Quasi-reversibility method for an optimal control of an ill-posed fractional diffusion equation

Claire Joseph

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Abstract

In this paper, an optimal control problem associated to an ill-posed fractional diffusion equation is considered. To study our initial problem, we use the quasi-reversibility method introduced by Lions and Lattès in 1969. More precisely, we consider an approximated optimal control problem of our initial problem. Then the new problem is associated to a well-posed state equation which approximate the ill-posed state equation. Firstly, we prove that the approximated optimal control problem admits a unique solution which we characterized using the Euler-Lagrange optimality conditions. Next, we show that the solution of the approximated optimal control problem converges to the solution of the initial optimal control problem. To finish, we characterize the optimal control of our initial problem by an optimality system.

1 Introduction

Let $d \in \mathbb{N}^*$ and Ω be a bounded open subset of $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^2. For $T > 0$, we set $Q = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$ and we consider the following fractional diffusion equation :

$$\left\{ \begin{array}{rcll} D^{\alpha}_{RL} y(x,t) - \Delta y(x,t) & = & v(x,t) & (x,t) \in Q, \\ y(\sigma,t) & = & 0 & (\sigma,t) \in \Sigma, \\ I^{1-\alpha} y(x,T) & = & y^T(x) & x \in \Omega, \end{array} \right. \tag{1.1}$$

where $3/4 < \alpha < 1$, $v \in L^2(Q)$, $y^T \in L^2(\Omega)$ and the integral $I^{1-\alpha}$ and the derivative D^{α}_{RL} of order α are understood in the Riemann-Liouville sense.

Fractional diffusion equation is obtained by replacing the first order time derivative with a time fractional derivative in the classical diffusion equation. Due to the fact that the Riemann-Liouville fractional derivatives are characterized by a convolution integral (see Definition 2.5), researchers speak about memory effect. This is why, many researchers have focused their attention on fractional calculus and there are many applications in other fields such as Physics, Economics and Biology. For more information about fractional calculus, we can refer to [22, 12, 23, 21, 16] and references therein.

Fractional diffusion equations are often used to model environmental phenomenon such as pollution problems. However, in this latter type of phenomenon, it is common to not have all the information of the problem. This is why, we decided to consider a problem where the initial condition is missing.

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Regarding studies of ill-posed fractional diffusion equation, we can refer to [10, 14, 20, 9, 11] and the references therein, for instance.

The main purpose of this paper is to solve an optimal control problem associated to equation (1.1). More precisely, we are interesting in solving of the following optimal control problem:

$$\inf_{(v,y)\in\mathcal{A}} J(v,y), \tag{1.2}$$

where

$$\mathcal{A} = \{(v,y) \;:\; v \in \mathbb{U}_{ad} \text{ and } y \text{ is solution of (1.1) in the sense of Definition 3.1}\}$$

$\mathbb{U}_{ad}$ being a given nonempty closed and convex subset of $L^2(Q)$ and J is the functional cost given by

$$J(v,y) = \frac{1}{2}\left\| I^{1-\alpha}y(\cdot,0) - z_d \right\|^2_{L^2(\Omega)} + \frac{N}{2}\|v\|^2_{L^2(Q)}, \tag{1.3}$$

where $z_d \in L^2(\Omega)$ is a given target and $N > 0$.

Model computed in (1.1) is an ill-posed problem, in Hadamard sense. Hence, the solution of the optimal control problem (1.1)-(1.2) is difficult to characterize. In this work, we decided to use the quasi-reversibility method that was introduced by Lions and Lattès in [13]. Moutamal et al. [10] used the quasi-boundary method, which is inspired by the quasi-reversibility method. More precisely, they approached Equation (1.1) by the well-posed problem :

$$\left\{\begin{array}{rcll} D^\alpha_{RL}y_\beta(x,t) - \Delta y_\beta(x,t) &=& f(x,t) & (x,t)\in Q,\\ y_\beta(\sigma,t) &=& 0 & (\sigma,t)\in\Sigma,\\ I^{1-\alpha}y_\beta(x,T) + \beta I^{1-\alpha}y_\beta(x,0^+) &=& y^1(x) & x\in\Omega, \end{array}\right.$$

where $1/2 < \alpha < 1$, $\beta > 0$ and $I^{1-\alpha}y_\beta(x,0^+) = \lim\limits_{t\downarrow 0} I^{1-\alpha}y_\beta(x,t)$. And they proved that when $y^1 \in H^1_0(\Omega)$, the solution of (1.4) converges in $L^2((0,T);H^1_0(\Omega))$ to the solution of the following equation :

$$\left\{\begin{array}{rcll} D^\alpha_{RL}y(x,t) - \Delta y(x,t) &=& f(x,t) & (x,t)\in Q,\\ y(\sigma,t) &=& 0 & (\sigma,t)\in\Sigma,\\ I^{1-\alpha}y(x,T) &=& y^1(x) & x\in\Omega, \end{array}\right.$$

under a certain condition.

In order to study our optimal control problem (1.1)-(1.2), we used the same approach. For this, we consider the associated approximated equation of (1.1) :

$$\left\{\begin{array}{rcll} D^\alpha_{RL}y^\varepsilon(x,t) - \Delta y^\varepsilon(x,t) &=& v(x,t) & (x,t)\in Q,\\ y^\varepsilon(\sigma,t) &=& 0 & (\sigma,t)\in\Sigma,\\ I^{1-\alpha}y^\varepsilon(x,T) + \varepsilon I^{1-\alpha}y^\varepsilon(x,0^+) &=& y^T(x) & x\in\Omega, \end{array}\right. \tag{1.4}$$

where $\varepsilon > 0$, $v \in L^2(Q)$, $y^T \in L^2(\Omega)$ and $I^{1-\alpha}y^\varepsilon(x,0^+) = \lim\limits_{t\downarrow 0} I^{1-\alpha}y^\varepsilon(x,t)$ and the associated approximated optimal control problem, given by:

$$\inf_{(v,y^\varepsilon)\in A} J^\varepsilon(v,y^\varepsilon), \tag{1.5}$$

where

$$A = \{(v,y^\varepsilon) \;:\; v\in\mathcal{U}_{ad} \text{ and } y^\varepsilon \text{ is the unique solution of (1.4)}\}.$$

$\mathcal{U}_{ad}$ being a given nonempty closed and convex subset of $L^2(Q)$ and J^ε is the functional cost given by

$$J^\varepsilon(v, y^\varepsilon) = \frac{1}{2}\left\|I^{1-\alpha} y^\varepsilon(\cdot, 0) - z_d\right\|^2_{L^2(\Omega)} + \frac{N}{2}\|v\|^2_{L^2(Q)}. \tag{1.6}$$

Recently, Mophou and Warma, in [19], used this latter method to study an optimal control problem associated to a non-well posed Cauchy problem for a general space-fractional diffusion equation. They approximated their problem by a well-posed problem and proved that the solution of the well-posed problem converges to the solution of the ill-posed problem. They also gave an optimality system which characterize their optimal control.

Over the past 10 years, optimal control problems associated to a well-posed fractional diffusion-wave equations have been studied extensively, see [1, 2, 3, 5, 6, 7, 18, 15] and references therein, for example. However, we have less studies about optimal control problem associated to ill-posed fractional diffusion-wave equations. For instance, in [4, 17], the authors used the concept of low-regret and no-regret controls to study optimal control problems associated to an ill-posed fractional diffusion-wave equations with incomplete data, where the derivative is understood in Riemann-Liouville sense. The best of the authors' knowledge, and, judging from the open literature available, this is the first application of the quasi-reversibility method to solve an optimal control problem associated to an ill-posed fractional diffusion equation.

This paper is structured as follows. In section 2, we firstly give some definitions and results on fractional calculus. After, we give some important existence and uniqueness results which are obtained using the spectral method. In section 3, we begin by the existence and the uniqueness of the approximated optimal control problem (1.4)-(1.5). Using the Euler-Lagrange optimality conditions, we characterized the solution of the approximated problem by a system. After, we proved that the solution of the approximated problem converges to the solution of the optimal control problem (1.1)-(1.2). To finish, we give the singular optimality system that characterizes the optimal control.

2 Preliminaries

In this section, we provide some basic definitions and results on fractional calculus. And we give some existence and uniqueness results of fractional diffusion equations.

Definition 2.1 *[22, 12] Let z a complex such as $Re(z) > 0$. Then the Gamma function, noted Γ, is given by*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Definition 2.2 *[22, 12] For $\alpha > 0$ and $\beta > 0$ we denote by,*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ z \in \mathbb{C} \tag{2.1}$$

the two-parameters Mittag-Leffler function and thus

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \alpha)}, \ z \in \mathbb{C}. \tag{2.2}$$

We set

$$E_{\alpha,1}(t) = E_\alpha(t) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{2.3}$$

Theorem 2.1 *[22] Let $0 < \alpha < 2$, $\beta \in \mathbb{R}$ be an arbitrary , and we suppose that μ as*

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}.$$

Then there exists a constant $C = C(\alpha, \beta, \mu) > 0$ such that

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \qquad \mu \leq |arg(z)| \leq \pi.$$

Definition 2.3 *[12, 8] Let $\alpha, \beta, \rho \in \mathbb{C}$ such that $Re(\alpha) > 0$ and $Re(\beta) > 0$ then the generalized Mittag-Leffler function is defined by*

$$\mathcal{E}^{\rho}_{\alpha,\beta}(t) = \sum_{n=0}^{+\infty} \frac{(\rho)_n t^n}{\Gamma(\alpha n + \beta) n!}, \qquad \textit{pour tout } t \in \mathbb{C},$$

where $(\rho)_n = \rho(\rho+1)\ldots(\rho+n-1)$.

Remark 2.1 *Note that, when $\rho = 1$ we get*

$$\mathcal{E}^{1}_{\alpha,\beta}(t) = E_{\alpha,\beta}(t),$$

where E is the classical Mittag-Leffler function defined in (2.1).

Definition 2.4 *[22, 12] The left and right Riemann–Liouville fractional integrals of order $\alpha \in (0,1)$ of f are defined, respectively, by:*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad (t > 0) \tag{2.4}$$

and

$$\mathcal{I}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s) ds, \quad (t < T), \tag{2.5}$$

provided that the integrals exist.

Definition 2.5 *[22, 12] The left and right Riemann–Liouville fractional derivatives of order $\alpha \in (0,1)$ of f are defined, respectively, by:*

$$D^{\alpha}_{RL} f(t) = \frac{d}{dt}(I^{1-\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds, \quad (t > 0) \tag{2.6}$$

and

$$\mathcal{D}^{\alpha}_{RL} f(t) = -\frac{d}{dt}(\mathcal{I}^{1-\alpha} f)(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (s-t)^{-\alpha} f(s) ds, \quad (t < T), \tag{2.7}$$

provided that the integrals exist.

Definition 2.6 *[22, 12] The left and right Caputo fractional derivative of order $\alpha \in (0,1)$ of f are defined respectively, by:*

$$D^{\alpha}_{C} f(t) = I^{1-\alpha} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \quad (t > 0) \tag{2.8}$$

and

$$\mathcal{D}^{\alpha}_{C} f(t) = \mathcal{I}^{1-\alpha} f'(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f'(s) ds \qquad (t < T) \tag{2.9}$$

provided that the integrals exist.

Now we give the following integration by parts formulas.

Lemma 2.1 *[18] Let $0<\alpha<1$, $y\in C^\infty(\bar{Q})$ and $\varphi\in C^\infty(\bar{Q})$. Then we have,*

$$
\begin{aligned}
&\int_0^T\int_\Omega (D^\alpha_{RL}y(x,t)-\Delta y(x,t))\varphi(x,t)dxdt = \\
&\int_\Omega \varphi(x,T)I^{1-\alpha}y(x,T)dx-\int_\Omega \varphi(x,0)I^{1-\alpha}y(x,0)dx+\int_0^T\int_{\partial\Omega} y(\sigma,t)\frac{\partial\varphi}{\partial v}(\sigma,t)d\sigma dt \\
&-\int_0^T\int_{\partial\Omega}\frac{\partial y}{\partial v}(\sigma,t)\varphi(\sigma,t)d\sigma dt+\int_\Omega\int_0^T y(x,t)(-\mathcal{D}^\alpha_C\varphi(x,t)-\Delta\varphi(x,t))dxdt,
\end{aligned}
\tag{2.10}
$$

where $\mathcal{D}^\alpha_C$ is the right Caputo fractional defined by (2.9)*.*

On other hand, since the embedding of $H^1_0(\Omega)$ in $L^2(\Omega)$ is compact and $(-\Delta)$ is a symmetric uniform elliptic operator, then $(-\Delta)$ admits real eigenvalues, $0<\lambda_1\leq\lambda_2\leq\lambda_3\leq ...$ with $\lambda_k\to\infty$ when $k\to\infty$. Moreover, there exists an orthonormal basis $\{w_k\}_{k=1}^\infty$ of $L^2(\Omega)$, where $w_k\in H^1_0(\Omega)$ is an eigenfunction corresponding to λ_k: $-\Delta w_k=\lambda_k w_k$. Further, we have,

$$\int_\Omega \nabla\varphi(x)\cdot\nabla\psi(x)dx=\lambda_k\int_\Omega\varphi(x)\psi(x)dx,\qquad \forall p\in H^1_0(\Omega). \tag{2.11}$$

In what follows, for all $\varphi,\psi\in L^2(\Omega)$, we denote

$$(\varphi,\psi)_{L^2(\Omega)}=\int_\Omega\varphi(x)\psi(x)dx,$$

as the inner product in $L^2(\Omega)$ and $\|\varphi\|_{L^2(\Omega)}$ as the associated norm.
We set

$$a(\varphi,\psi)=\int_\Omega\nabla\varphi(x)\cdot\nabla\psi(x)dx,\quad \forall\varphi,\psi\in H^1_0(\Omega). \tag{2.12}$$

Then, the bilinear functional $a(.,.)$ defines an inner product on $H^1_0(\Omega)$, and we have

$$\|\varphi\|^2_{H^1_0(\Omega)}=a(\varphi,\varphi), \tag{2.13}$$

which is a norm on $H^1_0(\Omega)$. Since $\left\{\dfrac{w_k}{\sqrt{\lambda_k}}\right\}_{k=1}^\infty$ is an orthonormal basis of $H^1_0(\Omega)$ for the inner product $a(.,.)$, we can write

$$||\phi||^2_{H^1_0(\Omega)}=\sum_{i=1}^{+\infty}\lambda_i(\phi,w_i)^2_{L^2(\Omega)},\qquad \forall\phi\in H^1_0(\Omega). \tag{2.14}$$

3 Existence results

In this section, we give some existence and uniqueness results for the fractional diffusion equations which are used in this paper.

We first have to give our notion of strong solution to the ill-posed problem (1.1):

Definition 3.1 *Let $v\in L^2(Q)$ and $y^T\in L^2(\Omega)$. A function $y\in L^2((0,T);H^1_0(\Omega))$ is said to be a strong solution of* (1.1)*, if the following assertions hold:*

- $I^{1-\alpha}y \in C([0,T]; L^2(\Omega))$,
- $D^{\alpha}_{RL}y(t) \in H^{-1}(\Omega)$, $y(\cdot,t) \in H^1_0(\Omega)$ *for a.e* $t \in (0,T)$ *and the first equation of* (1.1) *is satisfied for a.e.* $t \in (0,T)$.
- $I^{1-\alpha}y(\cdot,T) = y^T$.

