),\ i = 1,\dots,m \\ \beta^i(b_i)\mathbb{D}^{\alpha}_{a^+}p^i(b_i^-,\cdot) & = 0, & in \quad (0,T),\ i = m+1,\dots,n, \\ p^i(\cdot,T) & = \hat{y}^i_t(\cdot,T) - z_d^{1,i} & in \quad (a,b),\ i = 1,\dots,n \\ p_t(\cdot,T) & = -(\hat{y}^i(\cdot,T) - z_d^{0,i}) & in \quad (a,b_i),\ i = 1,\dots,n \end{cases} \tag{4.29}$$

and

$$\sum_{i=1}^n \int_a^{b_i} \left(u_i + p^i(\cdot,0)\right)(v_i - u_i)\,\mathrm{d}x \geq 0 \tag{4.30}$$

for all $v = (v_i)_i \in \mathcal{U}_{ad}$.

Proof. We have already proved (4.28). To complete the proof of the theorem, we introduce for every $p \in L^2((0,T);\mathbb{V})) \cap \mathcal{C}^1([0,T],\mathbb{L}^2)$ the Lagrangian

$$\begin{aligned} \mathcal{L}(y,v,p) \quad &= \quad \mathcal{J}(v) - \sum_{i=1}^n \int_{Q_i} (y^i_{tt} + \mathcal{D}^{\alpha}_{b^-}(\beta^i\,\mathbb{D}^{\alpha}_{a^+}y^i) + q^i\,y^i)p^i\,dx\,dt + \sum_{i=1}^n \int_{Q_i} f^i\,p^i\,dx\,dt \\ &= \quad \mathcal{J}(v) + \sum_{i=1}^n \int_{Q_i} f^i\,p^i\,dx\,dt + \sum_{i=1}^n \int_a^{b_i} y^i_t(\cdot,T)p^i(\cdot,T)\ dx - \sum_{i=1}^n \int_a^{b_i} v_i p^i(\cdot,0)\ dx \\ &+ \quad \sum_{i=m+1}^n \int_0^T h^i(t) I^{1-\alpha}_{a^+}p^i(b_i^-,t)\,dt - \sum_{i=2}^m \int_0^T g_i(t)(b_i^-,t)(\beta^i\mathbb{D}^{\alpha}_{a^+}p^i)(b_i^-,t)\,dt \\ &- \quad \sum_{i=1}^n \int_a^{b_i} y^i(\cdot,T)p^i_t(\cdot,T)\,dx + \sum_{i=1}^n \int_a^{b_i} y^{i0}p^i_t(\cdot,0)\ \mathrm{d}x. \end{aligned}$$

In order to derive the optimality conditions, we compute

$$\lim_{\lambda\to 0} \frac{\mathcal{L}(\hat{y} + \lambda(y-\hat{y}), u + \lambda(v-u), p) - \mathcal{L}(\hat{y},u,p)}{\lambda} \geq 0 \ \ \forall v = (v_i)_i \in \mathcal{U}_{ad},$$

where p solves (4.29) and $\hat{y}$ is the unique weak solution of (4.28). After some calculations, we obtain that for all $v = (v_i)_i \in \mathcal{U}_{ad}$,

$$\sum_{i=1}^{n} \int_a^{b_i} \left(u_i + p^i(\cdot, 0)\right) (v_i - u_i) \, \mathrm{d}x \geq 0$$

which gives (4.30). The proof is finished. □

5. Conclusion and open problems

We investigated an optimal control problem of a fractional hyperbolic type partial differential equation involving a fractional Sturm-Liouville operator in a space interval, and in a general star graph, where the Sturm-Liouville operator is obtained as a composition of a left fractional Caputo derivative, and a right fractional Riemann–Liouville derivative. Using the spectral theory, we proved that the considered fractional optimal control in an interval as well as in the graph has a unique weak solution. We then derived the optimality system that characterizes the control in an edge by means of the Euler-Lagrange optimality conditions, and those in the graph by using the method of Lagrange multipliers.

Among open questions for further research, there is, investigation on the case where the controls are taking at the boundaries. It would also be interesting to see what happen when we take $\alpha \in (1, 2]$. Another potentially interesting research direction would be to employ the fractional wave equation by replacing the second order derivative in time by the Caputo derivative or the Riemann-Liouville derivative.

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Disclosure statement

The authors report there are no competing interests to declare.

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