We have the following results:

Lemma 3.1 *Let* $3/4 < \alpha < 1$, $T > 0$, $v \in L^2(Q)$, $y^T \in L^2(\Omega)$ *and* y *satisfies* (1.1).
Then $y \in L^2((0,T); H^1_0(\Omega))$ *if*

$$\lim_{N\to\infty}\left(K_1\left(\sum_{i=1}^{N}\frac{|y_i^T|^2}{E^2_{\alpha}(-\lambda_i T^{\alpha})}\right)^{1/2} + K_2\left(\sum_{i=1}^{N}\frac{\displaystyle\int_0^T |v_i(s)|^2 ds}{E^2_{\alpha}(-\lambda_i T^{\alpha})}\right)^{1/2}\right) < \infty \tag{3.1}$$

where

$$K_1 = C^2\sqrt{\frac{2T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}} \quad \text{and} \quad K_2 = C^4\sqrt{\frac{2T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}}.$$

Proof. Set

$$V_N = \mathrm{Span}\,(w_1, w_2, \cdots, w_N)\,. \tag{3.2}$$

Then we look for

$$Y_N(x,t) = \sum_{i=1}^{N}(y(t), w_i)_{L^2(\Omega)} w_i(x) = \sum_{i=1}^{N} y_i(t) w_i(x), \tag{3.3}$$

solution of the following approximate problem :

$$\left\{\begin{array}{rcll} D^{\alpha}_{RL}Y_N(x,t) - \Delta Y_N(x,t) &=& v_N(x,t) & (x,t) \in Q, \\ Y_N(\sigma,t) &=& 0 & (\sigma,t) \in \Sigma, \\ I^{1-\alpha}Y_N(x,T) &=& y^T_N(x) & x \in \Omega, \end{array}\right. \tag{3.4}$$

where

$$v_N(x,t) = \sum_{i=1}^{N}(v(t), w_i)_{L^2(\Omega)} w_i(x) = \sum_{i=1}^{N} v_i(t) w_i(x), \tag{3.5}$$

and

$$y^T_N = \sum_{i=1}^{N}(y^T, w_i)_{L^2(\Omega)} w_i(x) = \sum_{i=1}^{N} y^T_i w_i(x). \tag{3.6}$$

Note that if Y_N converge then $\lim_{N\to\infty} Y_N = y$, where y satisfies (1.1).

If we replace Y_N in (3.4) by $\sum_{i=1}^{N} y_i(t)w_i(x)$, we obtain that y_i, $i = 1, \cdots, N$ is a solution of the ordinary differential equation

$$\left\{\begin{array}{rcll} D^{\alpha}_{RL}y_i(t) + \lambda_i y_i(t) &=& v_i(t), & t \in (0,T), \\ I^{1-\alpha}y_i(T) &=& y^T_i. & \end{array}\right. \tag{3.7}$$

Now, using the Laplace transform, we obtain from the first equation of (3.7) that,

$$\widehat{D}^{\alpha}_{RL}y_i(s) + \lambda_i \widehat{y}_i(s) = \widehat{v}_i(s), \tag{3.8}$$

where

$$\begin{array}{lcl} \widehat{D}^{\alpha}_{RL}y_i(s) & = & \mathcal{L}(D^{\alpha}_{RL}y_i(t))(s), \\ \widehat{y}_i(s) & = & \mathcal{L}(y_i(t))(s), \\ \widehat{v}_i(s) & = & \mathcal{L}(v_i(t))(s) \end{array}$$

and $\mathcal{L}$ denotes the Laplace transform operator. Then after some computations we obtain (see [10]):

$$y_i(t) = I^{1-\alpha}y_i(0)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i (t-s)^{\alpha})v_i(s)ds, \tag{3.9}$$

which implies that

$$I^{1-\alpha}y_i(t) = I^{1-\alpha}y_i(0)E_{\alpha}(-\lambda_i t^{\alpha}) + \int_0^t E_{\alpha}(-\lambda_i (t-s)^{\alpha})v_i(s)ds. \tag{3.10}$$

From the latter equality, we can deduce that

$$I^{1-\alpha}y_i(T) = I^{1-\alpha}y_i(0)E_{\alpha}(-\lambda_i T^{\alpha}) + \int_0^T E_{\alpha}(-\lambda_i (T-s)^{\alpha})v_i(s)ds,$$

which combining with the second equation of (3.7) gives

$$I^{1-\alpha}y_i(0) = \frac{y_i^T - \displaystyle\int_0^T E_{\alpha}(-\lambda_i (T-s)^{\alpha})v_i(s)ds}{E_{\alpha}(-\lambda_i T^{\alpha})}, \tag{3.11}$$

where $E_{\alpha}(-\lambda_i T^{\alpha}) > 0$ (see [24], for instance).

Therefore, combining (3.9) and (3.11), we obtain

$$\begin{array}{rcl} y_i(t) & = & \left[\dfrac{y_i^T - \displaystyle\int_0^T E_{\alpha}(-\lambda_i (T-s)^{\alpha})v_i(s)ds}{E_{\alpha}(-\lambda_i T^{\alpha})}\right] t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^{\alpha}) \\ & + & \displaystyle\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i (t-s)^{\alpha})v_i(s)ds. \end{array}$$

It then follows from (3.3) that

$$\begin{array}{rcl} Y_N(t) & = & \displaystyle\sum_{i=1}^{N} \left\{ \dfrac{y_i^T - \displaystyle\int_0^T E_{\alpha}(-\lambda_i (T-s)^{\alpha})v_i(s)ds}{E_{\alpha}(-\lambda_i T^{\alpha})} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^{\alpha}) \right\} w_i \\ & + & \displaystyle\sum_{i=1}^{N} \left\{ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i (t-s)^{\alpha})v_i(s)ds \right\} w_i. \end{array} \tag{3.12}$$

Set $a_i = \dfrac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{E_\alpha(-\lambda_i T^\alpha)}$. Then, we have that,

$$\begin{aligned}
a(Y_N(t), Y_N(t)) &= \sum_{i=1}^{N} \lambda_i [y_i(t)]^2 \\
&\leq 2\sum_{i=1}^{N} \lambda_i t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha)|a_i|^2 \\
&+ 2\sum_{i=1}^{N} \lambda_i \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)v_i(s)ds \right\}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|Y_N(t)\|^2_{L^2((0,T);H_0^1(\Omega))} &= \int_0^T a(Y_N(t), Y_N(t))dt \\
&\leq A_N + B_N,
\end{aligned}$$

with

$$\begin{aligned}
A_N &= 2\sum_{i=1}^{N} \lambda_i |a_i|^2 \int_0^T t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha)dt, \\
B_N &= 2\sum_{i=1}^{N} \int_0^T \lambda_i \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)v_i(s)ds \right\}^2 dt.
\end{aligned}$$

Note that from Theorem 2.1, we know that there exists a generic constant $C > 0$ such that

$$\begin{aligned}
A_N &= 2\sum_{i=1}^{N} \lambda_i |a_i|^2 \int_0^T t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha)dt \\
&\leq 2\sum_{i=1}^{N} \lambda_i |a_i|^2 \left(\int_0^T t^{4\alpha-4} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha)dt \right) \left(\int_0^T E_{\alpha,\alpha}^2(-\lambda_i t^\alpha)dt \right) \\
&\leq C^4 \sum_{i=1}^{N} |a_i|^2 \left(\int_0^T t^{4\alpha-4} dt \right) \left(\int_0^T t^{-\alpha} dt \right) \qquad (3.13) \\
&\leq C^4 \sum_{i=1}^{N} |a_i|^2 \left[\frac{t^{4\alpha-3}}{4\alpha-3} \right]_0^T \left[\frac{t^{1-\alpha}}{1-\alpha} \right]_0^T \\
&\leq \frac{C^4 T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)} \sum_{i=1}^{N} |a_i|^2
\end{aligned}$$

Therefore, we have

$$A_N \leq \frac{C^4 T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)} \sum_{i=1}^{N} |a_i|^2. \qquad (3.14)$$

Remark 3.1 *From the latter estimation, we see that we have to take $3/4 < \alpha < 1$ to give a sense to our computation.*

Using Theorem 2.1 and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\sum_{i=1}^{N}|a_i|^2 &\leq 2\sum_{i=1}^{N}\frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + 2\sum_{i=1}^{N}\frac{1}{E_\alpha^2(-\lambda_i T^\alpha)}\left|\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds\right|^2\\ &\leq 2\sum_{i=1}^{N}\frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + 2C^2\sum_{i=1}^{N}\frac{1}{E_\alpha^2(-\lambda_i T^\alpha)}\left(\int_0^T |v_i(s)|^2 ds\right)\end{aligned}$$

Therefore, we have

$$\sum_{i=1}^{N}|a_i|^2 \leq 2\sum_{i=1}^{N}\frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + 2C^2\sum_{i=1}^{N}\frac{\displaystyle\int_0^T |v_i(s)|^2 ds}{E_\alpha^2(-\lambda_i T^\alpha)}, \tag{3.15}$$

which combining with (3.14) gives

$$A_N \leq \frac{2C^4T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}\sum_{i=1}^{N}\frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + \frac{2C^6T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}\sum_{i=1}^{N}\frac{\displaystyle\int_0^T |v_i(s)|^2 ds}{E_\alpha^2(-\lambda_i T^\alpha)}. \tag{3.16}$$

Proceeding as in [10], we have

$$B_N \leq \frac{4C^2T^\alpha}{\alpha-\frac{1}{2}}\sum_{i=1}^{N}\int_0^T |v_i(s)|^2 ds. \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\begin{aligned}\|Y_N(t)\|^2_{L^2((0,T);H_0^1(\Omega))} &\leq \frac{2C^4T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}\sum_{i=1}^{N}\frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)}\\ &+ \frac{2C^6T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}\sum_{i=1}^{N}\frac{\displaystyle\int_0^T |v_i(s)|^2 ds}{E_\alpha^2(-\lambda_i T^\alpha)} + \frac{4C^2T^\alpha}{\alpha-\frac{1}{2}}\sum_{i=1}^{N}\int_0^T |v_i(s)|^2 ds.\end{aligned}$$

Hence, we can deduce that

$$\begin{aligned}\|Y_N(t)\|_{L^2((0,T);H_0^1(\Omega))} &\leq C^2\sqrt{\frac{2T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}}\left(\sum_{i=1}^{N}\frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)}\right)^{1/2}\\ &+ C^4\sqrt{\frac{2T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}}\left(\sum_{i=1}^{N}\frac{\displaystyle\int_0^T |v_i(s)|^2 ds}{E_\alpha^2(-\lambda_i T^\alpha)}\right)^{1/2} \\ &+ 2C\sqrt{\frac{T^\alpha}{\alpha-\frac{1}{2}}}\left(\sum_{i=1}^{N}\int_0^T |v_i(s)|^2 ds\right)^{1/2}.\end{aligned} \tag{3.18}$$

Passing to the limit, when $N \to \infty$, in (3.18), we have that $y \in L^2((0,T);H_0^1(\Omega))$ if (3.1) holds. Therefore,

$$\begin{aligned} y(t) &= \sum_{i=1}^{+\infty}\left\{\frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{E_\alpha(-\lambda_i T^\alpha)} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)\right\} w_i \\ &+ \sum_{i=1}^{+\infty}\left\{\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\} w_i. \end{aligned} \tag{3.19}$$

■

Lemma 3.2 *Let $3/4 < \alpha < 1$, $y^T \in L^2(\Omega)$ and $v \in L^2(Q)$. Then the problem* (1.1) *admits a strong solution if and only if the following two series converge:*

$$\left(\sum_{i=1}^{+\infty}\frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)}\right)^{1/2} \quad and \quad \left(\sum_{i=1}^{+\infty}\frac{\displaystyle\int_0^T |v_i(s)|^2 ds}{E_\alpha^2(-\lambda_i T^\alpha)}\right)^{1/2} \tag{3.20}$$

where λ_i is the eigenvalue of the operator $-\Delta$ corresponding to the eigenfunction w_i. $y_i^T = (y^T, w_i)_{L^2(\Omega)}$ and $v_i(t) = (v(t), w_i)_{L^2(\Omega)}$ are respectively the i-th component of y^T and $v(t)$ in the orthonormal basis $\{w_i\}_{i=1}^\infty$ of $L^2(\Omega)$.

Proof. Let $y \in L^2((0,T);H_0^1(\Omega))$ be a strong solution of (1.1). Then, (3.1) holds. Therefore taking successively in (3.1) $v = 0$ and $y^T = 0$, we obtain that the two series in (3.20) converge.
Conversely, assume that the two series in (3.20) converge, then $y \in L^2((0,T);H_0^1(\Omega))$.
Combining (3.10) and (3.11), we can write that

$$\begin{aligned} I^{1-\alpha}Y_N(t) &= \sum_{i=1}^{N}\left|\frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{E_\alpha(-\lambda_i T^\alpha)}\right| E_\alpha(-\lambda_i t^\alpha)w_i \\ &+ \sum_{i=1}^{N}\left\{\int_0^t E_\alpha(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\} w_i, \end{aligned}$$

which implies that,

$$\begin{aligned} \|I^{1-\alpha}Y_N(t)\|^2_{L^2((0,T);H_0^1(\Omega))} &= \int_0^T a(I^{1-\alpha}Y_N(t), I^{1-\alpha}Y_N(t))dt \\ &\leq C_N + Z_N, \end{aligned}$$

where

$$\begin{aligned} C_N &= 2\sum_{i=1}^{N}\lambda_i|a_i|^2\int_0^T E_\alpha^2(-\lambda_i t^\alpha)dt, \\ Z_N &= 2\sum_{i=1}^{N}\int_0^T \lambda_i\left\{\int_0^t E_\alpha(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\}^2 dt. \end{aligned}$$

Using Theorem 2.1, (3.15) and proceeding as in [10], we obtain

$$\begin{aligned} C_N &= 2\sum_{i=1}^{N}\lambda_i|a_i|^2\int_0^T E_\alpha^2(-\lambda_i t^\alpha)dt \\ &\leq C^2\sum_{i=1}^{N}|a_i|^2\int_0^T t^{-\alpha}dt \\ &\leq \frac{C^2T^{1-\alpha}}{1-\alpha}\left(2\sum_{i=1}^{N}\frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + 2C^2\sum_{i=1}^{N}\frac{\int_0^T|v_i(s)|^2ds}{E_\alpha^2(-\lambda_i T^\alpha)}\right), \end{aligned} \tag{3.21}$$

We can also write that

$$Z_N \leq \frac{C^2T^{1-\alpha}}{1-\alpha}\sum_{i=1}^{N}\left(\int_0^T|v_i(s)|^2ds\right).$$

Consequently, we have

$$\begin{aligned} \|I^{1-\alpha}Y_N(t)\|^2_{L^2((0,T);H_0^1(\Omega))} &\leq \frac{2C^2T^{1-\alpha}}{1-\alpha}\sum_{i=1}^{N}\frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)} + \frac{2C^4T^{1-\alpha}}{1-\alpha}\sum_{i=1}^{N}\frac{\int_0^T|v_i(s)|^2ds}{E_\alpha^2(-\lambda_i T^\alpha)} \\ &+ \frac{C^2T^{1-\alpha}}{1-\alpha}\sum_{i=1}^{N}\left(\int_0^T|v_i(s)|^2ds\right), \end{aligned}$$

which implies that

$$\begin{aligned} \|I^{1-\alpha}Y_N(t)\|_{L^2((0,T);H_0^1(\Omega))} &\leq C\sqrt{\frac{2T^{1-\alpha}}{1-\alpha}}\left(\sum_{i=1}^{N}\frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)}\right)^{1/2} \\ &+ C^2\sqrt{\frac{2T^{1-\alpha}}{1-\alpha}}\left(\sum_{i=1}^{N}\frac{\int_0^T|v_i(s)|^2ds}{E_\alpha^2(-\lambda_i T^\alpha)}\right)^{1/2} \\ &+ C\sqrt{\frac{T^{1-\alpha}}{1-\alpha}}\left(\sum_{i=1}^{N}\int_0^T|v_i(s)|^2ds\right)^{1/2}. \end{aligned} \tag{3.22}$$

As $v \in L^2(Q)$ and Series in (3.20) converge, we have

$$\lim_{N\to+\infty}\left(\sum_{i=1}^{N}\frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)}\right)^{1/2} < \infty, \quad \lim_{N\to+\infty}\left(\sum_{i=1}^{N}\frac{\int_0^T|v_i(s)|^2ds}{E_\alpha^2(-\lambda_i T^\alpha)}\right)^{1/2} < \infty,$$

and

$$\lim_{N\to+\infty}\left(\sum_{i=1}^{N}\int_0^T |v_i(s)|^2 ds\right)^{1/2} = \|v\|_{L^2(Q)}.$$

This implies that $I^{1-\alpha}y \in L^2((0,T);H_0^1(\Omega))$.
Therefore

$$I^{1-\alpha}y(t) = \sum_{i=1}^{+\infty}\left\{\frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{E_\alpha(-\lambda_i T^\alpha)}E_\alpha(-\lambda_i t^\alpha) + \int_0^t E_\alpha(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\}w_i. \tag{3.23}$$

Since $y \in L^2((0,T);H_0^1(\Omega))$ is solution to (1.1) and $v \in L^2(Q)$, we have that

$$D_{RL}^\alpha y(t) = \Delta y(t) + v(t) \in H^{-1}(\Omega) \text{ for almost every } t \in (0,T).$$

Let $\varphi \in H_0^1(\Omega)$. If we multiply the first equation in (1.1) by φ and integrate by parts, we have that:

$$\begin{aligned}\int_\Omega D_{RL}^\alpha y(t)\varphi dx &= \int_\Omega \nabla y(t)\cdot\nabla\varphi dx + \int_\Omega v(t)\varphi dx\\ &\leq \|\nabla y(t)\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}\\ &\leq \Big(\|y(t)\|_{H_0^1(\Omega)} + C(\Omega)\|v(t)\|_{L^2(\Omega)}\Big)\|\varphi\|_{H_0^1(\Omega)}.\end{aligned} \tag{3.24}$$

This implies that $D_{RL}^\alpha y(t) \in H^{-1}(\Omega)$.

Let $\varphi \in L^2((0,T);H_0^1(\Omega))$. If we multiply the first equation in (1.1) by φ and integrate by parts, then in view of (3.24), we have that:

Hence, using again Cauchy-Schwartz inequality, we deduce that

$$\begin{aligned}\int_0^T\int_\Omega \left|D_{RL}^\alpha y(x,t)\varphi(x,t)\right| dxdt &\leq \int_0^T \Big(\|y(t)\|_{H_0^1(\Omega)} + C(\Omega)\|v(t)\|_{L^2(\Omega)}\Big)\|\varphi(t)\|_{H_0^1(\Omega)}dt\\ &\leq \Big(\|y\|_{L^2((0,T);H_0^1(\Omega))} + C(\Omega)\|v\|_{L^2(Q)}\Big)\|\varphi\|_{L^2((0,T);H_0^1(\Omega))},\end{aligned}$$

This implies that $D_{RL}^\alpha y \in L^2((0,T);H^{-1}(\Omega))$.

Finally, we showed that $I^{1-\alpha}y \in L^2((0,T);H_0^1(\Omega))$ and $D_{RL}^\alpha y \in L^2((0,T);H^{-1}(\Omega))$, then we have $I^{1-\alpha}y \in C([0,T];L^2(\Omega))$.

Hence, using (3.23), we have that

$$\begin{aligned}I^{1-\alpha}y(T) &= \sum_{i=1}^{+\infty}\left\{\frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{E_\alpha(-\lambda_i T^\alpha)}E_\alpha(-\lambda_i T^\alpha) + \int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds\right\}\\ &= \sum_{i=1}^{+\infty} y_i^T w_i = y^T.\end{aligned}$$

Consequently, we can conclude that y is a strong solution of (1.1) in the sense of Definition 3.1. ■

Now, using again eigenfunctions expansions of the Laplace operator and proceeding as the proof of the latter Lemma, we prove the existence and uniqueness of solution to the approximated problem (1.4).

Let V_N the space given in (3.2). Proceeding as in the proof of Lemma 3.1, we look for

$$Y_N^{\varepsilon}(x,t) = \sum_{i=1}^{N} (y^{\varepsilon}(t), w_i)_{L^2(\Omega)} w_i(x) = \sum_{i=1}^{N} y_i^{\varepsilon}(t) w_i(x). \tag{3.25}$$

the solution of the following approximate problem of (1.4):

$$\left\{ \begin{array}{rcll} D_{RL}^{\alpha} Y_N^{\varepsilon}(x,t) - \Delta Y_N^{\varepsilon}(x,t) &=& v_N(x,t) & (x,t) \in Q, \\ Y_N^{\varepsilon}(\sigma,t) &=& 0 & (\sigma,t) \in \Sigma, \\ I^{1-\alpha} Y_N^{\varepsilon}(x,T) + \varepsilon I^{1-\alpha} Y_N^{\varepsilon}(x,0) &=& y_N^T(x) & x \in \Omega, \end{array} \right. \tag{3.26}$$

where v_N and y_N^T are given respectively in (3.5) and (3.6).
We recall that if Y_N^{ε} converge then $\lim\limits_{N\to\infty} Y_N^{\varepsilon} = y^{\varepsilon}$, where y^{ε} satisfies (1.4).
We have the following result :

Theorem 3.1 *Let $3/4 < \alpha < 1$, $T > 0$, $v \in L^2(Q)$, and $y^T \in L^2(\Omega)$. Then, the approximate problem (1.4) has a unique solution $y^{\varepsilon} \in L^2((0,T); H_0^1(\Omega))$ given by*

$$\begin{array}{rcl} y^{\varepsilon}(t) &=& \displaystyle\sum_{i=1}^{+\infty} \left\{ \frac{y_i^T - \displaystyle\int_0^T E_{\alpha}(-\lambda_i (T-s)^{\alpha}) v_i(s) ds}{\varepsilon + E_{\alpha}(-\lambda_i T^{\alpha})} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^{\alpha}) \right. \\ && \\ &+& \left. \displaystyle\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^{\alpha}) v_i(s) ds \right\} w_i. \end{array} \tag{3.27}$$

where λ_i is the eigenvalue of the operator $-\Delta$ corresponding to the eigenfunction w_i. $E_{\alpha,\alpha}$ as given in (2.2), $y_i^T = (y^T, w_i)_{L^2(\Omega)}$ and $v_i(t) = (v(t), w_i)_{L^2(\Omega)}$ are respectively, the i-th component of y^T and $v(t)$ in the orthonormal basis $\{w_i\}_{i=1}^{\infty}$ of $L^2(\Omega)$. Moreover, $I^{1-\alpha} y^{\varepsilon} \in C([0,T]; L^2(\Omega))$ and there exists a constant $C > 0$ such that,

$$\|y^{\varepsilon}\|_{L^2((0,T);H_0^1(\Omega))} \leq \Pi \left(\|y^T\|_{L^2(\Omega)} + \|v\|_{L^2(Q)} \right), \tag{3.28}$$

and

$$\left\| I^{1-\alpha} y^{\varepsilon} \right\|_{L^2((0,T);H_0^1(\Omega))} \leq \Theta \left(\|y^T\|_{L^2(\Omega)} + \|v\|_{L^2(Q)} \right), \tag{3.29}$$

where

$$\Pi = \max \left(\frac{C^2}{\varepsilon} \sqrt{\frac{2T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}}, \sqrt{\frac{2C^6 T^{3\alpha-2}}{\varepsilon^2 (4\alpha-3)(1-\alpha)} + \frac{4C^2 T^{\alpha}}{\alpha - \frac{1}{2}}} \right)$$

and

$$\Theta = \sup \left(\frac{C}{\varepsilon} \sqrt{\frac{T^{1-\alpha}}{1-\alpha}}, \sqrt{\frac{2C^4 T^{1-\alpha}}{\varepsilon^2 (1-\alpha)} + \frac{C^2 T^{2-2\alpha}}{\lambda_1 (1-\alpha)^2}} \right).$$

Proof. If we replace Y_N^ε in (3.26) by $\sum_{i=1}^{N} y_i^\varepsilon(t)w_i(x)$, we obtain that y_i^ε, $i = 1, \cdots, N$ is a solution of the ordinary differential equation

$$\left\{ \begin{array}{rcl} D_{RL}^\alpha y_i^\varepsilon(t) + \lambda_i y_i^\varepsilon(t) &=& v_i(t), \quad t \in (0,T), \\ I^{1-\alpha} y_i^\varepsilon(T) + \varepsilon I^{1-\alpha} y_i^\varepsilon(0) &=& y_i^T. \end{array} \right. \tag{3.30}$$

Now, using the Laplace transform, and proceeding as in [10], we obtain

$$y_i^\varepsilon(t) = I^{1-\alpha} y_i^\varepsilon(0^+) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) v_i(s) ds, \tag{3.31}$$

and

$$I^{1-\alpha} y_i^\varepsilon(t) = I^{1-\alpha} y_i^\varepsilon(0) E_\alpha(-\lambda_i t^\alpha) + \int_0^t E_\alpha(-\lambda_i (t-s)^\alpha) v_i(s) ds. \tag{3.32}$$

Therefore, we have

$$I^{1-\alpha} y_i^\varepsilon(0) = \frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i (T-s)^\alpha) v_i(s) ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)}, \tag{3.33}$$

which implies that

$$\begin{array}{rcl} y_i^\varepsilon(t) &=& \left[\dfrac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i (T-s)^\alpha) v_i(s) ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} \right] t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) \\ && + \displaystyle\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) v_i(s) ds. \end{array}$$

It then follows from (3.25) that

$$\begin{array}{rcl} Y_N^\varepsilon(t) &=& \displaystyle\sum_{i=1}^{N} \left\{ \dfrac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i (T-s)^\alpha) v_i(s) ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) \right\} w_i \\ && + \displaystyle\sum_{i=1}^{N} \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) v_i(s) ds \right\} w_i. \end{array} \tag{3.34}$$

Set $b_i = \dfrac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)}$. Then, we have that,

$$\begin{aligned} a(Y_N^\varepsilon(t), Y_N^\varepsilon(t)) &= \sum_{i=1}^N \lambda_i [y_i^\varepsilon(t)]^2 \\ &\leq 2\sum_{i=1}^N \lambda_i t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha)|b_i|^2 \\ &+ 2\sum_{i=1}^N \lambda_i \left\{\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\}^2. \end{aligned}$$

Hence,

$$\|Y_N^\varepsilon(t)\|^2_{L^2((0,T);H_0^1(\Omega))} = \int_0^T a(Y_N^\varepsilon(t), Y_N^\varepsilon(t))dt \leq A_N^\varepsilon + B_N^\varepsilon,$$

with

$$\begin{aligned} A_N^\varepsilon &= 2\sum_{i=1}^N \lambda_i |b_i|^2 \int_0^T t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha)dt, \\ B_N^\varepsilon &= 2\sum_{i=1}^N \int_0^T \lambda_i \left\{\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\}^2 dt. \end{aligned}$$

Proceeding as in (3.13), we know that there exists a generic constant $C > 0$ such that

$$A_N^\varepsilon \leq \frac{C^4 T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)} \sum_{i=1}^N |b_i|^2 \tag{3.35}$$

Using again Theorem 2.1, we obtain

$$\begin{aligned} \sum_{i=1}^N |b_i|^2 &= \sum_{i=1}^N \left| \frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} \right|^2 \\ &\leq \frac{2}{\varepsilon^2}\sum_{i=1}^N |y_i^T|^2 + \frac{2C^2}{\varepsilon^2}\sum_{i=1}^N \int_0^T |v_i(s)|^2 ds \end{aligned}$$

Consequently,

$$\sum_{i=1}^N |b_i|^2 \leq \frac{2}{\varepsilon^2}\sum_{i=1}^N |y_i^T|^2 + \frac{2C^2}{\varepsilon^2}\sum_{i=1}^N \int_0^T |v_i(s)|^2 ds. \tag{3.36}$$

and we have that

$$A_N^\varepsilon \leq \frac{2C^4 T^{3\alpha-2}}{\varepsilon^2(4\alpha-3)(1-\alpha)} \sum_{i=1}^N |y_i^T|^2 + \frac{2C^6 T^{3\alpha-2}}{\varepsilon^2(4\alpha-3)(1-\alpha)} \left[\sum_{i=1}^N \int_0^T |v_i(s)|^2 ds\right]. \tag{3.37}$$

On the other hand, using the Cauchy-Schwartz inequality and proceeding as in [10], we have

$$B_N^\varepsilon \le \frac{4C^2T^\alpha}{\alpha-\frac{1}{2}}\sum_{i=1}^N\int_0^T|v_i(s)|^2ds \tag{3.38}$$

Combining (3.37) and (3.38), we obtain

$$\begin{aligned}\|Y_N^\varepsilon(t)\|^2_{L^2((0,T);H_0^1(\Omega))} &\le \frac{2C^4T^{3\alpha-2}}{\varepsilon^2(4\alpha-3)(1-\alpha)}\sum_{i=1}^N|y_i^T|^2\\ &+ \left[\frac{2C^6T^{3\alpha-2}}{\varepsilon^2(4\alpha-3)(1-\alpha)}+\frac{4C^2T^\alpha}{\alpha-\frac{1}{2}}\right]\left[\sum_{i=1}^N\int_0^T|v_i(s)|^2ds\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\|Y_N^\varepsilon(t)\|_{L^2((0,T);H_0^1(\Omega))} &\le \frac{C^2}{\varepsilon}\sqrt{\frac{2T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}}\left(\sum_{i=1}^N|y_i^T|^2\right)^{1/2}\\ &+ \sqrt{\frac{2C^6T^{3\alpha-2}}{\varepsilon^2(4\alpha-3)(1-\alpha)}+\frac{4C^2T^\alpha}{\alpha-\frac{1}{2}}}\left(\sum_{i=1}^N\int_0^T|v_i(s)|^2ds\right)^{1/2}.\end{aligned} \tag{3.39}$$

In view of Equation (3.32) and (3.33), we have

$$I^{1-\alpha}Y_N^\varepsilon(t)=\sum_{i=1}^N|b_i|E_\alpha(-\lambda_it^\alpha)w_i+\sum_{i=1}^N\left\{\int_0^tE_\alpha(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\}w_i,$$

from which we deduce that,

$$\begin{aligned}\|I^{1-\alpha}Y_N^\varepsilon(t)\|^2_{L^2((0,T);H_0^1(\Omega))} &\le \int_0^Ta(I^{1-\alpha}Y_N^\varepsilon(t),I^{1-\alpha}Y_N^\varepsilon(t))dt\\ &\le 2\sum_{i=1}^N\lambda_i|b_i|^2\int_0^TE_\alpha^2(-\lambda_it^\alpha)dt\\ &+ 2\sum_{i=1}^N\int_0^T\lambda_i\left\{\int_0^tE_\alpha(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\}^2dt.\end{aligned}$$

If we set

$$\begin{aligned}C_N^\varepsilon &= 2\sum_{i=1}^N\lambda_i|b_i|^2\int_0^TE_\alpha^2(-\lambda_it^\alpha)dt,\\ Z_N^\varepsilon &= 2\sum_{i=1}^N\int_0^T\lambda_i\left\{\int_0^tE_\alpha(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\}^2dt.\end{aligned}$$

Proceeding as in (3.21), we obtain

$$C_N^\varepsilon \le \frac{C^2T^{1-\alpha}}{1-\alpha}\sum_{i=1}^N|b_i|^2, \tag{3.40}$$

which combining with (3.36), gives

$$C_N^\varepsilon \le \frac{2C^2T^{1-\alpha}}{\varepsilon^2(1-\alpha)}\sum_{i=1}^N |y_i^T|^2 + \frac{2C^4T^{1-\alpha}}{\varepsilon^2(1-\alpha)}\sum_{i=1}^N \int_0^T |v_i(s)|^2 ds. \tag{3.41}$$

On the other hand, we can write, using Theorem 2.1 and Cauchy-Schwartz inequality, that

$$\begin{aligned}
Z_N^\varepsilon &\le 2\sum_{i=1}^N \int_0^T \lambda_i \left\{\frac{C}{\lambda_i}\int_0^t (t-s)^{-\alpha} v_i(s)ds\right\}^2 dt \\
&\le \frac{C^2}{\lambda_1}\sum_{i=1}^N \int_0^T \left\{\int_0^t (t-s)^{-\frac{\alpha}{2}}(t-s)^{-\frac{\alpha}{2}} v_i(s)ds\right\}^2 dt \\
&\le \frac{C^2}{\lambda_1}\sum_{i=1}^N \int_0^T \left(\int_0^t (t-s)^{-\alpha}ds\right)\left(\int_0^t (t-s)^{-\alpha}|v_i(s)|^2 ds\right) dt \\
&\le \frac{C^2}{\lambda_1}\sum_{i=1}^N \int_0^T \left[-\frac{(t-s)^{1-\alpha}}{1-\alpha}\right]_0^t \left(\int_0^t (t-s)^{-\alpha}|v_i(s)|^2 ds\right) dt \\
&\le \frac{C^2T^{1-\alpha}}{\lambda_1(1-\alpha)}\sum_{i=1}^N \int_0^T |v_i(s)|^2 \left(\int_s^T (t-s)^{-\alpha}dt\right) ds \\
&\le \frac{C^2T^{2-2\alpha}}{\lambda_1(1-\alpha)^2}\sum_{i=1}^N \int_0^T |v_i(s)|^2 ds.
\end{aligned} \tag{3.42}$$

Combining (3.41) and the latter estimation of Z_N^ε, we finally obtain

$$\begin{aligned}
\|I^{1-\alpha}Y_N^\varepsilon(t)\|^2_{L^2((0,T);H_0^1(\Omega))} &\le \frac{2C^2T^{1-\alpha}}{\varepsilon^2(1-\alpha)}\sum_{i=1}^N |y_i^T|^2 + \frac{2C^4T^{1-\alpha}}{\varepsilon^2(1-\alpha)}\sum_{i=1}^N \int_0^T |v_i(s)|^2 ds \\
&+ \frac{C^2T^{2-2\alpha}}{\lambda_1(1-\alpha)^2}\sum_{i=1}^N \int_0^T |v_i(s)|^2 ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|I^{1-\alpha}Y_N^\varepsilon(t)\|_{L^2((0,T);H_0^1(\Omega))} &\le \frac{C}{\varepsilon}\sqrt{\frac{2T^{1-\alpha}}{1-\alpha}}\left(\sum_{i=1}^N |y_i^T|^2\right)^{1/2} \\
&+ \sqrt{\frac{2C^4T^{1-\alpha}}{\varepsilon^2(1-\alpha)} + \frac{C^2T^{2-2\alpha}}{\lambda_1(1-\alpha)^2}}\left(\sum_{i=1}^N \int_0^t |v_i(s)|^2 ds\right)^{1/2}.
\end{aligned} \tag{3.43}$$

As $y^T \in L^2(\Omega)$ and $v \in L^2(Q)$, we have

$$\lim_{N\to+\infty}\left(\sum_{i=1}^N |y_i^T|^2\right)^{1/2} < \infty \quad \text{and} \quad \lim_{N\to+\infty}\left(\sum_{i=1}^N \int_0^t |v_i(s)|^2 ds\right)^{1/2} < \infty.$$

Consequently, we have $y^\varepsilon \in L^2((0,T);H_0^1(\Omega))$ and $I^{1-\alpha}y^\varepsilon \in L^2((0,T);H_0^1(\Omega))$. And we have

$$
\begin{aligned}
y^\varepsilon(t) &= \sum_{i=1}^{+\infty}\left\{\frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)\right\} w_i \\
&+ \sum_{i=1}^{+\infty}\left\{\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\} w_i,
\end{aligned}
\tag{3.44}
$$

and

$$
\begin{aligned}
I^{1-\alpha}y^\varepsilon(t) &= \sum_{i=1}^{+\infty}\left\{\frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} E_\alpha(-\lambda_i t^\alpha)\right. \\
&+ \left.\int_0^t E_\alpha(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\} w_i.
\end{aligned}
\tag{3.45}
$$

Now, proceeding as in the proof of Lemma 3.2, we can say that $D_{RL}^\alpha y^\varepsilon \in L^2((0,T);H^{-1}(\Omega))$, which implies that $I^{1-\alpha}y^\varepsilon \in C([0,T];L^2(\Omega))$. Then we know that $I^{1-\alpha}y^\varepsilon(T)$ and $I^{1-\alpha}y^\varepsilon(0)$ exist and belong to $L^2(\Omega)$.
From (3.45), we have

$$
\begin{aligned}
I^{1-\alpha}y^\varepsilon(T) + \varepsilon I^{1-\alpha}y^\varepsilon(0) &= \sum_{i=1}^{+\infty}\left\{\frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} E_\alpha(-\lambda_i T^\alpha)\right. \\
&+ \left.\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds\right\} w_i \\
&+ \varepsilon\sum_{i=1}^{+\infty}\left\{\frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} E_\alpha(-\lambda_i 0^\alpha)\right\} w_i \\
&+ \left.\varepsilon\frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} + \int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds\right\} w_i \\
&= \sum_{i=1}^{+\infty}\left\{\frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} (\varepsilon + E_\alpha(-\lambda_i T^\alpha))\right. \\
&+ \left.\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)v_i(s)ds\right\} w_i \\
&= \sum_{i=1}^{+\infty} y_i^T w_i = y^T.
\end{aligned}
$$

Passing to the limit, when $N \to \infty$ in (3.39) and (3.43), we obtain (3.28) and (3.29).

■

We have the following remark.

Remark 3.2 *Let $3/4 < \alpha < 1$ and $y^\varepsilon \in L^2((0,T); H_0^1(\Omega))$ be the solution of (1.4). Then, there exists a constant $C > 0$ independent of ε such that,*

$$\|y^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq \Pi \left(\|I^{1-\alpha} y^\varepsilon(0)\|_{L^2(\Omega)} + \|v\|_{L^2(Q)} \right), \tag{3.46}$$

and

$$\|I^{1-\alpha} y^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq \Theta \left(\|I^{1-\alpha} y^\varepsilon(0)\|_{L^2(\Omega)} + \|v\|_{L^2(Q)} \right), \tag{3.47}$$

where

$$\Pi = \max \left(C^2 \sqrt{\frac{T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}}, 2C \sqrt{\frac{T^\alpha}{\alpha - \frac{1}{2}}} \right),$$

and

$$\Theta = \max \left(C \sqrt{\frac{T^{1-\alpha}}{1-\alpha}}, \frac{C T^{1-\alpha}}{\sqrt{\lambda_1}(1-\alpha)} \right).$$

From Theorem 3.1, we have $I^{1-\alpha} y^\varepsilon \in C([0,T]; L^2(\Omega))$ then we know that $I^{1-\alpha} y^\varepsilon(0)$ exists and belongs to $L^2(\Omega)$. Hence from (3.31) and (3.32), we can write that: $\forall t \in (0,T)$,

$$y^\varepsilon(t) = \sum_{i=1}^{+\infty} \left\{ I^{1-\alpha} y_i^\varepsilon(0) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) v_i(s) ds \right\} w_i, \tag{3.48}$$

and

$$I^{1-\alpha} y^\varepsilon(t) = \sum_{i=1}^{+\infty} \left\{ I^{1-\alpha} y_i^\varepsilon(0) E_\alpha(-\lambda_i t^\alpha) + \int_0^t E_\alpha(-\lambda_i (t-s)^\alpha) v_i(s) ds \right\} w_i. \tag{3.49}$$

Therefore, using Theorem 2.1 and the Cauchy-Schwartz inequality, we obtain from (3.48) and proceeding as in (3.13)

$$\begin{aligned}
\|y^\varepsilon\|^2_{L^2((0,T);H_0^1(\Omega))} &\leq 2 \sum_{i=1}^{+\infty} \lambda_i \left| I^{1-\alpha} y_i^\varepsilon(0) \right|^2 \int_0^T t^{2\alpha-2} E^2_{\alpha,\alpha}(-\lambda_i t^\alpha) dt \\
&+ 2 \sum_{i=1}^{+\infty} \int_0^T \lambda_i \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) v_i(s) ds \right\}^2 dt. \\
&\leq \frac{C^4 T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)} \left(\sum_{i=1}^{+\infty} \left| I^{1-\alpha} y_i^\varepsilon(0) \right|^2 \right) + \frac{4C^2 T^\alpha}{\alpha - \frac{1}{2}} \left(\sum_{i=1}^{+\infty} \int_0^T |v_i(s)|^2 ds \right).
\end{aligned}$$

Thus

$$\|y^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq C^2 \sqrt{\frac{T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}} \|I^{1-\alpha} y^\varepsilon(0)\|_{L^2(\Omega)} + 2C \sqrt{\frac{T^\alpha}{\alpha - \frac{1}{2}}} \|v\|_{L^2(Q)}.$$

Moreover, from (3.49) *and proceeding as in* (3.21) *and* (3.42), *we obtain*

$$
\begin{aligned}
\|I^{1-\alpha}y^{\varepsilon}(t)\|^2_{L^2((0,T);H_0^1(\Omega))} &\le 2\sum_{i=1}^{+\infty}\lambda_i|I^{1-\alpha}y^{\varepsilon}(0)|^2\int_0^T E^2_{\alpha,1}(-\lambda_i t^{\alpha})dt \\
&+ 2\sum_{i=1}^{+\infty}\int_0^T \lambda_i\left\{\int_0^t E_{\alpha,1}(-\lambda_i(t-s)^{\alpha})v_i(s)ds\right\}^2 dt \\
&\le \frac{C^2T^{1-\alpha}}{1-\alpha}\left(\sum_{i=1}^{+\infty}|I^{1-\alpha}y^{\varepsilon}(0)|^2\right)+\frac{C^2T^{2-2\alpha}}{\lambda_1(1-\alpha)^2}\left(\sum_{i=1}^{+\infty}\int_0^T|v_i(s)|^2ds\right)
\end{aligned}
$$

Hence, we have

$$
\|I^{1-\alpha}y^{\varepsilon}(t)\|_{L^2((0,T);H_0^1(\Omega))}\le C\sqrt{\frac{T^{1-\alpha}}{1-\alpha}}\|I^{1-\alpha}y^{\varepsilon}(0)\|_{L^2(\Omega)}+\frac{CT^{1-\alpha}}{\sqrt{\lambda_1}(1-\alpha)}\|v\|_{L^2(Q)}.
$$

We also have the following results, which are useful for characterizing our approximate optimal control:

Theorem 3.2 *Let $\varepsilon > 0$, $0 < \alpha < 1$ and $p^T \in L^2(\Omega)$. Then the problem*

$$
\left\{
\begin{array}{rcll}
D_C^{\alpha}p^{\varepsilon}(x,t)-\Delta p^{\varepsilon}(x,t) &=& 0 & (x,t)\in Q,\\
p^{\varepsilon}(\sigma,t) &=& 0 & (\sigma,t)\in\Sigma,\\
p^{\varepsilon}(x,T)+\varepsilon p^{\varepsilon}(x,0^+) &=& p^T(x) & x\in\Omega,
\end{array}
\right.
\tag{3.50}
$$

has a unique solution $p^{\varepsilon}\in C([0,T];L^2(\Omega))$ given by

$$
p^{\varepsilon}(t) = \sum_{i=1}^{+\infty}\left[\frac{p_i^T}{\varepsilon+E_{\alpha}(-\lambda_i T^{\alpha})}\right]E_{\alpha}(-\lambda_i t^{\alpha})w_i. \tag{3.51}
$$

where E_α is given by (2.3), *λ_i is the eigenvalue of the operator $-\Delta$ corresponding to the eigenfunction w_i. $p_i^T = (p^T, w_i)_{L^2(\Omega)}$ is the i-th component of p^T in the orthonormal basis $\{w_i\}_{i=1}^{\infty}$ of $L^2(\Omega)$. Moreover, there exists a constant $C > 0$ such that,*

$$
\|p^{\varepsilon}\|_{C([0,T];L^2(\Omega))}\le\frac{C}{\varepsilon}\|p^T\|_{L^2(\Omega)}. \tag{3.52}
$$

Proof. To prove this theorem, we decide to use again the spectral method.
Let V_N the space given in (3.2), we look for

$$
P_N^{\varepsilon}(x,t)=\sum_{i=1}^{N}(p^{\varepsilon}(t),w_i)_{L^2(\Omega)}w_i(x)=\sum_{i=1}^{N}p_i^{\varepsilon}(t)w_i(x). \tag{3.53}
$$

the solution of the following approximate of (3.50)

$$
\left\{
\begin{array}{rcll}
D_C^{\alpha}P_N^{\varepsilon}(x,t)-\Delta P_N^{\varepsilon}(x,t) &=& 0 & (x,t)\in Q,\\
P_N^{\varepsilon}(\sigma,t) &=& 0 & (\sigma,t)\in\Sigma,\\
P_N^{\varepsilon}(x,T)+\varepsilon P_N^{\varepsilon}(x,0) &=& p_N^T(x) & x\in\Omega,
\end{array}
\right.
\tag{3.54}
$$

where

$$p_N^T = \sum_{i=1}^{N} (p^T, w_i)_{L^2(\Omega)} w_i(x) = \sum_{i=1}^{N} p_i^T w_i(x). \tag{3.55}$$

We know that if P_N^ε converge then $\lim_{N\to\infty} P_N^\varepsilon = p^\varepsilon$, where p^ε satisfies (3.50).

If we replace P_N^ε in (3.54) by $\sum_{i=1}^{N} p_i^\varepsilon(t) w_i(x)$, we obtain that y_i, $i = 1, \cdots, N$ is a solution of the ordinary differential equation

$$\left\{ \begin{array}{rcl} D_C^\alpha p_i^\varepsilon(t) + \lambda_i p_i^\varepsilon(t) & = & 0, \quad t \in (0,T), \\ p_i^\varepsilon(T) + \varepsilon p_i^\varepsilon(0^+) & = & p_i^T. \end{array} \right. \tag{3.56}$$

Using the Laplace transform and proceeding as in the proof of Lemma (3.1) , we obtain

$$p_i^\varepsilon(t) = p_i^\varepsilon(0^+) E_\alpha(-\lambda_i t^\alpha).$$

Therefore, we have

$$p_i^\varepsilon(T) = p_i^\varepsilon(0^+) E_\alpha(-\lambda_i T^\alpha),$$

which gives from $(3.56)_2$

$$p_i^\varepsilon(0) = \frac{p_i^T}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)}, \tag{3.57}$$

and, we obtain

$$p_i^\varepsilon(t) = \left\{ \frac{p_i^T}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} \right\} E_\alpha(-\lambda_i t^\alpha).$$

Therefore, from (3.53) we can write that

$$P_N^\varepsilon(t) = \sum_{i=1}^{N} \left\{ \frac{p_i^T}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} \right\} E_\alpha(-\lambda_i t^\alpha) w_i. \tag{3.58}$$

Using Theorem 2.1, we obtain

$$\|P_N^\varepsilon(t)\|_{L^2(\Omega)}^2 = \sum_{i=1}^{N} \left| \frac{p_i^T}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} E_\alpha(-\lambda_i t^\alpha) \right|^2 \leq \frac{C^2}{\varepsilon^2} \sum_{i=1}^{N} |p_i^T|^2,$$

then, we can deduce that

$$\|P_N^\varepsilon(t)\|_{C([0,T];L^2(\Omega))} = \sup_{t\in[0,T]} \|P_N^\varepsilon(t)\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon} \left(\sum_{i=1}^{N} |p_i^T|^2 \right)^{1/2}. \tag{3.59}$$

As $p^T \in L^2(\Omega)$, we know that

$$\lim_{N\to+\infty} \left(\sum_{i=1}^{N} |p_i^T|^2 \right)^{1/2} < \infty.$$

Therefore, we have $p^\varepsilon \in C([0,T];L^2(\Omega))$ and we can write that

$$p^\varepsilon(t) = \sum_{i=1}^{+\infty} \left\{ \frac{p_i^T}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} \right\} E_\alpha(-\lambda_i t^\alpha) w_i. \tag{3.60}$$

Moreover, from (3.60), we have

$$\begin{array}{rcl} p^\varepsilon(T) + \varepsilon p^\varepsilon(0) & = & \displaystyle\sum_{i=1}^{+\infty} \left\{ \frac{p_i^T}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} \right\} E_\alpha(-\lambda_i T^\alpha) w_i + \varepsilon \sum_{i=1}^{+\infty} \left\{ \frac{p_i^T}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} \right\} w_i \\ & = & \displaystyle\sum_{i=1}^{+\infty} \left\{ \frac{p_i^T}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} \right\} (\varepsilon + E_\alpha(-\lambda_i T^\alpha)) w_i \\ & = & \displaystyle\sum_{i=1}^{+\infty} p_i^T = p^T. \end{array}$$

To finish, passing to the limit, when $N \to +\infty$ in (3.59), we can deduce (3.52). ■
From the latter theorem, we can deduce the following result:

Corollary 3.1 *Let $0 < \alpha < 1$ and $p^T \in L^2(\Omega)$. Then problem*

$$\left\{ \begin{array}{rcll} -\mathcal{D}_C^\alpha p(x,t) - \Delta p(x,t) & = & 0 & (x,t) \in Q, \\ p(\sigma,t) & = & 0 & (\sigma,t) \in \Sigma, \\ \varepsilon p(x,T) + p(x,0^+) & = & p^T(x) & x \in \Omega, \end{array} \right. \tag{3.61}$$

where $\mathcal{D}_C^\alpha$ is the right Caputo fractional of order $0 < \alpha < 1$, admits a unique solution $p \in C([0,T];L^2(\Omega))$. Moreover, there exists a constant $C > 0$ such that,

$$\|p\|_{C([0,T];L^2(\Omega))} \leq \frac{C}{\varepsilon} \|p^T\|_{L^2(\Omega)}. \tag{3.62}$$

Proof. Making the change of variable $t \to T - t$ in (3.50), we obtain the following equivalent problem

$$\left\{ \begin{array}{rcll} -\mathcal{D}_C^\alpha \psi(x,t) - \Delta \psi(x,t) & = & 0 & (x,t) \in Q, \\ \psi(\sigma,t) & = & 0 & (\sigma,t) \in \Sigma, \\ \varepsilon \psi(x,T) + \psi(x,0^+) & = & p^T(x) & x \in \Omega, \end{array} \right.$$

where $\psi(x,t) = p(x,T-t)$. Therefore, using theorem 3.2, we can say that the latter equation has a unique solution $\psi \in C([0,T];L^2(\Omega))$. Moreover there exists a constant $C > 0$ such that

$$\|\psi\|_{C([0,T];L^2(\Omega))} \leq \frac{C}{\varepsilon} \|p^T\|_{L^2(\Omega)}.$$

■

4 Optimal control problems

In this section, we assume that $y^T \in L^2(\Omega)$ and $v \in L^2(Q)$ such that Series in (3.20) converge. Our goal is to solve the non-well posed problem (1.1)-(1.2). Let $\mathbb{U}_{ad}$ be a suitable nonempty closed and convex subset of $L^2(Q)$ and $\mathcal{A}$ be defined as in (1). For instance, we can consider the following nonempty closed and convex subset of $L^2(Q)$:

$$\mathbb{U}_{ad} := \Big\{ v \in L^2(Q) \text{ such that Series (3.20) converge} \Big\}. \tag{4.1}$$

Remark 4.1 *We have the following observations:*

1. *From Lemma* (3.2)*, we know that Equation* (1.1) *admits a strong solution in the sense of Definition 3.1. Therefore,* $\mathcal{A} \neq \emptyset$*.*

2. *Let* y *be a strong solution of Equation* (1.1)*, then we know that* $I^{1-\alpha}y \in C([0,T]; L^2(\Omega))$*, which implies that* $I^{1-\alpha}y(\cdot, 0)$ *exists and belongs to* $L^2(\Omega)$*. Therefore, the cost function* J *which we defined in* (1.3) *has a sense.*

3. *We can prove that optimal control problem* (1.1)-(1.2) *admits a unique solution* $(u, y) \in \mathcal{A}$*, using minimizing sequences, the structure of the functional* J *and estimations given in the proof of Lemma 3.2. Moreover, using the Euler-Lagrange optimality condition, we can give the following result:*

$$\int_\Omega (I^{1-\alpha}y(u,0) - z_d) I^{1-\alpha}y(v-u,0)dx + N \int_\Omega \int_0^T u(v-u)dtdx \geq 0 \quad \forall (v, y) \in \mathcal{A}.$$

   However, as mentioned in the introduction, Equation (1.1) *is not well-posed in the Hadamard sense, then the increase of the state and the control in the latter estimation are linked. This is why, we decided to use the quasi-reversibility method.*

Let's start with the following existence and uniqueness result for the approximated problem:

Theorem 4.1 *For every* $\varepsilon > 0$*, there exists a unique control* $u^\varepsilon \in \mathfrak{U}_{ad}$ *such that* (1.4)-(1.5) *holds.*

Proof. Let $(v_n) \in \mathfrak{U}_{ad}$ be a minimizing sequence such that

$$\lim_{n \to +\infty} J^\varepsilon(v_n) = \inf_{v \in \mathfrak{U}_{ad}} J^\varepsilon(v). \tag{4.2}$$

Then, there exists a constant $C > 0$ such that $J^\varepsilon(v_n) \leq C$. Hence, we obtain

$$\|v_n\|_{L^2(Q)} \leq C, \tag{4.3a}$$

$$\|I^{1-\alpha}y_n^\varepsilon(\cdot, 0)\|_{L^2(\Omega)} \leq C. \tag{4.3b}$$

Moreover, let $y_n^\varepsilon = y^\varepsilon(v_n; x, t)$ be solution of the following equation

$$D_{RL}^\alpha y_n^\varepsilon(x,t) - \Delta y_n^\varepsilon(x,t) = v_n(x,t), \tag{4.4a}$$

$$y_n^\varepsilon(\sigma, t) = 0, \tag{4.4b}$$

$$I^{1-\alpha}y_n^\varepsilon(x,T) + \varepsilon I^{1-\alpha}y_n^\varepsilon(x, 0^+) = y^T(x). \tag{4.4c}$$

From (4.4c), we have

$$I^{1-\alpha}y_n^\varepsilon(\cdot, T) = y^T - \varepsilon I^{1-\alpha}y_n^\varepsilon(\cdot, 0), \tag{4.5}$$

and combining (4.3b) and (4.5), we obtain

$$\begin{aligned}
\|I^{1-\alpha}y_n^\varepsilon(\cdot, T)\|_{L^2(\Omega)} &= \|y^T - \varepsilon I^{1-\alpha}y_n^\varepsilon(\cdot, 0)\|_{L^2(\Omega)} \\
&\leq \|y^T\|_{L^2(\Omega)} + \|\varepsilon I^{1-\alpha}y_n^\varepsilon(\cdot, 0)\|_{L^2(\Omega)} \\
&\leq \|y^T\|_{L^2(\Omega)} + \varepsilon C.
\end{aligned}$$

Then we have

$$\|I^{1-\alpha}y_n^\varepsilon(\cdot,T)\|_{L^2(\Omega)} \le \|y^T\|_{L^2(\Omega)} + \varepsilon C. \tag{4.6}$$

From Theorem 3.1, we know there exists a constant $C > 0$ such that

$$\|y_n^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \le C\|y^T\|_{L^2(\Omega)}, \tag{4.7a}$$

$$\|I^{1-\alpha}y_n^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \le C\|y^T\|_{L^2(\Omega)}. \tag{4.7b}$$

Combining (4.4a) and (4.3a), we obtain

$$\|D_{RL}^\alpha y_n^\varepsilon - \Delta y_n^\varepsilon\|_{L^2(Q)} \le C. \tag{4.8}$$

It follows from (4.3a), (4.3b), (4.6), (4.7a), (4.7b) and (4.8) that there exist $u^\varepsilon \in L^2(Q)$, $y^\varepsilon \in L^2((0,T);H_0^1(\Omega)), \gamma \in L^2((0,T);H_0^1(\Omega))$, $\delta \in L^2(Q)$, $\pi_1 \in L^2(\Omega)$, $\pi_2 \in L^2(\Omega)$ and we can extract subsequences of (v_n) and (y_n^ε) (still called (v_n) and (y_n^ε)), such that

$$v_n \rightharpoonup u^\varepsilon \quad \text{weakly in} \quad L^2(Q), \tag{4.9a}$$

$$y_n^\varepsilon \rightharpoonup y^\varepsilon \quad \text{weakly in} \quad L^2((0,T);H_0^1(\Omega)), \tag{4.9b}$$

$$I^{1-\alpha}y_n^\varepsilon \rightharpoonup \gamma \quad \text{weakly in} \quad L^2((0,T);H_0^1(\Omega)), \tag{4.9c}$$

$$D_{RL}^\alpha y_n^\varepsilon - \Delta y_n^\varepsilon \rightharpoonup \delta \quad \text{weakly in} \quad L^2(Q), \tag{4.9d}$$

$$I^{1-\alpha}y_n^\varepsilon(\cdot,0) \rightharpoonup \pi_1 \quad \text{weakly in} \quad L^2(\Omega), \tag{4.9e}$$

$$I^{1-\alpha}y_n^\varepsilon(\cdot,T) \rightharpoonup \pi_2 \quad \text{weakly in} \quad L^2(\Omega). \tag{4.9f}$$

$\mathcal{U}_{ad}$ being a closed subset of $L^2(Q)$, we can write

$$u^\varepsilon \in \mathcal{U}_{ad}. \tag{4.10}$$

Set $\mathbb{D}(Q)$, the set of C^∞ function on Q with compact support and denote by $\mathbb{D}'(Q)$ its dual. Then multiplying (4.4a) by $\varphi \in \mathbb{D}(Q)$ and integrating by part over Q, we obtain

$$\int_0^T \int_\Omega (D_{RL}^\alpha y_n^\varepsilon(x,t) - \Delta y_n^\varepsilon(x,t))\varphi(x,t)dxdt = \int_0^T \int_\Omega v_n(x,t)\varphi(x,t)dxdt. \tag{4.11}$$

Using Lemma 2.1, we can write

$$\int_0^T \int_\Omega (D_{RL}^\alpha y_n^\varepsilon(x,t) - \Delta y_n^\varepsilon(x,t))\varphi(x,t)dxdt =$$
$$\int_0^T \int_\Omega y_n^\varepsilon(x,t)(-\mathcal{D}_C^\alpha\varphi(x,t) - \Delta\varphi(x,t))dxdt.$$

Then passing to the limit in the latter equality when $n \to +\infty$ and using (4.9b), we obtain

$$\lim_{n\to+\infty} \int_0^T \int_\Omega (D_{RL}^\alpha y_n^\varepsilon(x,t) - \Delta y_n^\varepsilon(x,t))\varphi(x,t)dxdt =$$
$$\int_0^T \int_\Omega y^\varepsilon(x,t)(-\mathcal{D}_C^\alpha\varphi(x,t) - \Delta\varphi(x,t))dxdt.$$

Hence, using again Lemma 2.1, we have

$$\lim_{n\to+\infty}\int_0^T\int_\Omega (D^\alpha_{RL}y^\varepsilon_n(x,t)-\Delta y^\varepsilon_n(x,t))\varphi(x,t)dxdt = \int_0^T\int_\Omega (D^\alpha_{RL}y^\varepsilon(x,t)-\Delta y^\varepsilon(x,t))\varphi(x,t)dxdt.$$

This implies that

$$D^\alpha_{RL}y^\varepsilon_n-\Delta y^\varepsilon_n \rightharpoonup D^\alpha_{RL}y^\varepsilon-\Delta y^\varepsilon \quad \text{weakly in } \mathbb{D}'(Q), \tag{4.12}$$

which combining with (4.9d) gives

$$D^\alpha_{RL}y^\varepsilon-\Delta y^\varepsilon=\delta \quad \text{in } Q. \tag{4.13}$$

Therefore, we have

$$D^\alpha_{RL}y^\varepsilon_n-\Delta y^\varepsilon_n \rightharpoonup D^\alpha_{RL}y^\varepsilon-\Delta y^\varepsilon \quad \text{weakly in } L^2(Q) \tag{4.14}$$

Passing to the limit in (4.11), using (4.9a) and (4.14), we obtain

$$D^\alpha_{RL}y^\varepsilon-\Delta y^\varepsilon=u^\varepsilon \quad \text{in} \quad Q. \tag{4.15}$$

Now, we know that

$$\int_\Omega\int_0^T I^{1-\alpha}y^\varepsilon_n(x,t)\varphi(x,t)dtdx = \int_\Omega\int_0^T y^\varepsilon_n(x,s)\left(\frac{-1}{\Gamma(1-a)}\int_s^T (t-s)^{-\alpha}\varphi(x,t)dt\right)dsdx,\ \forall\varphi\in\mathbb{D}(Q),$$

and passing to the limit in the latter equality and using (4.9c) and (4.9b), we obtain

$$\begin{aligned}\int_\Omega\int_0^T \gamma\varphi(x,t)dtdx &= \int_\Omega\int_0^T y^\varepsilon(x,s)\left(\frac{-1}{\Gamma(1-a)}\int_s^T (t-s)^{-\alpha}\varphi(x,t)dt\right)dsdx\\ &= \int_\Omega\int_0^T I^{1-\alpha}y^\varepsilon(x,t)\varphi(x,t)dtdx, \quad \forall\varphi\in\mathbb{D}(Q).\end{aligned}$$

Thus,

$$I^{1-\alpha}y^\varepsilon=\gamma \quad \text{in} \quad Q,$$

which combining with (4.9c) gives

$$I^{1-\alpha}y^\varepsilon_n \rightharpoonup I^{1-\alpha}y^\varepsilon \quad \text{weakly in } L^2((0,T);H^1_0(\Omega)). \tag{4.16}$$

We have $y^\varepsilon\in L^2((0,T);H^1_0(\Omega))$ and $I^{1-\alpha}y^\varepsilon\in L^2((0,T);H^1_0(\Omega))$, then

$$y^\varepsilon=0 \quad \text{on} \quad \Sigma. \tag{4.17}$$

On the other hand, $y^\varepsilon\in L^2((0,T);H^1_0(\Omega))$ then $\Delta y^\varepsilon\in L^2((0,T);H^{-1}(\Omega))$, which implies that

$$\frac{\partial}{\partial t}I^{1-\alpha}y^\varepsilon = D^\alpha_{RL}y^\varepsilon = u^\varepsilon+\Delta y^\varepsilon\in L^2((0,T);H^{-1}(\Omega)).$$

Finally, we have

$$I^{1-\alpha}y^{\varepsilon} \in C([0,T];L^2(\Omega)).$$

This means that $I^{1-\alpha}y^{\varepsilon}(0)$ and $I^{1-\alpha}y^{\varepsilon}(T)$ exist and belong to $L^2(\Omega)$.
Now, multiplying (4.4a) by a function $\varphi \in C^{\infty}(\bar{Q})$ with $\varphi_{|\partial\Omega} = 0$ and integrating by part over Q, we obtain using Lemma 2.1

$$\begin{aligned}
&\int_0^T \int_\Omega (D^\alpha_{RL} y^\varepsilon_n(x,t) - \Delta y^\varepsilon_n(x,t))\varphi(x,t)dxdt = \\
&\int_\Omega \varphi(x,T) I^{1-\alpha} y^\varepsilon_n(x,T)dx - \int_\Omega \varphi(x,0) I^{1-\alpha} y^\varepsilon_n(x,0)dx + \\
&\int_\Omega \int_0^T y^\varepsilon_n(x,t)(-\mathcal{D}^\alpha_C \varphi(x,t) - \Delta\varphi(x,t))dxdt.
\end{aligned}$$

Passing to the limit when $n \to +\infty$ in the latter result and using (4.14), (4.9e), (4.9f) and (4.9b), we have

$$\begin{aligned}
&\int_0^T \int_\Omega (D^\alpha_{RL} y^\varepsilon(x,t) - \Delta y^\varepsilon(x,t))\varphi(x,t)dxdt = \\
&\int_\Omega \varphi(x,T)\pi_2 dx - \int_\Omega \varphi(x,0)\pi_1 dx + \\
&\int_\Omega \int_0^T y^\varepsilon(x,t)(-\mathcal{D}^\alpha_C \varphi(x,t) - \Delta\varphi(x,t))dxdt,
\end{aligned} \tag{4.18}$$

which using again Lemma 2.1 gives

$$\int_\Omega \varphi(x,T)[\pi_2 - I^{1-\alpha}y^\varepsilon(x,T)]dx - \int_\Omega \varphi(x,0)[\pi_1 - I^{1-\alpha}y^\varepsilon(x,0)]dx = 0.$$

Now, choose φ such that $\varphi(\cdot,0) = 0$ in Ω, we obtain

$$I^{1-\alpha}y^\varepsilon(\cdot,T) = \pi_2,$$

and finally, we have

$$I^{1-\alpha}y^\varepsilon(\cdot,0) = \pi_1.$$

Therefore, we have

$$I^{1-\alpha}y^\varepsilon_n(\cdot,0) \rightharpoonup I^{1-\alpha}y^\varepsilon(\cdot,0) \quad \text{weakly in} \quad L^2(\Omega), \tag{4.19a}$$

$$I^{1-\alpha}y^\varepsilon_n(\cdot,T) \rightharpoonup I^{1-\alpha}y^\varepsilon(\cdot,T) \quad \text{weakly in} \quad L^2(\Omega). \tag{4.19b}$$

Hence, passing to the limit when $n \to +\infty$ in (4.4c) and using (4.19) we can write that

$$I^{1-\alpha}y^\varepsilon(x,T) + \varepsilon I^{1-\alpha}y^\varepsilon(x,0) = y^T \quad \text{in} \quad \Omega. \tag{4.20}$$

From (4.15), (4.17) and (4.20), we can deduce that $y^\varepsilon(u^\varepsilon)$ is solution of Equation (1.4).
It follows from the lower semi-continuity of the functional J^ε, (4.9a) and (4.19a) that

$$J^\varepsilon(u^\varepsilon) \le \lim_{n\to+\infty} \inf J^\varepsilon(v_n),$$

which combining with (4.2) gives

$$J^\varepsilon(u^\varepsilon) = \inf_{v\in\mathcal{U}_{ad}} J^\varepsilon(v_n).$$

And from the strict convexity of J^ε, we have the uniqueness of the optimal control u^ε. ■

Theorem 4.2 *Let u^ε be solution of (1.4)-(1.5). Then there exists $p^\varepsilon \in C([0,T];L^2(\Omega))$ such that $(u^\varepsilon, y^\varepsilon, p^\varepsilon)$ verifies the following optimality systems:*

$$\left\{\begin{array}{rcll} D^\alpha_{RL} y^\varepsilon(x,t) - \Delta y^\varepsilon(x,t) &=& u^\varepsilon(x,t) & (x,t)\in Q,\\ y^\varepsilon(\sigma,t) &=& 0 & (\sigma,t)\in\Sigma,\\ I^{1-\alpha}y^\varepsilon(x,T) + \varepsilon I^{1-\alpha}y^\varepsilon(x,0^+) &=& y^T(x) & x\in\Omega, \end{array}\right. \tag{4.21}$$

$$\left\{\begin{array}{rcll} -\mathcal{D}^\alpha_C p^\varepsilon(x,t) - \Delta p^\varepsilon(x,t) &=& 0 & (x,t)\in Q,\\ p^\varepsilon(\sigma,t) &=& 0 & (\sigma,t)\in\Sigma,\\ \varepsilon p^\varepsilon(x,T) + p^\varepsilon(x,0) &=& I^{1-\alpha}y^\varepsilon(u^\varepsilon;x,0) - z_d & x\in\Omega, \end{array}\right. \tag{4.22}$$

and

$$\int_0^T \int_\Omega (Nu^\varepsilon(x,t) - p^\varepsilon(x,t))(v(x,t) - u^\varepsilon(x,t))dxdt \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \tag{4.23}$$

Proof. From (4.15), (4.17) and (4.20), we have (4.21). To prove (4.22) and (4.23), we use the Euler-Lagrange optimality conditions

$$\lim_{k\to 0} \frac{J^\varepsilon(u^\varepsilon + k(v-u^\varepsilon)) - J^\varepsilon(u^\varepsilon)}{k} \geq 0, \quad \forall v\in\mathcal{U}_{ad}, \tag{4.24}$$

which characterize the control u^ε.
We set $w = v - u^\varepsilon$, then from (1.6) we have

$$\begin{array}{rcl} J^\varepsilon(u^\varepsilon + kw) &=& \dfrac{1}{2}\left\|I^{1-\alpha}y^\varepsilon(u^\varepsilon,0) - z_d\right\|^2_{L^2(\Omega)} + \dfrac{k^2}{2}\left\|I^{1-\alpha}y^\varepsilon(w,0)\right\|^2_{L^2(\Omega)}\\ &+& k\left(I^{1-\alpha}y^\varepsilon(u^\varepsilon,0) - z_d, I^{1-\alpha}y^\varepsilon(w,0)\right)_{L^2(\Omega)} + \dfrac{N}{2}\left\|u^\varepsilon\right\|^2_{L^2(Q)}\\ &+& \dfrac{k^2N}{2}\left\|w\right\|^2_{L^2(Q)} + Nk(u^\varepsilon,w)_{L^2(Q)}, \end{array}$$

which implies that

$$\lim_{k\to 0}\frac{J^\varepsilon(u^\varepsilon + k(v-u^\varepsilon)) - J^\varepsilon(u^\varepsilon)}{k} =$$
$$\left(I^{1-\alpha}y^\varepsilon(u^\varepsilon,0) - z_d, I^{1-\alpha}y^\varepsilon(w,0)\right)_{L^2(\Omega)} + N(u^\varepsilon,w)_{L^2(Q)}.$$

Combining the latter result with (4.24), we obtain

$$\int_\Omega (I^{1-\alpha}y^\varepsilon(u^\varepsilon,0) - z_d)I^{1-\alpha}y^\varepsilon(v-u^\varepsilon,0)dx + N\int_\Omega\int_0^T u^\varepsilon(v-u^\varepsilon)dtdx \geq 0 \quad \forall v\in\mathcal{U}_{ad}. \tag{4.25}$$

From Corollary 3.1, we know that Equation (4.22) admits a unique solution $p^\varepsilon \in C([0,T];L^2(\Omega))$. Let $z^\varepsilon(v-u^\varepsilon)$ the associated state of $v - u^\varepsilon \in L^2(Q)$. Then from Theorem 3.1, $z^\varepsilon \in L^2((0,T);H^1_0(\Omega))$ verifies the following equation

$$D^\alpha_{RL} z^\varepsilon(x,t) - \Delta z^\varepsilon(x,t) = v(x,t) - u^\varepsilon(x,t), \tag{4.26a}$$
$$z^\varepsilon(\sigma,t) = 0, \tag{4.26b}$$
$$I^{1-\alpha}z^\varepsilon(x,T) + \varepsilon I^{1-\alpha}z^\varepsilon(x,0^+) = 0. \tag{4.26c}$$

Moreover, we know that $I^{1-\alpha}z^\varepsilon \in C([0,T];L^2(\Omega))$. Note that, from (4.26c), we have

$$I^{1-\alpha}z^\varepsilon(x,T) = -\varepsilon I^{1-\alpha}z^\varepsilon(x,0). \tag{4.27}$$

Multiplying (4.26a) by the solution p^ε of (4.22), and integrating over Q, we obtain from Lemma 2.1 and (4.27):

$$\begin{aligned}
&\int_0^T\int_\Omega (D^\alpha_{RL}z^\varepsilon(x,t) - \Delta z^\varepsilon(x,t))p^\varepsilon(x,t)dxdt \\
&= \int_\Omega p^\varepsilon(x,T)I^{1-\alpha}z^\varepsilon(x,T)dx - \int_\Omega p^\varepsilon(x,0)I^{1-\alpha}z^\varepsilon(x,0)dx \\
&= -\int_\Omega I^{1-\alpha}z^\varepsilon(x,0)[\varepsilon p^\varepsilon(x,T) + p^\varepsilon(x,0)]dx \\
&= -\int_\Omega I^{1-\alpha}z^\varepsilon(x,0)[I^{1-\alpha}y^\varepsilon(u^\varepsilon,0) - z_d]dx \\
&= \int_0^T\int_\Omega (v-u^\varepsilon)(x,t)p^\varepsilon(x,t)dxdt.
\end{aligned}$$

Hence we have

$$-\int_\Omega I^{1-\alpha}z^\varepsilon(x,0)(I^{1-\alpha}y^\varepsilon(u^\varepsilon,0) - z_d)dx = \int_0^T\int_\Omega (v-u^\varepsilon)(x,t)p^\varepsilon(x,t)dxdt,$$

which combining with (4.25), gives

$$\int_0^T\int_\Omega (v-u^\varepsilon)(x,t)p^\varepsilon(x,t)dxdt \leq N\int_\Omega\int_0^T u^\varepsilon(x,t)(v-u^\varepsilon)(x,t)dtdx.$$

And after some computations, we obtain (4.23). ■

We proved that our approximated optimal control problem has a unique solution and we gave an optimality system which characterize it. Now, we want to prove that the solution of the approximated optimal control problem (1.4) - (1.5) converges to the solution of our initial optimal control problem (1.1)-(1.2).

We have the following result:

Theorem 4.3 *Let* $(u^\varepsilon, y^\varepsilon)$ *be the solution of Problem* (1.4) - (1.5). *Then* $u^\varepsilon \in \mathbb{U}_{ad}$.

Proof. Let $(u^\varepsilon, y^\varepsilon)$ be the solution of Problem (1.4) - (1.5). Then from Theorem 3.1, $y^\varepsilon(u^\varepsilon) \in L^2((0,T);H^1_0(\Omega))$ and we have

$$\begin{aligned}
y^\varepsilon(t) &= \sum_{i=1}^{+\infty}\left\{\frac{y_i^T - \displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)u_i^\varepsilon(s)ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)\right. \\
&+ \left.\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)u_i^\varepsilon(s)ds\right\}w_i.
\end{aligned} \tag{4.28}$$

On the one hand, if we take $u^\varepsilon = 0$ in (4.28), we have

$$y_1^\varepsilon(t) = \sum_{i=1}^{+\infty} \left\{ \frac{y_i^T}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) \right\} w_i.$$

Proceeding as in (3.13), we have

$$\begin{aligned}
\|y_1^\varepsilon\|^2_{L^2((0,T);H_0^1(\Omega))} &= \int_0^T a(y_1^\varepsilon(t), y_1^\varepsilon(t))dt \\
&\leq 2\sum_{i=1}^{+\infty} \lambda_i \left| \frac{y_i^T}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} \right|^2 \int_0^T t^{2\alpha-2} E^2_{\alpha,\alpha}(-\lambda_i t^\alpha)dt \\
&\leq \frac{C^4 T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)} \sum_{i=1}^{+\infty} \frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)},
\end{aligned}$$

which implies that

$$\|y_1^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq C^2 \sqrt{\frac{T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}} \left(\sum_{i=1}^{+\infty} \frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)} \right)^{1/2}.$$

We know that $y_1^\varepsilon \in L^2((0,T);H_0^1(\Omega))$, then we have

$$\left(\sum_{i=1}^{+\infty} \frac{|y_i^T|^2}{E_\alpha^2(-\lambda_i T^\alpha)} \right)^{1/2} < \infty. \tag{4.29}$$

On the other hand, if we take $y^T = 0$ in (4.28), we have

$$\begin{aligned}
y_2^\varepsilon(t) &= \sum_{i=1}^{+\infty} \left\{ \frac{-\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha) u_i^\varepsilon(s)ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) \right. \\
&+ \left. \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha) u_i^\varepsilon(s)ds \right\} w_i.
\end{aligned}$$

Thus, proceeding as in (3.13) and in [10], we can write that

$$
\begin{aligned}
\|y_2^\varepsilon\|^2_{L^2((0,T);H_0^1(\Omega))} &= \int_0^T a(y_2^\varepsilon(t), y_2^\varepsilon(t))dt \\
&\leq 2\sum_{i=1}^{+\infty} \lambda_i \left| \frac{-\displaystyle\int_0^T E_\alpha(-\lambda_i(T-s)^\alpha)u_i^\varepsilon(s)ds}{\varepsilon + E_\alpha(-\lambda_i T^\alpha)} \right|^2 \int_0^T t^{2\alpha-2}E^2_{\alpha,\alpha}(-\lambda_i t^\alpha)dt \\
&+ 2\sum_{i=1}^{+\infty} \lambda_i \int_0^T \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)u_i^\varepsilon(s)ds \right\}^2 dt. \\
&\leq \frac{C^6 T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)} \sum_{i=1}^{+\infty} \frac{\displaystyle\int_0^T |u_i^\varepsilon(s)|^2 ds}{E^2_\alpha(-\lambda_i T^\alpha)} + \frac{4C^2T^\alpha}{\alpha-\frac{1}{2}} \sum_{i=1}^{+\infty} \int_0^T |u_i^\varepsilon(s)|^2 ds.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\|y_2^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} &\leq C^4 \sqrt{\frac{T^{3\alpha-2}}{(4\alpha-3)(1-\alpha)}} \left(\sum_{i=1}^{+\infty} \frac{\displaystyle\int_0^T |u_i^\varepsilon(s)|^2 ds}{E^2_\alpha(-\lambda_i T^\alpha)} \right)^{1/2} \\
&+ 2C\sqrt{\frac{T^\alpha}{\alpha-\frac{1}{2}}} \left(\sum_{i=1}^{+\infty} \int_0^T |u_i^\varepsilon(s)|^2 ds \right)^{1/2}.
\end{aligned}
$$

As $y_2^\varepsilon \in L^2((0,T);H_0^1(\Omega))$, we have

$$
\left(\sum_{i=1}^{+\infty} \frac{\displaystyle\int_0^T |u_i^\varepsilon(s)|^2 ds}{E^2_\alpha(-\lambda_i T^\alpha)} \right)^{1/2} < \infty. \tag{4.30}
$$

Finally, from (4.29) and (4.30), we have $u^\varepsilon \in \mathbb{U}_{ad}$. ■

We assume that y^T and v are such that Series in (3.20) converge. Therefore, we give the following result:

Theorem 4.4 *Let (u, y) be a solution of the problem (1.1)-(1.2). Let $(u^\varepsilon, y^\varepsilon)$ be the solution of Problem (1.4) - (1.5) and let p^ε be the solution of Equation (4.22). Then, there exists $p \in L^2(Q)$ such that, as $\varepsilon \to 0$, we have the following convergences:*

$$
\begin{aligned}
u^\varepsilon \to u \quad & \textit{strongly in} \quad L^2(Q) \quad \textit{and} \quad u \in \mathbb{U}_{ad}, && \text{(4.31a)} \\
y^\varepsilon \rightharpoonup y \quad & \textit{weakly in} \quad L^2((0,T);H_0^1(\Omega)), && \text{(4.31b)} \\
I^{1-\alpha}y^\varepsilon(\cdot,T) \to y^T \quad & \textit{strongly in} \quad L^2(\Omega), && \text{(4.31c)} \\
I^{1-\alpha}y^\varepsilon(\cdot,0) \rightharpoonup I^{1-\alpha}y(\cdot,0) \quad & \textit{weakly in} \quad L^2(\Omega), && \text{(4.31d)} \\
p^\varepsilon \rightharpoonup p \quad & \textit{weakly in} \quad L^2(Q). && \text{(4.31e)}
\end{aligned}
$$

Proof. For this proof, we proceed in three steps.
Step 1: Let $(u^\varepsilon, y^\varepsilon)$ be the solution of Problem (1.4) - (1.5). Combining (4.3a) and (4.9a), we know that there exists a constant $C > 0$ independent of ε such that

$$\|u^\varepsilon\|_{L^2(Q)} \leq C, \tag{4.32}$$

which implies that, there exists $\bar{u} \in L^2(Q)$ such that

$$u^\varepsilon \rightharpoonup \bar{u} \quad \text{weakly in} \quad L^2(Q), \quad \text{when } \varepsilon \to 0. \tag{4.33}$$

From Theorem 4.3, we have $u^\varepsilon \in \mathbb{U}_{ad}$ and $\mathbb{U}_{ad}$ being a closed subset of $L^2(Q)$, we have $\bar{u} \in \mathbb{U}_{ad}$.
From (4.3b) and (4.19a), we can deduce that there exists a constant $C > 0$ independent of ε such that

$$\|I^{1-\alpha} y^\varepsilon(\cdot, 0)\|_{L^2(\Omega)} \leq C, \tag{4.34}$$

which combining with (4.21), gives

$$\|y^T - I^{1-\alpha} y^\varepsilon(\cdot, T)\|_{L^2(\Omega)} \leq \varepsilon C. \tag{4.35}$$

Thus, we obtain

$$I^{1-\alpha} y^\varepsilon(\cdot, T) \to y^T \quad \text{strongly in } L^2(\Omega), \quad \text{when } \varepsilon \to 0, \tag{4.36}$$

and we proved (4.31c).
Moreover, from Remark 3.2, we know that there exists constants $K_1 > 0$ and $K_2 > 0$, independent of ε, such that

$$\|y^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq K_1 \left(\|I^{1-\alpha} y^\varepsilon(0)\|_{L^2(\Omega)} + \|u^\varepsilon\|_{L^2(Q)} \right). \tag{4.37}$$

$$\|I^{1-\alpha} y^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq K_2 \left(\|I^{1-\alpha} y^\varepsilon(0)\|_{L^2(\Omega)} + \|u^\varepsilon\|_{L^2(Q)} \right). \tag{4.38}$$

Thus, using (4.32) and (4.34), we can say that there exists a constant $C > 0$ independent of ε such that

$$\|y^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq C, \tag{4.39}$$

and

$$\|I^{1-\alpha} y^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq C. \tag{4.40}$$

Therefore, there exists $\bar{y} \in L^2((0,T); H_0^1(\Omega))$ and $\gamma \in L^2((0,T); H_0^1(\Omega))$ such that

$$y^\varepsilon \rightharpoonup \bar{y} \quad \text{weakly in} \quad L^2((0,T); H_0^1(\Omega)), \quad \text{when } \varepsilon \to 0. \tag{4.41}$$

$$I^{1-\alpha} y^\varepsilon \rightharpoonup \gamma \quad \text{weakly in} \quad L^2((0,T); H_0^1(\Omega)), \quad \text{when } \varepsilon \to 0. \tag{4.42}$$

From (4.34), we also know that there exists $\pi \in L^2(\Omega)$ such that

$$I^{1-\alpha} y^\varepsilon(\cdot, 0) \rightharpoonup \pi \quad \text{weakly in } L^2(\Omega), \quad \text{when } \varepsilon \to 0.$$

And proceeding as in the proof of Theorem 4.1, we obtain

$$I^{1-\alpha} y^\varepsilon(\cdot, 0) \rightharpoonup I^{1-\alpha} \bar{y}(\cdot, 0) \quad \text{weakly in } L^2(\Omega), \quad \text{when } \varepsilon \to 0. \tag{4.43}$$

Now, let us prove that $(\bar{u}, \bar{y})$ is solution of (1.1), in the sense of Definition 3.1.
Combining the first equation of (4.21) and (4.32), we obtain

$$\|D_{RL}^\alpha y^\varepsilon - \Delta y^\varepsilon\|_{L^2(Q)} \leq C.$$

Then, there exists $\delta \in L^2(Q)$ such that

$$D^\alpha_{RL} y^\varepsilon - \Delta y^\varepsilon \rightharpoonup \delta \quad \text{weakly in } L^2(Q), \quad \text{when } \varepsilon \to 0.$$

Proceeding as the proof of (4.15), we prove that

$$D^\alpha_{RL} y^\varepsilon - \Delta y^\varepsilon \rightharpoonup D^\alpha_{RL} \bar{y} - \Delta \bar{y} \quad \text{weakly in } L^2(Q), \quad \text{when } \varepsilon \to 0. \tag{4.44}$$

Finally,

$$D^\alpha_{RL} \bar{y} - \Delta \bar{y} = \bar{u} \quad \text{in } Q. \tag{4.45}$$

We know that

$$\begin{aligned} &\int_\Omega \int_0^T I^{1-\alpha} y^\varepsilon(x,t)\varphi(x,t)dtdx = \\ &\int_\Omega \int_0^T y^\varepsilon(x,s) \left(\frac{-1}{\Gamma(1-a)} \int_s^T (t-s)^{-\alpha} \varphi(x,t)dt \right) dsdx, \ \forall \varphi \in \mathbb{D}(Q), \end{aligned}$$

and passing to the limit in the latter equality and using (4.41) and (4.42), we obtain

$$\begin{aligned} \int_\Omega \int_0^T \gamma \varphi(x,t)dtdx &= \int_\Omega \int_0^T \bar{y}(x,s) \left(\frac{-1}{\Gamma(1-a)} \int_s^T (t-s)^{-\alpha} \varphi(x,t)dt \right) dsdx \\ &= \int_\Omega \int_0^T I^{1-\alpha} \bar{y}(x,t)\varphi(x,t)dtdx, \quad \forall \varphi \in \mathbb{D}(Q). \end{aligned}$$

Thus,

$$I^{1-\alpha} \bar{y} = \gamma \quad \text{in} \quad Q,$$

which combining with (4.42) gives

$$I^{1-\alpha} y^\varepsilon \rightharpoonup I^{1-\alpha} \bar{y} \quad \text{weakly in } L^2((0,T); H^1_0(\Omega)). \tag{4.46}$$

Combining (4.45) and the fact that $I^{1-\alpha}\bar{y} \in L^2((0,T); H^1_0(\Omega))$, we can say, proceeding as in (3.24), that $D^\alpha_{RL}\bar{y}(t) \in H^{-1}(\Omega)$. This implies that $I^{1-\alpha}\bar{y} \in C([0,T]; L^2(\Omega))$.
Now, multiplying the first equation of (4.21) by a function $\varphi \in C^\infty(\bar{Q})$ with $\varphi_{|\partial\Omega} = 0$ and $\varphi(\cdot, 0) = 0$ in Ω and integrating by part over Q, we obtain using Lemma 2.1

$$\begin{aligned} &\int_0^T \int_\Omega (D^\alpha_{RL} y^\varepsilon(x,t) - \Delta y^\varepsilon(x,t))\varphi(x,t)dxdt = \\ &\int_\Omega \varphi(x,T) I^{1-\alpha} y^\varepsilon(x,T)dx + \int_\Omega \int_0^T y^\varepsilon(x,t)(-\mathcal{D}^\alpha_C \varphi(x,t) - \Delta\varphi(x,t))dxdt. \end{aligned} \tag{4.47}$$

Passing to the limit when $\varepsilon \to 0$ in (4.47) and using (4.44), (4.36) and (4.41), we have

$$\begin{aligned} &\int_0^T \int_\Omega (D^\alpha_{RL} \bar{y}(x,t) - \Delta \bar{y}(x,t))\varphi(x,t)dxdt = \\ &\int_\Omega \varphi(x,T) y^T dx + \int_\Omega \int_0^T \bar{y}(x,t)(-\mathcal{D}^\alpha_C \varphi(x,t) - \Delta\varphi(x,t))dxdt. \end{aligned} \tag{4.48}$$

From Lemma 2.1, we can write

$$
\begin{aligned}
&\int_\Omega\int_0^T \bar{y}(x,t)(-\mathcal{D}_C^\alpha\varphi(x,t)-\Delta\varphi(x,t))dxdt = \\
&\int_0^T\int_\Omega (D_{RL}^\alpha\bar{y}(x,t)-\Delta\bar{y}(x,t))\varphi(x,t)dxdt - \int_\Omega \varphi(x,T)I^{1-\alpha}\bar{y}(x,T)dx \\
&-\left\langle \bar{y}(\sigma,t),\frac{\partial\varphi}{\partial\nu}(\sigma,t)\right\rangle_{H^{-1}((0,T);H^{-1/2}(\partial\Omega)),H_0^1((0,T);H^{1/2}(\partial\Omega))}
\end{aligned}
\tag{4.49}
$$

$$\forall\varphi\in C^\infty(\bar{Q}) \text{ with } \varphi_{|\partial\Omega}=0 \text{ and } \varphi(\cdot,0)=0 \text{ in } \Omega.$$

Combining (4.48) and (4.49), we obtain

$$
\begin{aligned}
-\int_\Omega \varphi(x,T)y^T dx &= \int_\Omega \varphi(x,T)I^{1-\alpha}\bar{y}(x,T)dx \\
&- \left\langle \bar{y}(\sigma,t),\frac{\partial\varphi}{\partial\nu}(\sigma,t)\right\rangle_{H^{-1}((0,T);H^{-1/2}(\partial\Omega)),H_0^1((0,T);H^{1/2}(\partial\Omega))}
\end{aligned}
\tag{4.50}
$$

$$\forall\varphi\in C^\infty(\bar{Q}) \text{ with } \varphi_{|\partial\Omega}=0 \text{ and } \varphi(\cdot,0)=0 \text{ in } \Omega.$$

Now, choose φ such that $\varphi(\cdot,T)=0$ in Ω, then we have

$$-\left\langle \bar{y}(\sigma,t),\frac{\partial\varphi}{\partial\nu}(\sigma,t)\right\rangle_{H^{-1}((0,T);H^{-1/2}(\partial\Omega)),H_0^1((0,T);H^{1/2}(\partial\Omega))}=0$$

which implies that

$$\bar{y}=0 \quad \text{on} \quad \Sigma. \tag{4.51}$$

Finally, we have

$$\int_\Omega \varphi(x,T)y^T dx = \int_\Omega \varphi(x,T)I^{1-\alpha}\bar{y}(x,T)dx$$

Thus

$$I^{1-\alpha}\bar{y}(x,T)=y^T \quad \text{in} \quad \Omega. \tag{4.52}$$

From (4.45), (4.51), (4.52) and the fact that $D_{RL}^\alpha\bar{y}(t)\in H^{-1}(\Omega)$ and $I^{1-\alpha}\bar{y}\in C([0,T];L^2(\Omega))$, we can deduce that $(\bar{u},\bar{y})$ is a strong solution of (1.1), in the sense of Definition 3.1.
Step 2: We prove that $(\bar{u},\bar{y})$ is solution of problem (1.1)-(1.2), and the functional J^ε converges to the functional J, when ε tends to 0.
Let (u,y) be the solution of problem (1.1) - (1.2). From $\bar{y}(\bar{u})$ be the solution of (1.1) and $\bar{u}\in\mathbb{U}_{ad}$, we have

$$J(u,y)\leq J(\bar{u},\bar{y}) \tag{4.53}$$

As u^ε is the solution of (1.5), and $u\in\mathbb{U}_{ad}$, we can write that

$$J^\varepsilon(u^\varepsilon)\leq J^\varepsilon(u).$$

Hence

$$\frac{1}{2}\left\|I^{1-\alpha}y^\varepsilon(\cdot,0)-z_d\right\|_{L^2(\Omega)}^2+\frac{N}{2}\left\|u^\varepsilon\right\|_{L^2(Q)}^2\leq\frac{1}{2}\left\|I^{1-\alpha}y(\cdot,0)-z_d\right\|_{L^2(\Omega)}^2+\frac{N}{2}\left\|u\right\|_{L^2(Q)}^2, \tag{4.54}$$

and passing to the limit in the latter estimation, using (4.43) and (4.33), we obtain

$$
\begin{aligned}
&J(\bar{u},\bar{y}) = \frac{1}{2}\left\| I^{1-\alpha}\bar{y}(\cdot,0) - z_d \right\|^2_{L^2(\Omega)} + \frac{N}{2}\left\|\bar{u}\right\|^2_{L^2(Q)} \leq \\
&\frac{1}{2}\left\| I^{1-\alpha}y(\cdot,0) - z_d \right\|^2_{L^2(\Omega)} + \frac{N}{2}\left\|u\right\|^2_{L^2(Q)} = J(u,y),
\end{aligned} \tag{4.55}
$$

Therefore, combining (4.53) and (4.55), we have

$$
J(u,y) \leq J(\bar{u},\bar{y}) \leq J(u,y),
$$

which implies that

$$
(\bar{u},\bar{y}) = (u,y). \tag{4.56}
$$

Moreover, as $J(\bar{u},\bar{y}) = \lim\limits_{\varepsilon\to 0} J^{\varepsilon}(u^{\varepsilon})$, we have

$$
J^{\varepsilon} \to J \quad \text{ when } \varepsilon \to 0. \tag{4.57}
$$

Thus, combining (4.41), (4.43) and (4.56), we obtain (4.31b) and (4.31d).
Combining again (4.33) and (4.56), we have as $\varepsilon \to 0$

$$
u^{\varepsilon} \rightharpoonup u \quad \text{ weakly in } \quad L^2(Q). \tag{4.58}
$$

From (4.57), we can write that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\left(\frac{1}{2}\left\| I^{1-\alpha}y^{\varepsilon}(\cdot,0) - z_d \right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|u^{\varepsilon}\right\|^2_{L^2(Q)}\right) = \\
&\frac{1}{2}\left\| I^{1-\alpha}y(\cdot,0) - z_d \right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|u\right\|^2_{L^2(Q)}.
\end{aligned} \tag{4.59}
$$

Using (4.31d) and (4.58), we obtain

$$
\left\| I^{1-\alpha}y(\cdot,0) - z_d \right\|^2_{L^2(\Omega)} \leq \lim_{\varepsilon\to 0}\left\| I^{1-\alpha}y^{\varepsilon}(\cdot,0) - z_d \right\|^2_{L^2(\Omega)}
$$

and

$$
\left\|u\right\|^2_{L^2(Q)} \leq \lim_{\varepsilon\to 0}\left\|u^{\varepsilon}\right\|^2_{L^2(Q)},
$$

which combining with (4.59) gives

$$
\left\| I^{1-\alpha}y(\cdot,0) - z_d \right\|^2_{L^2(\Omega)} = \lim_{\varepsilon\to 0}\left\| I^{1-\alpha}y^{\varepsilon}(\cdot,0) - z_d \right\|^2_{L^2(\Omega)} \tag{4.60a}
$$

$$
\left\|u\right\|^2_{L^2(Q)} = \lim_{\varepsilon\to 0}\left\|u^{\varepsilon}\right\|^2_{L^2(Q)}. \tag{4.60b}
$$

We have

$$
\left\|u^{\varepsilon} - u\right\|^2_{L^2(Q)} = \left\|u^{\varepsilon}\right\|^2_{L^2(Q)} + \left\|u\right\|^2_{L^2(Q)} - 2\int_Q u^{\varepsilon}u\,dxdt.
$$

Passing to the limit in the latter equality and using (4.58) and (4.60b), we obtain when $\varepsilon \to 0$

$$
\lim_{\varepsilon\to 0}\left\|u^{\varepsilon} - u\right\|^2_{L^2(Q)} = 0.
$$

Therefore, we have

$$u^\varepsilon \to u \quad \text{strongly in} \quad L^2(Q),$$

and we proved (4.31a).

Step 3: From Theorem 4.3, we have $u^\varepsilon \in \mathbb{U}_{ad}$ then $Int(\mathbb{U}_{ad}) \neq \emptyset$ and we know that there are $w \in Int(\mathbb{U}_{ad})$ and $r > 0$, such that

$$\|v - w\|_{L^2(Q)} < r \quad \text{implies that } v \in \mathbb{U}_{ad}.$$

Since $\mathbb{U}_{ad}$ is a closed and convex subset of $L^2(Q)$ with nonempty interior , we have $\mathbb{U}_{ad} = \overline{Int(\mathbb{U}_{ad})}$. Thus, there exists a minimizing sequence $\{v_n^\varepsilon\}_{n\in\mathbb{N}}$ of J^ε in $Int(\mathbb{U}_{ad})$ associated with a state Φ_n^ε which verifies the following equation

$$\left\{\begin{array}{rcl} D_{RL}^\alpha \Phi_n^\varepsilon(x,t) - \Delta\Phi_n^\varepsilon(x,t) &=& v_n^\varepsilon(x,t) \\ \Phi_n^\varepsilon(\sigma,t) &=& 0 \\ I^{1-\alpha}\Phi_n^\varepsilon(x,T) + \varepsilon I^{1-\alpha}\Phi_n^\varepsilon(x,0^+) &=& y^T, \end{array}\right. \tag{4.61}$$

since $v_n^\varepsilon \in \mathbb{U}_{ad}$.

Proceeding as in the proof of Theorem 4.1, we know that there exists a constant $C > 0$ independent of n and ε such that

$$\|v_n^\varepsilon\|_{L^2(Q)} \leq C, \tag{4.62}$$

$$\|I^{1-\alpha}\Phi_n^\varepsilon(\cdot,0)\|_{L^2(\Omega)} \leq C, \tag{4.63}$$

$$\|\Phi_n^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq C\|y^T\|_{L^2(\Omega)}. \tag{4.64}$$

From (4.32) and (4.62), we obtain

$$\|v_n^\varepsilon - u^\varepsilon\|_{L^2(Q)} \leq C, \tag{4.65}$$

where $C > 0$ is a constant independent of n and ε. Let y^ε be the state associated to the optimal control u^ε. Set $z_n^\varepsilon := y^\varepsilon - \Phi_n^\varepsilon$, then using (4.61) and (4.21), we can say that z_n^ε is solution of

$$\left\{\begin{array}{rcl} D_{RL}^\alpha z_n^\varepsilon(x,t) - \Delta z_n^\varepsilon(x,t) &=& u^\varepsilon(x,t) - v_n^\varepsilon(x,t) \\ z_n^\varepsilon(\sigma,t) &=& 0 \\ I^{1-\alpha}z_n^\varepsilon(x,T) + \varepsilon I^{1-\alpha}z_n^\varepsilon(x,0^+) &=& 0, \end{array}\right. \tag{4.66}$$

From (4.63) and (4.3b), we obtain

$$\|I^{1-\alpha}z_n^\varepsilon(\cdot,0)\|_{L^2(\Omega)} = \|I^{1-\alpha}y^\varepsilon(\cdot,0) - I^{1-\alpha}\Phi_n^\varepsilon(\cdot,0)\|_{L^2(\Omega)} \leq C \tag{4.67}$$

where $C > 0$ is a constant independent of n and ε.

Multiplying the first equation of (4.66) by p^ε be the solution of (4.22), and integrating by parts over Q we have, using Lemma 2.1

$$\begin{aligned} &\int_0^T\int_\Omega (D_{RL}^\alpha z_n^\varepsilon(x,t) - \Delta z_n^\varepsilon(x,t))p^\varepsilon(x,t)dxdt \\ &= \int_\Omega p^\varepsilon(x,T)I^{1-\alpha}z_n^\varepsilon(x,T)dx - \int_\Omega p^\varepsilon(x,0)I^{1-\alpha}z_n^\varepsilon(x,0)dx \\ &= \int_\Omega\int_0^T (u^\varepsilon(x,t) - v_n^\varepsilon(x,t))p^\varepsilon(x,t))dxdt. \end{aligned} \tag{4.68}$$

However, from (4.66) and (4.22) we have

$$I^{1-\alpha}z_n^\varepsilon(x,T) = -\varepsilon I^{1-\alpha}z_n^\varepsilon(x,0) \quad \text{and} \quad p^\varepsilon(x,T) = \frac{1}{\varepsilon}(I^{1-\alpha}y^\varepsilon(u^\varepsilon;x,0) - z_d - p^\varepsilon(x,0))$$

which combining with (4.68) gives

$$\int_\Omega I^{1-\alpha}z_n^\varepsilon(x,0)(z_d - I^{1-\alpha}y^\varepsilon(u^\varepsilon;x,0))dx = \int_\Omega\int_0^T(u^\varepsilon(x,t) - v_n^\varepsilon(x,t))p^\varepsilon(x,t))dxdt$$

Using the Cauchy-Schwartz inequality, (4.67) and (4.34),we can write that

$$\left|\int_\Omega\int_0^T(u^\varepsilon(x,t) - v_n^\varepsilon(x,t))p^\varepsilon(x,t))dxdt\right| \leq C, \tag{4.69}$$

where $C > 0$ is a constant independent of n and ε.
Let $v \in \mathbb{U}_{ad}$ be such that $\|v - v_n^\varepsilon\|_{L^2(Q)} \leq r$. Therefore, we have

$$\begin{aligned}
&\int_Q(Nu^\varepsilon(x,t) - p^\varepsilon(x,t))(u(x,t) - u^\varepsilon(x,t))dxdt \\
&= \int_Q(Nu^\varepsilon(x,t) - p^\varepsilon(x,t))(u(x,t) - v(x,t))dxdt \\
&+ \int_Q(Nu^\varepsilon(x,t) - p^\varepsilon(x,t))(v(x,t) - u^\varepsilon(x,t))dxdt \\
&= \int_Q Nu^\varepsilon(x,t)(u(x,t) - v(x,t))dxdt - \int_Q p^\varepsilon(x,t)(u(x,t) - v(x,t))dxdt \\
&+ \int_Q(Nu^\varepsilon(x,t) - p^\varepsilon(x,t))(v(x,t) - u^\varepsilon(x,t))dxdt.
\end{aligned}$$

Setting

$$X_\varepsilon := \int_Q Nu^\varepsilon(x,t)(u(x,t) - v(x,t))dxdt + \int_Q(Nu^\varepsilon(x,t) - p^\varepsilon(x,t))(v(x,t) - u^\varepsilon(x,t))dxdt,$$

we can write

$$\int_Q(Nu^\varepsilon(x,t) - p^\varepsilon(x,t))(u(x,t) - u^\varepsilon(x,t))dxdt = X_\varepsilon - \int_Q p^\varepsilon(x,t)(u(x,t) - v(x,t))dxdt. \tag{4.70}$$

Taking $v = v_n^\varepsilon$ in (4.70), we obtain

$$\int_Q(Nu^\varepsilon(x,t) - p^\varepsilon(x,t))(u(x,t) - u^\varepsilon(x,t))dxdt = X_\varepsilon^1 - \int_Q p^\varepsilon(x,t)(u(x,t) - v_n^\varepsilon(x,t))dxdt, \tag{4.71}$$

where

$$\begin{aligned}
X_\varepsilon^1 \quad &:= \quad \int_Q Nu^\varepsilon(x,t)(u(x,t) - v_n^\varepsilon(x,t))dxdt \\
&+ \quad \int_Q(Nu^\varepsilon(x,t) - p^\varepsilon(x,t))(v_n^\varepsilon(x,t) - u^\varepsilon(x,t))dxdt \\
&:= \quad \int_Q Nu^\varepsilon(x,t)(u(x,t) - v_n^\varepsilon(x,t))dxdt + \int_Q Nu^\varepsilon(x,t)(v_n^\varepsilon(x,t) - u^\varepsilon(x,t))dxdt \\
&- \quad \int_Q p^\varepsilon(x,t)(v_n^\varepsilon(x,t) - u^\varepsilon(x,t))dxdt.
\end{aligned}$$

However, from (4.65), (4.69) and (4.32), we have

$$|X_\varepsilon^1| \leq 2NCr + r \quad \text{with} \quad \|u - v_n^\varepsilon\|_{L^2(Q)} \leq r, \ \forall u \in \mathbb{U}_{ad}. \tag{4.72}$$

Hence, combining (4.71), (4.23) and (4.72), we obtain

$$\left| \int_Q p^\varepsilon(x,t)(v_n^\varepsilon(x,t) - u^\varepsilon(x,t))dxdt \right| \leq |X_\varepsilon^1| \leq C(N,r),$$

where $C(N,r) = 2NCr + r$ and $\|u - v_n^\varepsilon\|_{L^2(Q)} \leq r, \ \forall u \in \mathbb{U}_{ad}$.

Thus, from the latter estimation, we know that there exists a constant $C > 0$ independent of ε such that

$$\|p^\varepsilon\|_{L^2(Q)} \leq C. \tag{4.73}$$

Then there exists $p \in L^2(Q)$ such that

$$p^\varepsilon \rightharpoonup p \quad \text{weakly in} \quad L^2(Q), \quad \text{as } \varepsilon \to 0,$$

and we proved (4.31e). ■

Theorem 4.5 *Let (u,y) be the solution of the problem* (1.1)-(1.2)*. Then there exists $p \in L^2(Q)$ such that (u,y,p) verifies the following optimality systems:*

$$\left\{ \begin{array}{rcll} D_{RL}^\alpha y(x,t) - \Delta y(x,t) & = & u(x,t) & (x,t) \in Q, \\ y(\sigma,t) & = & 0 & (\sigma,t) \in \Sigma, \\ I^{1-\alpha} y(x,T) & = & y^T(x) & x \in \Omega, \end{array} \right. \tag{4.74}$$

$$\left\{ \begin{array}{rcll} -\mathcal{D}_C^\alpha p(x,t) - \Delta p(x,t) & = & 0 & (x,t) \in Q, \\ p(\sigma,t) & = & 0 & (\sigma,t) \in \Sigma, \end{array} \right. \tag{4.75}$$

and

$$\int_0^T \int_\Omega (Nu(x,t) - p(x,t))(v(x,t) - u(x,t))dxdt \geq 0, \quad \forall v \in \mathbb{U}_{ad}. \tag{4.76}$$

Proof. Combining (4.45), (4.51), (4.52) and (4.56), we obtain (4.74).
Now, multiplying the first equation of (4.22), by a function $\varphi \in \mathbb{D}(Q)$ and integrating by part over Q we obtain using Lemma 2.1

$$\begin{array}{l} \displaystyle\int_0^T (-\mathcal{D}_C^\alpha p^\varepsilon(x,t) - \Delta p^\varepsilon(x,t))\varphi(x,t)dxdt = \\ \displaystyle\int_0^T \int_\Omega (D_{RL}^\alpha \varphi(x,t) - \Delta\varphi(x,t))p^\varepsilon(x,t)dxdt = 0. \end{array}$$

Passing to the limit in the latter result, using (4.31e), we have

$$\begin{array}{l} \displaystyle\int_0^T \int_\Omega (-\mathcal{D}_C^\alpha p^\varepsilon(x,t) - \Delta p^\varepsilon(x,t))\varphi(x,t)dxdt = \\ \displaystyle\int_0^T \int_\Omega (D_{RL}^\alpha \varphi(x,t) - \Delta\varphi(x,t))p(x,t)dxdt = 0. \end{array}$$

Using again Lemma 2.1, we can write

$$\int_0^T \int_\Omega (-\mathcal{D}_C^\alpha p^\varepsilon(x,t) - \Delta p^\varepsilon(x,t))\varphi(x,t)dxdt = \int_0^T (-\mathcal{D}_C^\alpha p(x,t) - \Delta p(x,t))\varphi(x,t)dxdt = 0,$$

which implies that

$$-\mathcal{D}_C^\alpha p(x,t) - \Delta p(x,t) = 0 \quad \text{in} \quad Q. \tag{4.77}$$

We have $p \in L^2((0,T); L^2(\Omega))$ then $p_t = \dfrac{\partial p}{\partial t} \in H^{-1}((0,T); L^2(\Omega))$. Therefore, from (4.77), we have

$$\Delta p = -\mathcal{D}_C^\alpha p = I_+^{1-\alpha} p_t \in H^{-1}((0,T); L^2(\Omega)).$$

Thus $p(t) \in L^2(\Omega)$ and $\Delta p(t) \in L^2(\Omega)$ then $p_{|\partial\Omega}$ exists and belong to $H^{-1/2}(\Omega)$.
Combining the latter result with the second equation of (4.22) and (4.31e), we obtain

$$p = 0 \quad \text{on } \Sigma. \tag{4.78}$$

Then from (4.77) and (4.78), we have (4.75).
Finally, passing to the limit when $\varepsilon \to 0$ in (4.23) and using (4.31a) and (4.31e), we deduce (4.76). ■

5 Conclusion

In this paper, we have studied an optimal control problem associated to an ill-posed fractional diffusion equation, where the derivative is understood in Riemann-Liouville sense. To solve our problem, we used the quasi-reversibility method. This work is our first application of the results which we obtained in our recent work [10]. Using Euler-Lagrange optimality conditions, we characterized our optimal control by an optimality system. Then, it would be interesting to verify this latter optimality system by numerical experiments.

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